JACK DE WET COMPETITION 2013

The Centre of Theoretical and Mathematical Physics at the University of Cape Town announces the Jack de Wet Student Competition 2013. The Department of Physics and the Department of Mathematics at the University of Cape Town sponsor the competition with R2000 to be awarded for the best and most elegant solution to the problem set out below. The winner will be invited to UCT for a presentation and award ceremony.

Science students up to and including Masters Level who are registered at a university or a comparable institution of tertiary education in South Africa are eligible to participate. Contributions must be submitted by 4pm Saturday, 15 February 2013, by email, preferably in pdf form, to the secretary of the physics department of UCT, Margaret Maich at margaret.maich@uct.ac.za. We require the entrant to provide his or her study record to allow us to verify academic affiliation. Candidates have to attest that their contributions represent own work.

Holographic QCD: strong interactions from a gravity dual

: Set by Dr. W. A. Horowitz, Department of Physics, UCT, wa.horowitz@uct.ac.za:

"Those who are not shocked when they first come across quantum theory cannot possibly have understood it." - Niels Bohr

1 Statement of the Problem

Compute the energy momentum tensor $T^{\mu\nu}$ for a glueball –a bound state of gluons, here used generically to refer to the interaction carriers of a Yang Mills theory– in strongly-coupled $\mathcal{N} = 4$ super-Yang-Mills in an arbitrary number of spatial dimensions D.

2 Introduction

The AdS/CFT correspondence [1, 2, 3], also known as the Maldacena conjecture, is the most exciting development in theoretical physics of the past 30 years. The correspondence relates the physics of a field theory in D dimensions to a string theory in $D + 1$ dimensions (really the product of a $D + 1$ dimensional space with a compact manifold). The miracle of the conjecture is that the dual descriptions swap strong and weak coupling limits: when the string theory is strongly-coupled the field theory is weakly-coupled, and *vice versa*. This exchange of limits is extraordinarily useful because physicists often have very few tools to explore the physics in strongly-coupled regimes; however, there are well-defined means of computation in the limit of weak-coupling. For instance, Feynman diagram techniques

are extremely useful when a field theory is in the weakly-coupled limit; on the other hand, the weakly-coupled limit of string theory is classical gravity (i.e. Einsteinian general relativity). We can therefore use the AdS/CFT correspondence to quantitatively derive the properties of objects in limits heretofore inaccessible to physics. One uses a "dictionary" to translate between the dual pictures.
In this problem, you will determine some characteristics of a glueball in strongly-coupled $\mathcal{N} = 4$ super-Yang-Mills (SYM). SYM is a cousin of quantum chromodynamics, the theory of the strong force. A glueball is a particle consisting only of gluons and no quarks; glueballs have been conjectured to exist in QCD (and some of their properties predicted using lattice gauge theory) but have yet to be observed experimentally.

3 Detailed Description of the Problem

According to the AdS/CFT dictionary, a glueball in $D + 1$-dimensional SYM corresponds to a point particle in $D + 2$-dimensional anti-de-Sitter space[1],

$$ds^2 = g_{mn}^{AdS^{D+2}} dx^m dx^n = \frac{L^2}{z^2}\Big(dt^2 + \sum_{i=1}^{D+1}(dx^i)^2\Big). \tag{2}$$

According to holographic renormalization [5, 6], the energy-momentum tensor $T_{\mu\nu}$ in the strongly-coupled field theory is related to the metric perturbations $h_{mn} = g_{mn} - g_{mn}^{AdS^{D+2}}$ on the boundary of AdS^{D+2} space,

$$T_{\mu\nu} = \frac{(D+1)L^D}{16\pi G_{D+2}} \lim_{z\to 0} \frac{h_{\mu\nu}}{z^{D+1}}. \tag{3}$$

Note that we use Greek indices for the $D + 1$ dimensions in the field theory and Roman indices for the $D + 2$ dimensional AdS gravity theory and that the boundary of the AdS space is the usual $\mathbb{R}^{D,1}$ Minkowski space,

$$ds^2 = g_{\mu\nu}^{\mathbb{R}^{D,1}} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu = -(dt)^2 + \sum_{i=1}^{D}(dx^i)^2. \tag{4}$$

Your task, then, is to compute the full $g_{mn} = g_{mn}^{AdS^{D+2}} + h_{mn}$ from Einstein's equations coming from coupling a probe point particle to gravity. It turns out that this is one of the few exactly solvable problems

[1] You can see that (2) solves Einstein's equations in empty space by extremizing the action

$$S_{AdS} = \frac{1}{16\pi G_{D+2}} \int d^{D+2}x \sqrt{g}\left[R + \frac{D(D+1)}{L^2}\right] \tag{1}$$

with respect to the metric g_{mn}, where R is the usual Ricci scalar [4].

in general relativity, where one takes a massless point particle p of energy E travelling at a fixed depth $z \equiv x_{D+1} = z_0$. Fig. 1 shows a visual representation of the problem: the particle p moves in the AdS^{D+2} bulk, emitting gravitons that slightly perturb the metric; one then uses Eq. (3) of the AdS/CFT dictionary to translate these perturbations in the string theory to the energy-momentum tensor that we are interested in that exists in the dual field theory.

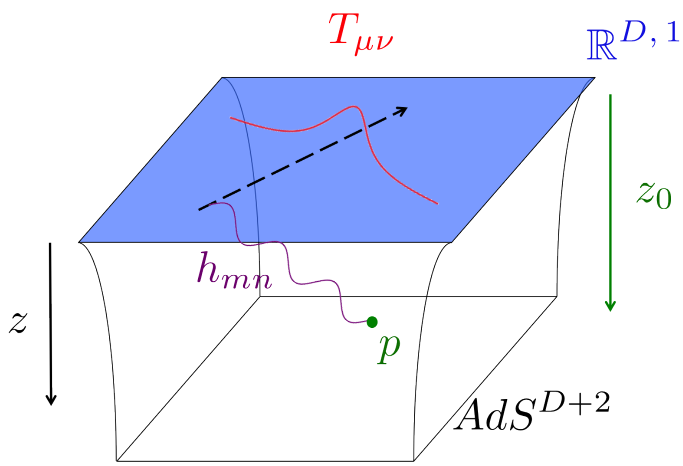

Figure 1: Schematic setup of the problem.

To compute the full g_{mn}, first derive the motion of the point particle p of energy E and constant depth $z = z_0$ in empty AdS^{D+2} space; i.e., compute the motion of the point particle in the metric given by (2). (Not simultaneously and self-consistently solving for the metric and the motion of the point particle as we are doing here is known as the probe limit.) One may do this by extremizing the action

$$S_p = \int d\eta \left[\frac{1}{2e} g_{mn}^{AdS^{D+2}} \frac{dX^m}{d\eta} \frac{dX^n}{d\eta} - \frac{e}{2} m^2 \right], \tag{5}$$

with respect to X^m and e where $g_{mn}^{AdS^{D+2}}$ is fixed and given by (2), e is the einbein on the point particle worldline, and the $X^m(\eta) : \mathbb{R} \to AdS^{D+2}$ are the embedding functions which parameterize the point particle worldline in AdS space [7, 8].

Then compute the perturbation to the metric caused by the presence of the point particle propagating with the motion you just found (this perturbation is known as the backreaction of the particle on the metric). One may compute this backreaction from the coupling of the probe to gravity by extremizing the action

$$S = S_{AdS} + S_p = \frac{1}{16\pi G_{D+2}} \int d^{D+2}x \sqrt{g} \left[R + \frac{D(D+1)}{L^2} \right] + \int d\eta \left[\frac{1}{2e} g_{mn} \frac{dX^m}{d\eta} \frac{dX^n}{d\eta} - \frac{e}{2} m^2 \right] \tag{6}$$

with respect to g_{mn}. Once you have the equations of motion for the full g_{mn}, take $g_{mn} = g_{mn}^{AdS^{D+2}} + h_{mn}$ and solve for the h_{mn} perturbation of the metric.

Once you have the full $T^{\mu\nu}$, confirm that $\int d^D x\, T^{00} = E$. Compute the apparent size of the glueball from $\int d^D x\, r^2 T^{00}$.

Possibly useful hints:

- the calculation may be easier in places if lightcone coordinates are used, $x^{\pm} = t \pm x$
- the symmetries of the generalized Laplace equation that you will have to solve are perhaps best represented/found from the geodesic of the (sub)manifold involved, which might be a hyperboloid.

References

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