Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #8

1. Convince yourself that

$$G_{0R}(\boldsymbol{x},\, \boldsymbol{x}';\, E \,=\, k^2/2m) = -\,\frac{2m}{4\pi}\frac{e^{ikr}}{r}, \tag{1}$$

where $r \,=\, |\boldsymbol{x} - \boldsymbol{x}'|$, is the Green's function for the Helmholtz wave equation

$$(\Delta_{\boldsymbol{x}} + k^2)G_{0R}(\boldsymbol{x},\, \boldsymbol{x}';\, E) \,=\, 2m\delta^n(\boldsymbol{x} - \boldsymbol{x}') \tag{2}$$

in 3 spatial dimensions. To do so, first take $\boldsymbol{x}' \,=\, 0$ and check that for $r \neq 0$ $\Delta \exp(ikr)/r \,=\, -k^2 \exp(ikr)/r$ (Mathematica is a quite useful tool here; include a printout if you do all the steps within the software). Next show that the volume integral of an epsilon ball centered around $r \,=\, 0$ yields the correct normalization for the delta functions on the LHS and RHS of Eq. (2).

Bonus: show that in n dimensions the retarded Green's function of the Helmholtz wave equation is

$$G_{0R}(\boldsymbol{x},\, \boldsymbol{x}'; E) = \frac{i^3 m}{2}\left(\frac{\pi k}{2r}\right)^{\frac{n}{2}-1} H^{(1)}_{\frac{n}{2}-1}(kr), \tag{3}$$

where $r \,=\, |\boldsymbol{x} - \boldsymbol{x}'|$. To do this, start with Eq. (2) and take $\boldsymbol{x}' \,=\, 0$. In spherical coordinates in n dimensions the Laplacian is

$$\Delta = \frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right) + \frac{1}{r^2}\Delta_{S^{n-1}}, \tag{4}$$

where $\Delta_{S^{n-1}}$ is the Laplace (or Laplace-Beltrami) operator on the $n-1$ dimensional sphere; i.e. $\Delta_{S^{n-1}}$ takes care of the angular derivatives of the Laplacian in spherical coordinates. However, notice the spherical symmetry that the original equation displays: we are searching for a G_{0R} over all space with a δ-function at the origin. Therefore $G_{0R}(\boldsymbol{x};\, E) \,=\, G_{0R}(r;\, E)$ and all the angular derivatives in the problem drop out. For $r \neq 0$ the equation we wish to solve is

$$\left(\frac{1}{r^{n-1}}\partial_r\left(r^{n-1}\partial_r\right) + k^2\right)G(r) = 0 \tag{5}$$

$$\Rightarrow \quad G''(r) + \frac{n-1}{r}G'(r) + k^2 G(r) = 0 \tag{6}$$

where I've temporarily dropped some of the notation on G for convenience. Notice that this looks very similar to Bessel's Equation,

$$x^2 y'' + x y' + (x^2 - a^2) y = 0, \tag{7}$$

which implies that $y(x)$ is a Bessel function of order a. The difference is that Eq. (6) is not dimensionless and there is an $n-1$ multiplying the second term. It is generally a good idea to make your differential equations dimensionless; we will do so here by defining a new variable, $\rho = k\,r$, and function, $P(\rho) \equiv G(r)$. Show that Eq. (6) is equivalent, then, to the differential equation

$$\rho^2 P'' + (n-1)\rho P' + \rho^2 P = 0 \tag{8}$$

for P. Now let's take care of that pesky $n-1$ by defining a final function $P(\rho) \equiv \rho^\alpha R(\rho)$. Show that by taking $\alpha = (2-n)/2$ $R(\rho)$ is a solution to Bessel's Equation.

For the physical problem we are interested in, we seek solutions that, at large r, are outgoing waves; i.e. $G_{0R} \sim \exp(ir)$. The asymptotic form of the Hankel function of the first kind, a solution of Bessel's Equation, is

$$H_\nu^{(1)}(z) \sim \sqrt{\frac{2}{\pi\, z}} e^{i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}, \tag{9}$$

is exactly of this type (see, e.g., Abramowitz and Stegun)[1]. Therefore we have that

$$G_{0R}(r;\, E) \propto (kr)^{1-\frac{n}{2}} H^{(1)}_{\frac{n}{2}-1}(kr). \tag{11}$$

In order to set the proportionality constant, integrate an ϵ ball centered around the origin (see, e.g., Abramowitz and Stegun for the small z expansion of the Hankel function of the first kind).

2. The Yukawa potential describes a physical setup in which a normal Coulomb charge is screened, such as in a plasma, or when a force is transmitted by a massive scalar particle (such as the pion). Suppose that one scatters particles off of scattering centers described by a Yukawa potential,

$$V(\boldsymbol{x}) = \frac{e^{-\mu|\boldsymbol{x}|}}{|\boldsymbol{x}|} = \frac{e^{-\mu\, r}}{r}. \tag{12}$$

[1]Solutions of the form $\sim \exp(-ir)$ are incoming waves. One may see the incoming ($-$) and outgoing ($+$) nature of these solutions by including the time dependence. Then these solutions look like $\exp\left(i(\pm r - Et)\right)$. A point of constant amplitude on the wave requires that the argument of the exponential is constant, $\pm r - Et = \text{const.}$, implies that for the $+$ solution r increases with increasing t whereas for the $-$ solution r decreases with increasing t. Note that the asymptotic expansion of the Hankel function of the second kind is

$$H_\nu^{(2)}(z) \sim \sqrt{\frac{2}{\pi\, z}} e^{-i\left(z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right)}, \tag{10}$$

which, therefore, corresponds to incoming spherical waves.

Depending on the physics, the factor μ in the Yukawa potential is interpreted as the mass of the scalar particle or $1/\mu$ as the characteristic screening distance for the Coulomb charge. Let's compute the differential cross section for this very important potential in the First Born Approximation. Recall that we need to compute $f(\boldsymbol{x},\, \boldsymbol{k'})$, which, to leading order, is related to the Fourier transform of the potential,

$$f(\boldsymbol{k},\, \boldsymbol{k'}) = -\frac{2m}{4\pi}\tilde{V}(\boldsymbol{q}), \tag{13}$$

where $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$ and

$$\tilde{V}(\boldsymbol{q}) = \int d^3x'\, e^{i\boldsymbol{x}'\cdot\boldsymbol{q}} V(\boldsymbol{x}'). \tag{14}$$

In order to evaluate the Fourier transform, we employ a common trick used when integrating an exponential $\exp(i\boldsymbol{x}'\cdot\boldsymbol{q})$ over a dummy variable over all space: since we are integrating over all space a dummy variable we can choose the orientation of the dummy variable coordinate system however we want. In particular, we can choose a smart coordinate system in which the z' direction of the $\boldsymbol{x}'$ system is oriented along the z direction of the $\boldsymbol{q}$ vector. In this case $\boldsymbol{x}'\cdot\boldsymbol{q} = r' q \cos(\theta')$. Integrating over spherical coordinates in the $\{\boldsymbol{x}'\} = (r',\, \theta',\, \phi')$ system, we can evaluate the integrals analytically if we do the θ' integral then the r' integral (the ϕ' integral is trivial). Show that

$$\tilde{V}(\boldsymbol{q}) = \frac{4\pi}{q^2 + \mu^2}, \tag{15}$$

where, as usual, $q = |\boldsymbol{q}|$. If we take our incoming particles to be along the z direction, then the $\boldsymbol{k}'$ vector will make an angle θ with respect to the $\boldsymbol{k}$ vector. Show that in this coordinate system

$$|\boldsymbol{k} - \boldsymbol{k}'| = \sqrt{2k^2\big(1 - \cos(\theta)\big)}. \tag{16}$$

Putting the pieces together, then, show that the differential cross section for the Yukawa potential in 3D is

$$\frac{d\sigma}{d\Omega} = \frac{4m^2}{\Big[k^2\big(1 - \cos(\theta)\big) + \mu^2\Big]^2}. \tag{17}$$