Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #6

1. Time-independent non-degenerate perturbation theory in a spin system. Suppose one has an electron in a magnetic field with $\boldsymbol{B} = B_0\hat{\imath} + B\hat{k}$. Compute the first and second order corrections to the energy levels and the first order correction to the wavefunctions for $B_0 \ll B$. For those of you who feel a little weak on the theory of spin, some further optional work might be useful:

   (a) Let's calculate ourselves the wavefunctions of spin. First recall that the fundamental commutation relation for the spin operators is

   $$[S_i,\, S_j] = i\,\epsilon_{ijk} S_k. \tag{1}$$

   Note that the Pauli spin matrices $\sigma_i/2$ satisfy these commutation relations (show this explicitly for $[\sigma_1,\, \sigma_2]$). Therefore the Pauli spin matrices are a representation of the spin operators on the two dimensional vector space $\mathbb{C}^2$; their eigenvalues are the eigenvalues of their corresponding spin operators, and their eigenvectors correspond to the eigenvectors of the abstract operators. Compute the eigenvalues and eigenvectors of σ_i for $i = 1,\, 2,\, 3$. Note that the eigenvectors form a CON basis for $\mathbb{C}^2$ for each i as they must (why?). What is the overlap of $|+,\, z\rangle$ with $|-,\, x\rangle$; $|-,\, y\rangle$ with $|+,\, x\rangle$? What is the probability of observing an electron polarized in the positive z direction as polarized in the negative x direction; negative y in the positive x direction?

   Bonus: Compute the exact energy eigenvalues and eigenvectors for $\boldsymbol{B}$. Expand your exact solutions for the energy levels and wavefunctions to the same order in B_0/B as you found using perturbation theory and show they are the same.

2. Consider a simple harmonic oscillator in two dimensions,

   $$\hat{H} = -\frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2) + \frac{1}{2}m\omega^2(\hat{x}^2 + \hat{y}^2). \tag{2}$$

   Show that energy levels are given by $E = (n_x + n_y + 1)\omega$; what are the state vectors and wavefunctions in terms of the known 1D SHO solutions? What are the degeneracies of the $E = \omega,\, 2\omega,\, 3\omega$ energy levels? Suppose we add a perturbation $\hat{H}_1 = \lambda m\omega^2 xy$. What are the first order energy corrections and zeroth order state vectors and wavefunctions for the $E = \omega$ and 2ω energy levels? Bonus: Compute the first order energy corrections and zeroth order state vectors and wavefunctions for the $E = 3\omega$ energy level. Solve the problem exactly (hint: change coordinates to $X = (x + y)/2$ and $Y = (y - x)/2$) and compare to your perturbative results for $E = \omega,\, 2\omega,\, 3\omega$.

3. Using $\phi(a,\, x) = \exp(-a x^2)$ as your trial wavefunction use the Ritz Variational Method for the simple harmonic oscillator, $\hat{H} = \hat{p}^2/2m + m\omega^2\hat{x}^2/2$; i.e., find a_0 and its corresponding ground state energy $E(a_0)$ and compare your results to the exact answer.

4. Suppose that one suddenly expands (or contracts) the size of an infinite potential well of length L by a factor of α; i.e., the new length of the infinite potential well is $\alpha\, L$. For $\alpha > 1$, what is the probability that the system will be measured in the ground state should it have started in the ground state? Optional: What is the probability that a state vector in the n^{th} eigenstate of the original Hamiltonian will be found in the m^{th} eigenstate of the final Hamiltonian? If one starts off in the ground state, what is the probability of remaining in the ground state for an expansion (contraction)?

5. Suppose that a particle is entangled in the first three energy levels with equal probability and the potential in a 1D SHO potential of characteristic length $x_0 = (m\,\omega)^{-1/2}$. What is the expectation value for the measured energy? Suppose the harmonic oscillator potential is very slowly changed to an infinite potential well of width x_0. What is the expectation value for the measured energy now?