Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #5

1. Let's put together a number of techniques that we've learned, and also learn a few more, in order to completely solve Poisson's Equation in an arbitrary, not even necessarily integral, number of dimensions when the problem demonstrates rotational symmetry. Recall Poisson's Equation,

$$\Delta\Phi = \rho, \tag{1}$$

where $\Delta = \vec{\nabla}\cdot\vec{\nabla}$. In spherical coordinates in n dimensions the Laplacian is

$$\Delta = r^{1-n}\partial_r\big(r^{n-1}\partial_r\big) + \frac{1}{r^2}\Delta_{S^{n-1}}, \tag{2}$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on the $n-1$ sphere, which can be thought of as the angular momentum operator in n dimensions.

Derive the GF $\phi(r)$, where $\Delta\phi(r) = -\delta(r)$, for PE for spherically symmetric problems in n dimensions. Follow the usual procedure whereby you solve $\Delta\phi = 0$ for $r \neq 0$ first, then set the normalization by integrating over an ϵ ball centered at the origin. The Divergence Theorem, whereby one converts volume integrals of divergences into surface integrals of fluxes,

$$\int_\Omega d^n x\,\Delta\phi = \int_\Omega d^n x\,\vec{\nabla}\cdot\vec{\nabla}\phi = \int_{\partial\Omega} d^{n-1}x\,\hat{n}\cdot\vec{\nabla}\phi \tag{3}$$

might prove useful. When the dust settles you should find that

$$\phi(r) = \frac{1}{n-2}\frac{1}{\Omega_n}\left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right), \tag{4}$$

where Ω_n is the solid angle in n dimensions and R is the radial distance at which we choose to take the potential to be 0 (for $n > 2$ we may safely take $R \to \infty$).

One may find Ω_n by exploiting the knowledge of Gaussian integrals. First, derive the result for a single Gaussian: $\int_{-\infty}^{\infty} dx\,\exp(-x^2) = \sqrt{\pi}$. One may do this by defining an integral $I = \int_{-\infty}^{\infty} dx\,\exp(-x^2)$ and then computing I^2 by going into polar coordinates. Next, see that if we have n Gaussian integrals, compare the result by direct evaluation in Cartesian coordinates and the result when in spherical coordinates. You should find that

$$\Omega_n = \frac{2\,\pi^{n/2}}{\Gamma(n/2)}, \tag{5}$$

where

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1}\, e^{-t} \tag{6}$$

is the Gamma function. Show that this formula for arbitrary (not necessarily integer) $n > 0$ yields the usual solid angle in 2 and 3 dimensions. What is Ω_1 and Ω_4? Note that you can determine $\Gamma(1/2)$ by relating Eq. (6) to a known Gaussian integral; also note that from Eq. (6) one can find that $\Gamma(z+1) = z\Gamma(z)$.

Putting the pieces together, what is the spherically symmetric GF for the PE in 3D; i.e., what is the potential for a point charge in 3D? What about for 4D? 1D? The formula may also be used to find the GF in 2D. The technique we will employ is commonly used in dimensional regularization in field theory. Let $n = 2 + \epsilon$ and take the limit as $\epsilon \to 0$. For instance

$$\pi^{-n/2} = \pi^{-1-\epsilon} = \frac{1}{\pi} e^{-\epsilon \log \pi} = \frac{1}{\pi}\big(1 - \epsilon \log \pi\big) + \mathcal{O}(\epsilon^2). \tag{7}$$

You should find that

$$\phi(r) = -\frac{1}{2\pi} \log\big(r/R\big). \tag{8}$$

Note that this is the same result as the potential of an infinite line of charge in 3D, as it must be by symmetry.

2. The propagator for the simple harmonic oscillator, $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}$. First, write down an expression for the propagator $K(x,\, t;\, x_0,\, t_0)$ in terms of the energy eigenstates in the position basis,

$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{(2^n n!)(\pi\tilde{x})}} H_n(x/\tilde{x}) e^{-\frac{1}{2}\left(\frac{x}{\tilde{x}}\right)^2}, \tag{9}$$

where $\tilde{x} = (m\omega)^{-1/2}$; you do not have to evaluate the integral or sum. Now let's evaluate the propagator using the Van-Vleck formula. First, show that there are initial and final positions and times, x_0 and t_0 and x and t, such that there is a unique classical path, no path, and more than one path (note that this is contrary to what Sakurai writes on p116). Show that for a unique classical path the classical action for reaching x at t from x_0 at t_0 is

$$S(x,\, t;\, x_0,\, t_0) = \frac{1}{2\tilde{x}^2 \sin\omega\Delta t} \left[\big(x^2 + x_0^2\big)\cos\omega\Delta t - 2x\, x_0\right], \tag{10}$$

where $\Delta t = t - t_0$. Compute, as done in class, the spectrum of eigenvalues of the second variation operator $\hat{S}_2$. Show that for $t < \pi/\omega$ all the eigenvalues are positive and the classical path minimizes the action. Then show that for general t the number of negative eigenvalues is the largest integer smaller than $\omega t/\pi$. Thus for $t > \pi/\omega$ there is at least one negative eigenvalue and the classical path is a saddle point of the action, not a minimum (note that this is contrary to Sakurai's claims on p117 that the classical path minimizes the action; rather, Hamilton's principle is that the classical path extremizes the action). As an aside, show that a condition in terms of ωt such that the action for a particle that is in a SHO potential well but is forced to move at constant velocity is smaller than a particle that moves along the usual oscillatory path is (taking $x_0 = t_0 = 0$ for simplicity)

$$1 + \frac{(\omega t)^2}{3} < (\omega t)\cot(\omega t). \tag{11}$$

Continuing the aside, plot separately (on the same graph) the LHS and RHS of the above and compute these two actions explicitly for $\omega t = 7\pi/6$ (give an answer to two decimal places; which is larger?). Back on track, put the pieces together and write down the propagator for the simple harmonic oscillator (you might want to work towards a known answer, such as Eq. 2.5.18 in Sakurai, which is correct for small times). Now recall that we showed that, via the resolvent, through K we have obtained the full solution to the quantum mechanical problem. Let's show this explicitly by using K to find the first few energy levels and position-basis eigenfunctions for the SHO. We will skip the complex analysis since for a time independent Hamiltonian we may write the time evolution operator as

$$e^{-i\hat{H}t} = \sum_n e^{-iE_n t}|E_n\rangle\langle E_n| \tag{12}$$

$$\Rightarrow \qquad K(x,\, x_0;\, t) = \sum_n e^{-iE_n t}\psi_n(x)\psi_n^*(x_0). \tag{13}$$

Now take your small time expression for the propagator (i.e. one in which all the eigenvalues of the second variation of the action are positive) and expand K out to the first two terms in powers of $\exp(i\omega t)$ (again taking $t_0 = 0$ for simplicity); show that you reproduce $E_n = (n+1/2)\omega$ for $n = 1,\, 2$ and that you reproduce the first two energy eigen-wavefunctions (see, e.g., Sakurai Eq. 2.3.32 for the generating function for $\psi_n(x)$). Finally, analytically continue the propagator into the complex plane take $t = -i\beta = -i/T$, $x = x_0$, and integrate over x_0 to find the partition function $Z(\beta)$ for the simple harmonic oscillator.

3. Compute the propagator for a particle in a constant gravitational field using the Van Vleck formula. Don't forget to check how many negative eigenvalues you have!