Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #4

1. We've spent a lot of effort evaluating time evolution problems in homework in which the Hamiltonian is constant in time. Let's examine the results when the Hamiltonian is time dependent.

   (a) In the last HW set you found $\langle \hat{\boldsymbol{S}} \rangle(t)$ for an electron initially polarized along the positive x direction exposed to a constant, time-independent magnetic field $\boldsymbol{B} = B_0 \hat{z}$, where, as usual, the $\hat{z}$ is a direction and not an operator. What is the probability of measuring the electron to be aligned in the positive x direction as a function of time?

   (b) Compute $\langle \hat{\boldsymbol{S}} \rangle(t)$ for an electron initially polarized along the positive x direction exposed to a constant, time-dependent magnetic field $\boldsymbol{B} = B_0 \cos(\omega' t)\hat{z}$. Does your result agree with that which you found in the previous homeworks for $\omega' t \ll 1$? What is the probability of measuring the electron to be aligned in the positive x direction as a function of time?

   (c) Compute $\langle \hat{\boldsymbol{S}} \rangle(t)$ to leading order in ω/ω' for an electron initially polarized along the positive x direction exposed to a constant, time-dependent magnetic field $\boldsymbol{B} = B_0\big(\cos(\omega' t)\hat{z} + \sin(\omega' t)\hat{y}\big)$, where, as usual, hats indicate a direction and not an operator. What is the leading order in ω/ω' probability of measuring the electron to be aligned in the positive x direction as a function of time? Bonus: compute the next-to-leading order correction to the expectation value and probability. Double bonus: solve the expectation value problem to all orders (which then of course automatically solves the original expectation value problem and the first bonus expectation value, too). Do so by finding that in the Heisenberg picture

$$\hat{\boldsymbol{S}}_H(t) = \mathbf{M}(t)\, \hat{\boldsymbol{S}}_H(0), \tag{1}$$

where

$$\mathbf{M}(t) = \begin{pmatrix} c_\omega(t) & -c(t)\, s_\omega(t) & s(t)\, s_\omega(t) \\ c(t)\, s_\omega(t) & s^2(t) + c^2(t)\, c_\omega(t) & s(t)\, c(t)\,\big(1 - c_\omega(t)\big) \\ -s(t)\, s_\omega(t) & s(t)\, c(t)\,\big(1 - c_\omega(t)\big) & c^2(t) + s^2(t)\, c_\omega(t) \end{pmatrix}, \tag{2}$$

and

$$\begin{aligned} s(t) &\equiv \sin(\frac{\omega' t}{2}) \qquad s_\omega(t) \equiv \sin\big(2\frac{\omega}{\omega'}\sin(\frac{\omega' t}{2})\big) \\ c(t) &\equiv \cos(\frac{\omega' t}{2}) \qquad c_\omega(t) \equiv \cos\big(2\frac{\omega}{\omega'}\sin(\frac{\omega' t}{2})\big) \end{aligned} \tag{3}$$

2. Let's do a problem that illustrates the power of Green's functions. We'll first derive the differential equation and boundary conditions we'd like for $G(x,\, x')$ such that the problem is readily solved, then we'll use our Green's function to find some particular solutions. Consider the problem of the *displacement* function $u(x)$ of a taut string with fixed endpoints placed under a load $f(x)$. The differential equation that governs this process is

$$\frac{d^2u(x)}{dx^2} = f(x) \tag{4}$$

for $x \in (0,\, 1)$, and $u(0) = u(1) = 0$. Show that if $G(x,\, x')$ satisfies the usual Green's function differential equation, in this case for our operator d^2/dx^2,

$$\frac{d^2G(x,\, x')}{dx^2} = \delta(x - x') \tag{5}$$

and has the boundary conditions $G(0,\, x') = G(1,\, x') = 0$, then

$$u(x) = \int_0^1 dx'\, G(x,\, x')\, f(x'). \tag{6}$$

OK, let's solve for $G(x,\, x')$ now. Note that $0 < x' < 1$ plays the role of a parameter throughout, so that we have to solve

$$\frac{d^2G_1(x,\, x')}{dx^2} = 0 \quad \text{for} \quad 0 < x < x', \tag{7}$$

$$\frac{d^2G_2(x,\, x')}{dx^2} = 0 \quad \text{for} \quad x' < x < 1 \tag{8}$$

subject to the boundary conditions $G_1(0,\, x') = G_2(1,\, x') = 0$, $G_1(x',\, x') = G_2(x',\, x')$ and there's a discontinuity in the first derivative between G_1 and G_2 whose normalization is given by Eq. (5) (remember to integrate over an epsilon ball!). Find that the Green's function is

$$G(x,\, x') = \begin{cases} G_1(x,\, x') & \text{for} \quad x < x' \\ G_2(x,\, x') & \text{for} \quad x > x' \end{cases}$$

$$= \begin{cases} x\,(x' - 1) & \text{for} \quad x < x' \\ x'\,(x - 1) & \text{for} \quad x > x' \end{cases} \tag{9}$$

Using this Green's function, solve the catenary problem: what is $u(x)$ for $f(x) = g$ (i.e. what is the shape of a taut string bent under its own weight due to gravity)? What is $u(x)$ if I pull a massless string down with a force f_0 at a point x_0? Draw a figure of the latter. With the Green's function finding $u(x)$ for any 'ole load $f(x)$ is easy. Just for fun, find $u(x)$ for a massless string when $f(x) = f_0 \sin(n\pi x)$, $n \in \mathbb{N}$.

Note that one often uses the notation $x_<$ and $x_>$ (see, e.g. Jackson) where

$$x_< \equiv \begin{cases} x & \text{for} \quad x < x' \\ x' & \text{for} \quad x' < x \end{cases} \tag{10}$$

$$x_> \equiv \begin{cases} x & \text{for} \quad x > x' \\ x' & \text{for} \quad x' > x. \end{cases} \tag{11}$$

With this notation we see that $G(x,\, x') = x_< (x_> - 1)$.

3. Let's get comfortable with propagators by computing some examples.

   (a) What is the retarded propagator $K_R(\theta,\, t;\, \theta_0,\, t_0)$ for a point particle on a circle? You are free to use results you have obtained from previous homework sets. Don't forget the $\theta(t - t_0)$!

   (b) What is the propagator $K(x,\, t;\, x_0,\, t_0)$ for a point particle in an infinite potential well of size L?

   (c) Calculate the propagator $K(x,\, t;\, x_0)$ for a particle of mass m under the influence of a constant gravitational field, $V(x) = mgx$ (or, equivalently, a charged particle in a constant electric field; I've taken $t_0 = 0$ for simplicity). Show that

$$K(x,\, t;\, x_0) = \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left\{\frac{i}{2}\left(\frac{m}{t}(x - x_0)^2 \right.\right.$$
$$\left.\left. - mgt(x + x_0) - \frac{1}{12} m g^2 t^3\right)\right\} \tag{12}$$

   Suggestion: insert complete sets of states $\hat{\mathbf{1}} = \int dp\, |p\rangle\langle p|$ and $\hat{\mathbf{1}} = \int dE |E\rangle\langle E|$ into the definition of the propagator; you will find that you need to compute $\langle p|E\rangle$. One could insert another complete set of states and go into the position basis, but then one would be dealing with Airy functions. I would suggest sticking it out and computing the momentum representations of the energy eigenstates, $H(p,\, id/dp)\phi_E(p) = E\phi_E(p)$, where $\hat{H}(\hat{p},\, \hat{x}) = \hat{p}^2/2m + mg\hat{x}$.

4. What is the partition function for a particle in a 1D infinite potential well of size L? Bonus: explicitly evaluate the sum for an n dimensional well. You might find the Abel-Plana summation formula useful:

$$\sum_0^\infty f(n) = \int_0^\infty f(x)\, dx + \frac{1}{2} f(0) + i \int_0^\infty \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1} dy. \tag{13}$$

   Notice that the Abel-Plana formula tells you the difference between the sum of a function and the integral of the function. Neat! Now compare the sum you have found with the usual partition function that one finds in Statistical Mechanics classes, where the sum is approximated by an integral over a density of states. (In 3D Baierlein gives $Z = (2\pi m T)^{3/2} L^3$ in formula 5.32.)