Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #1

1. Problem 4 (e) from HW#0.

2. Using the properties of an inner product discussed in class, show that $\langle \alpha\, v_j + \beta\, v_k | v_i \rangle \,=\, \alpha^* \langle v_j | v_i \rangle + \beta^* \langle v_k | v_i \rangle$.

3. Show that in a finite or countably infinite dimensional vector space that $\langle v^i | v^j \rangle \equiv \sum_n a_n^{i*}\, a_n^j$, where $|v^i\rangle \,=\, \sum_n a_n^i |e_n\rangle$, is an inner product. Show that for the square-integrable (possibly complex-valued) functions on the real line $\langle f_i | f_j \rangle \,=\, \int_{-\infty}^{\infty} dx\, f_i^*(x)\, f_j(x)$ is an inner product.

4. Delta "function"s are common and extremely useful. I put function in quotation marks because rigorously the delta is a distribution; it only makes rigorous mathematical sense when integrated over a suitably well-behaved function (i.e. a function that has a Taylor expansion at the point where the delta distribution is infinite). One can find delta distributions as the limit of a sequence of true functions $\delta(x;\, a)$, $\delta(x) \,=\, \lim_{\epsilon\to 0} \delta(x;\, \epsilon)$. Show that if

   (a) $\lim_{\epsilon\to 0} \delta(x;\, \epsilon) \,=\, 0$ for $x \,\neq\, 0$ (technically for $x \,\neq\, 0$ almost everywhere), and

   (b) $\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx\, \delta(x;\, \epsilon) \,=\, 1$

   then $\int_{-\infty}^{\infty} dx\, \delta(x;\, \epsilon)\, f(x) \,=\, f(0)$. One approach is to use a physicist's proof and make an argument based on dimensionality; i.e. what dimensionful quantity do you have left after integrating over x, and therefore what must $\int_{-\infty}^{\infty} dx\, \delta(x;\, \epsilon)\, x^n$ be proportional to? What are the dimensions of a delta distribution? Show that

$$\delta(x;\, a) \equiv \frac{1}{a}\theta(a/2 - x)\theta(a/2 + x), \tag{1}$$

   where $\theta(x - a)$ is the Heaviside step function,

$$\theta(x-a) = \begin{cases} 0, & x \,<\, a \\ 1, & x \,>\, a \end{cases}, \tag{2}$$

   converges to a delta distribution. One can also define the derivative of a delta distribution. Using integration by parts (IBP), show that $\int_{-\infty}^{\infty} dx\, \delta'(x)\, f(x) \,=\, -f'(0)$. Show by a change of variables that if there is one solution x_0 such that $g(x_0) \,=\, 0$ then

$$\int_{-\infty}^{\infty} dx\, \delta\big(g(x)\big)\, f(x) = f(x_0) \Big/ \left|\frac{\partial g}{\partial x}\right|_{x\,=\,x_0}. \tag{3}$$

Note that in general if $g(x) = 0$ has multiple solutions x_i then

$$\int_{-\infty}^{\infty} dx\, \delta\big(g(x)\big)\, f(x) = \sum_i \frac{f(x_i)}{\left|\frac{\partial g}{\partial x}\right|_{x=x_i}}. \tag{4}$$

Finally, show that $\lim_{\epsilon\to 0} \delta(x;\,\epsilon) = \delta(x)$ for

$$\delta(x;\,\epsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{i\,k\,x - |k|\,\epsilon}. \tag{5}$$

Hint: evaluate the k integral first then check to see if your result satisfies the above conditions. (Note that adding a small imaginary part to make integrals converge is a common technique.) Use this result to set the normalization to the stationary wavefunction solutions of the free particle on the infinite line (remember that we want $\langle p|p'\rangle = \delta(p-p')$ and $\langle x|x'\rangle = \delta(x-x')$; judicious insertion of 1 is useful). Write down the stationary wavefunction solutions in n dimensions.

5. The representation of $\hat{x}$ in the p basis.

   (a) Show that $[\hat{A},\, \hat{B}\hat{C}] = [\hat{A},\, \hat{B}]\,\hat{C} + \hat{B}\,[\hat{A},\, \hat{C}]$.

   (b) Show that $[\hat{p},\, \hat{x}^n] = -i\,n\,\hat{x}^{n-1}$.

   (c) Show that
   $$\hat{\boldsymbol{p}}\, e^{i\,\hat{\boldsymbol{x}}\cdot\boldsymbol{l}}\,|\boldsymbol{p}\rangle = (\boldsymbol{p}+\boldsymbol{l})\, e^{i\,\hat{\boldsymbol{x}}\cdot\boldsymbol{l}}\,|\boldsymbol{p}\rangle; \tag{6}$$
   therefore $e^{i\,\hat{\boldsymbol{x}}\cdot\boldsymbol{l}}\,|\boldsymbol{p}\rangle = |\boldsymbol{p}+\boldsymbol{l}\rangle$. Use this to show that
   $$\langle\boldsymbol{p}|\, e^{i\,\hat{\boldsymbol{x}}\cdot\boldsymbol{l}}\,|\psi\rangle = \psi(\boldsymbol{p}-\boldsymbol{l}) = e^{-\boldsymbol{\nabla}_{\boldsymbol{p}}\cdot\boldsymbol{l}}\,\psi(\boldsymbol{p}), \tag{7}$$
   and, therefore, $\langle\boldsymbol{p}|\hat{\boldsymbol{x}}|\psi\rangle = i\,\boldsymbol{\nabla}_{\boldsymbol{p}}\,\psi(\boldsymbol{p})$.

   (d) Show that $[\hat{\boldsymbol{x}},\, f(\hat{\boldsymbol{p}})] = i\,\boldsymbol{\nabla}_{\boldsymbol{p}} f(\boldsymbol{p})$ holds as an operator equation; i.e. show that $\langle\boldsymbol{p}|\,[\hat{\boldsymbol{x}},\, f(\hat{\boldsymbol{p}})]\,|\psi\rangle = \psi(\boldsymbol{p})\, i\,\boldsymbol{\nabla}_{\boldsymbol{p}} f(\boldsymbol{p})$.

Bonus

6. Remember Problem 3 from HW Set 0? We're not done with it just yet. Suppose we have a linear combination of two stationary states at $t = 0$,

$$|\psi(\theta,\, t=0)\rangle = c_1|n_1\rangle + c_2|n_2\rangle, \qquad n_1 \neq n_2, \tag{8}$$

and where we have chosen to represent the orthonormal stationary states by

$$\langle\theta|n_i\rangle = \frac{1}{\sqrt{2\pi R}} e^{i\,n_i\,\theta}, \qquad n_i \in \mathbb{Z}. \tag{9}$$

What is $|\psi(t)\rangle$? What are $P(n = n_1,\, t)$ and $P(n = n_2,\, t)$, the probabilities of measuring $n = n_1$ and $n = n_2$ as a function of time, respectively? Suppose for the moment that $c_1 = 0$; what is $\langle\hat{\theta}\rangle(t)$? Similarly for $c_2 = 0$. Compute $\langle\hat{\theta}\rangle$ for general c_i and see that a nontrivial t dependence emerges (make sure your answer for general c_i reduces to the results you found when an individual $c_i = 0$). Compute also

$$\langle \widehat{\Delta\theta}^2 \rangle = \langle \left(\hat{\theta} - \langle \hat{\theta} \rangle \right)^2 \rangle = \langle \hat{\theta}^2 \rangle - \langle \hat{\theta} \rangle^2, \tag{10}$$

and do the same for $\langle \widehat{\Delta p_\theta}^2 \rangle$. Does $\langle \widehat{\Delta\theta}^2 \rangle \langle \widehat{\Delta p_\theta}^2 \rangle$ satisfy the uncertainty principle? What values of n_1 and n_2 minimize the uncertainty? Note that you will have to compute integrals of the form $I_j(m) \equiv \int_0^{2\pi} d\theta\, \theta^j \exp(i\, m\, \theta)$ where j and m are integers. Ordinarily you could simply quote the result of these definite integrals from canned software. However a nice way of shortcutting the painstaking integration by parts necessary should one do these analytically is through the use of Feynman's trick of differentiating under the integral sign. Feynman's trick is a useful tool to know, and you are required to use it here, at least for the $\langle \widehat{\Delta\theta} \rangle$ case. Define a general

$$I_j(\alpha) \equiv \int_0^{2\pi} d\theta\, \theta^j\, e^{i\,\alpha\,\theta}. \tag{11}$$

Then one sees that

$$I_j(\alpha) = (-i)^j \partial_\alpha^j I_0(\alpha). \tag{12}$$

Then our desired integral is

$$I_j(m) = (-i)^j \partial_\alpha^j I_0(\alpha)\big|_{\alpha = m}. \tag{13}$$

Note that it is only at the end of the calculation that we set $\alpha = m \in \mathbb{Z}$.

Analytically evaluate $\int_0^1 dt\, (t^a - t^b)/\log(t)$, where $a,\, b \in \mathbb{R}$. Hint: do the opposite of Feynman's Trick and integrate under the integral.