Name:________________________ Date:____________

Time for Completion:____________

Honours QM HW #0

Welcome to Honours QM! This rather wordy problem set is designed as a warm-up to get your brains going again after a nice, long summer. Turn your homework in with this sheet stapled to the front. In order to alleviate possible biases in grading, do not write your name, etc., on the work you attach to this sheet. Feel free to work in groups (I recommend it!), but hand in your own written (or typed) out work. Be sure to clearly justify all nontrivial steps in a calculation. This set is meant to remind you of the physics you should already know, e.g. Griffiths; his book will be your friend if you hit any snags. I *strongly* suggest you follow the lead of problem 1 and stop using dimensionful constants that can be restored at the end of the calculation if one so desires (it will not be necessary in this class); physics is hard enough without having to track down $\hbar$'s in the middle of a derivation. Also, feel free to quote results for definite integrals from canned software such as *Mathematica*; however, be sure to cite your source.

1. We will work in natural units, $\hbar = c = 1$. In this case all relevant units are in terms of energy, $[E]$. By multiplying by as many $\hbar$'s and c's as necessary, show that length has units of $[L] = [E]^{-1}$, time has units $[t] = [E]^{-1}$, time and space derivatives have units $[E]$, and show that wavefunctions in n dimensions have units $[\psi] = [E]^{n/2}$.

2. Classical Mechanics vs. Quantum Mechanics. In CM, all particle dynamics may be described using only two quantities: position and momentum (and time). This is the Hamiltonian formalism, which is equivalent to the Newtonian or Lagrangian formalisms, for which the fundamental quantities are position and velocity. Usually we denote these quantities as q and p to allow for more generalized coordinates; however, we will stick with the usual x and p. The crucial point is that these fundamental quantities are treated as parameters. It turns out that the fundamental object for determining the dynamics in CM is the Poisson Bracket,

$$\{f(x,p),\, g(x,p)\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x}.$$

   (a) Show that $\{x,p\} = 1$.

   In QM, on the other hand, all observables (such as position and momentum) are elevated to operators, and the fundamental object for determining dynamics is the commutator,

$$[\hat{O}_1,\, \hat{O}_2] \equiv \hat{O}_1\hat{O}_2 - \hat{O}_2\hat{O}_1.$$

   The value of the commutator of two operators is then postulated to be i times the Poisson Bracket of the CM analogue.

(b) Show that $[\partial_x,\, f(x)] = \big(\partial_x f(x)\big)$, where $\partial_x \equiv \partial/\partial x$.

(c) Show that $\hat{p} = -i\partial_x$ satisfies our QM postulate.

3. Solve the time-dependent Schrödinger equation for a time-independent potential using the separation of variables technique; i.e. postulate that a solution $\Psi(\boldsymbol{x},t) = \psi(\boldsymbol{x})\,T(t)$ exists, solve for $T(t)$, and find the differential equation that $\psi(\boldsymbol{x})$ must satisfy.

4. The Schrödinger equation on S^1. Suppose a particle of mass m lives in a 1-D circular world of radius R.

   (a) What parameters will a free particle's energy depend on? Based on your intuition from the infinite potential well, what will the energy levels be proportional to?

   (b) Using the de Broglie wavelength determine the momenta and the energy levels of the system.

   (c) What is the differential equation for the spatial component of static solutions, $\psi(\theta)$, in this world? Specifically, start with the Laplacian operator in terms of derivatives of x and y, then perform a coordinate transformation to find the Laplacian on the circle. From this deduce the momentum operator. Using the Laplacian you derived, write down the time-independent Schrödinger equation. What are the boundary conditions for this differential equation?

   (d) Solve the SE for a free particle of mass m on a circle of radius R. Write your eigenfunctions of the Hamiltonian such that they are eigenfunctions of the momentum operator (i.e. choose your solutions $\psi(\theta)$ to be proportional to exponentials of θ and not to sin's and cos's); show that for $E \neq 0$ you have found right-moving and left-moving solutions. What are their momenta and energy levels (and are they the same as from 4. (b))? Do the units of your wavefunctions satisfy the condition you found in 1.? Note that, by Dirichlet's Theorem, you now have a complete, orthonormal set of functions on $[0,\, 2\pi)$ (and the $E = 0$ solution is necessary for completing the set). This is what you expect from Sturm-Liouville Theory (to be discussed in class).

   (e) Suppose we introduce a potential $V(\theta) = a\delta(\theta)$ to the free particle system. Compute the wavefunctions and energy levels for the bounded and unbounded states of a particle of mass m as a function of a. In general one solves differential equations that have a δ-function in them by finding the solution to the differential equation where the δ-function is zero and then appropriately matching the boundary conditions. What are the boundary conditions for our case? (Hint: one of the boundary conditions is found by integrating an ϵ ball around $\theta = 0$, thus giving a condition on the discontinuity in $\psi'(\theta)$.) Specifically, show that for a bound state to exist $a < 0$ and for a generic wavefunction solution to the differential equation when $\theta \neq 0$ $\psi(\theta) = A_\kappa \exp(\kappa\theta) + B_\kappa \exp(-\kappa\theta)$, where A_κ and B_κ are constants with respect to θ, $\kappa = \sqrt{-2mE}$, and the energy is found from the equation

$$\frac{\kappa}{mR^2} \sinh(\pi\,\kappa) = -a\,\cosh(\pi\,\kappa). \tag{1}$$

Properly normalize the wavefunctions, thus solving for A_κ and B_κ. One can find half of the unbound states simply by taking $\psi(\theta) = A_k \exp(ik\theta) + B_k \exp(-ik\theta)$ where $k = -i\kappa$ and Eq. (1) still holds (write this out explicitly in terms of sin's and cos's). The other half of the unbound state solutions are proportional to sin functions; find them and explain why Eq. (1) does not hold for them.

Congratulations! You have used a very general method to solve a very important class of differential equations, in which a delta function is involved. We will see these again (both here and in E&M) when it comes to solving for Green's functions. (In the case of Green's functions, the delta function is not multiplying the solution of the DE, but the technique of solving the homogeneous equation and then using the delta function to enforce boundary conditions is very powerful and general.)