Some Delta Function Representations

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1 Delta Functions

1.1 Principal Value Representation

The Sokhotsky-Weierstrauss Theorem states that, when considered as a distribution,

$$\lim_{\epsilon\to 0^+}\frac{1}{x\pm i\epsilon} = PV\frac{1}{x} \mp i\,\pi\,\delta(x). \tag{1}$$

To see this consider

$$f(x) \equiv \lim_{\epsilon\to 0^+}\frac{1}{x\pm i\epsilon} = \lim_{\epsilon\to 0^+}\frac{x}{x^2+\epsilon^2} \mp i\frac{\epsilon}{x^2+\epsilon^2}. \tag{2}$$

For $x\neq 0$, $f(x)=1/x$. For $x=0$, $f(x)=\mp i\,\infty$. So it already looks like the form of Eq. (1); we really need only check the normalization. If we integrate $f(x)$ we get, for $a<0<b$,

$$\int_a^b dx\, f(x) = PV\int_a^b dx\,\frac{1}{x} \mp i \lim_{\epsilon\to 0^+} \tan^{-1}\left(\frac{b}{\epsilon}\right) - \tan^{-1}\left(\frac{a}{\epsilon}\right) \tag{3}$$

$$= PV\int_a^b dx\,\frac{1}{x} \mp i\,\pi, \tag{4}$$

where the Cauchy principal value is defined as the exclusion of a singular value from a region of integration; e.g. for $a<0<b$

$$PV\int_a^b dx\,\frac{1}{x} \equiv \lim_{\epsilon\to 0^+}\int_a^{-\epsilon} dx\,\frac{1}{x} + \int_\epsilon^b dx\,\frac{1}{x}. \tag{5}$$

To be specific as to the application of Eq. (1), for a smooth function $g(x)$ and $a<0<b$

$$\lim_{\epsilon\to 0^+}\int_a^b dx\, g(x)\,\frac{1}{x\pm i\epsilon} = PV\int_a^b g(x)\frac{1}{x} \mp i\,\pi\, g(0). \tag{6}$$

1.2 Integral Representation

To get the normalization correct for the integral representation of a delta function note that:

$$\begin{aligned} \int_{-\infty}^{\infty} dx e^{ikx} f(x) &= \tilde{f}(k) \\ f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ikx} \tilde{f}(k). \end{aligned} \tag{7}$$

Therefore taking $f(x) = \delta(x)$, $\tilde{f}(k) = 1$ and, we have that

$$\delta(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx e^{ikx}. \tag{8}$$

We also get, for free, another form of the delta function:

$$\begin{aligned} \delta(k) &= \lim_{\tau \to 0^+} \frac{1}{2\pi} \left[\int_{-\infty}^{0} dx e^{i(k-i\tau)x} + \int_{0}^{\infty} dx e^{i(k+i\tau)x} \right] \\ &= \lim_{\tau \to 0^+} \frac{1}{2\pi} \left[\frac{1}{i(k-i\tau)} - \frac{1}{i(k+i\tau)} \right] \\ &= \lim_{\tau \to 0^+} \frac{1}{2\pi} \left[\frac{1}{\tau + ik} + \frac{1}{\tau - ik} \right] \\ &= \lim_{\tau \to 0^+} \frac{1}{\pi} \frac{\tau}{\tau^2 + k^2}. \end{aligned} \tag{9}$$