Active and Passive Transformations

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1 Active and Passive Transformations

1.1 1D Scalar Transformations

Suppose $\phi(x) = e^{-x^2}$, Fig. 1.

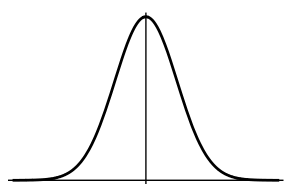

Figure 1: $\phi(x)$.

1.1.1 Passive

Now suppose we have a Passive Transformation, Fig. 2. We're taking $\phi(x) \to$

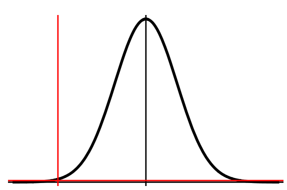

Figure 2: Visualization of a Passive Transformation.

$\phi'(y)$, where x are our original coordinates and y are the new coordinates. Suppose, for instance, that $y = x + a = f(x)$. Then

$$\phi'(y) = e^{-(y-a)^2} = \phi\big(x(y)\big) = \phi\big(f^{-1}(y)\big). \tag{1}$$

So we see that

$$\phi'_{\text{Passive}}(y) = \phi\big(f^{-1}(y)\big). \tag{2}$$

1.1.2 Active

An Active Transformation takes $\phi(x) \to \phi'(x)$, Fig. 3. Then

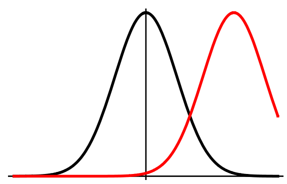

Figure 3: Visualization of an Active Transformation.

$$\phi'_{\text{Active}}(x) = \phi\big(f^{-1}(x)\big). \tag{3}$$

We can see that the Active and Passive give the exact same result when we change labels $x \leftrightarrow y$:

$$\phi'_{\text{Active}}(y) = \phi'_{\text{Passive}}(y); \qquad \phi'_{\text{Active}}(x) = \phi'_{\text{Passive}}(x). \tag{4}$$

1.2 2D Scalar Transformation

Suppose we have $\phi(x_1, x_2) = x_1^2 + x_2^2$ and we undertake a transformation with $y_1 = x_1 + a_1$ and $y_2 = x_2 + a_2$. Then the Passive Transformation gives

$$\phi'(y_1, y_2) = (y_1 - a_1)^2 + (y_2 - a_2)^2. \tag{5}$$

In general for $\vec{y} = \vec{f}(\vec{x})$, and hence $\vec{x} = \vec{f}^{-1}(\vec{y})$, we have that

$$\phi'_{\text{Passive}}(\vec{y}) = \phi\big(\vec{f}^{-1}(\vec{y})\big). \tag{6}$$

The Active Transformation gives

$$\phi'_{\text{Active}}(\vec{x}) = \phi\big(\vec{f}^{-1}(\vec{x})\big). \tag{7}$$

Suppose we do a less trivial example. Let $\phi(x_1, x_2) = x_1 + x_1^2 + x_2^2$ and let's effect the coordinate transformation $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$. This is necessarily Passive, as it is a coordinate transformation. Then

$$\phi'(r, \theta) = r\cos(\theta) + r^2. \tag{8}$$

However I can still think of this as an active transformation:

$$\begin{aligned} \phi'(\vec{x}) &= \phi\big(\vec{f}^{-1}(\vec{x})\big) & (9) \\ &= x_1 \cos(x_2) + x_1^2. & (10) \end{aligned}$$

We can then distinguish between Passive and Active transformations by

$$\begin{aligned} \phi(\vec{x}) &\to \phi'(\vec{x}') \qquad \text{[Passive]} & (11) \\ \phi(\vec{x}) &\to \phi'(\vec{x}) \qquad \text{[Active]}. & (12) \end{aligned}$$

In fact for a Passive transformation

$$\phi'(\vec{x}') \equiv \phi(\vec{x}). \tag{13}$$

1.3 Derivatives

Let's first brush up on the chain rule. If I take the derivative of a composition of functions I get

$$\frac{\partial}{\partial x^\mu}\left[f\big(\vec{g}(\vec{x})\big)\right] = \left(\frac{\partial g^\nu}{\partial x^\mu}\right)(\vec{x})\left(\frac{\partial f}{\partial g^\nu}\right)\big(\vec{g}(\vec{x})\big). \tag{14}$$

Our somewhat confusing notation for the RHS should be interpreted as $\partial_\mu g^\nu$ evaluated at $\vec{x}$ multiplied by $\partial_\nu f$ evaluated at $\vec{g}(\vec{x})$.

As an example take the scalar function $f(z,w) = z^2 + w^2$ acting on $\vec{z} = \vec{g}(x,y) = (xy, x^3)$. Then

$$\partial_x f\big(\vec{g}(\vec{x})\big) = \partial_x(x^2y^2 + x^6) = 2xy^2 + 6x^5. \tag{15}$$

This is the same result as from the Chain Rule (again with somewhat confusing notation):

$$\begin{aligned}
\partial_x f\big(\vec{g}(\vec{x})\big) &= \left(\frac{\partial g_z}{\partial x}\right)(\vec{x})\left(\frac{\partial f}{\partial g_z}\right)\big(\vec{g}(\vec{x})\big) + \left(\frac{\partial g_w}{\partial x}\right)(\vec{x})\left(\frac{\partial f}{\partial g_w}\right)\big(\vec{g}(\vec{x})\big) \\
&= \left(\frac{\partial z}{\partial x}\right)(\vec{x})\left(\frac{\partial f}{\partial z}\right)\big(\vec{z}(\vec{x})\big) + \left(\frac{\partial w}{\partial x}\right)(\vec{x})\left(\frac{\partial f}{\partial w}\right)\big(\vec{z}(\vec{x})\big) \\
&= (y)\,\Big(2z\Big)\Big|_{z=xy} + (3x^2)\,\Big(2w\Big)\Big|_{w=x^3} \\
&= 2xy^2 + 6x^5.
\end{aligned} \tag{16}$$

In general we have that

$$\left.\frac{\partial}{\partial x^\mu}\right|_{\vec{x}} = \left.\frac{\partial y^\nu}{\partial x^\mu}\right|_{\vec{x}}\left.\frac{\partial}{\partial y^\nu}\right|_{\vec{y}=\vec{y}(\vec{x})}. \tag{17}$$

Let's consider the Passive and Active transformations of the derivative of a scalar function.

1.3.1 Passive

$$\begin{aligned}
\partial_{\mu'}\left[\phi'(\vec{x}')\right] &= \frac{\partial}{\partial y^\mu}\left[\phi(\vec{x})\right] \\
&= \frac{\partial x^\nu}{\partial y^\mu}(\vec{y})\frac{\partial}{\partial x^\nu}\left[\phi(\vec{x})\right] \\
&= \frac{\partial x^\nu}{\partial y^\mu}(\vec{y})\big(\partial_\nu\phi\big)(\vec{x}) \\
&= \frac{\partial x^\nu}{\partial y^\mu}(\vec{y})\Big(\partial_\nu\phi\Big)\big(\vec{f}^{-1}(\vec{y})\big).
\end{aligned} \tag{18}$$

Take as an example our previous scalar function, $\phi(x_1, x_2) = x_1^2x_2^2 + x_1^6$ and the transformation to $(y_1, y_2) = (x_1^2, x_2)$. Note how this can be confusing because of the double usage of $\vec{x}$ in both the argument of ϕ and in $\vec{y}$. We can check Eq. (18) by evaluating $\partial_{y_1}\phi'(\vec{y})$ two ways. First we evaluate by direct subsitution:

$$\begin{aligned} \partial_{y_1}\phi'(\vec{y}) &= \partial_{y_1}\phi\big(\vec{f}^{-1}(\vec{y})\big) \\ &= \partial_{y_1}\left[\phi(\sqrt{y_1}, y_2)\right] \\ &= \partial_{y_1}\left[y_1 y_2^2 + y_1^3\right] \\ &= y_2^2 + 3y_1^2. \end{aligned} \tag{19}$$

Now let's use our transformation machinery with Eq. (18):

$$\begin{aligned} \partial_{y_1}\phi'(\vec{y}) &= \frac{\partial x^\nu}{\partial y_1}(\vec{y})\Big(\partial_\nu\phi\Big)\big(\vec{f}^{-1}(\vec{y})\big) \\ &= \frac{1}{2}\frac{1}{\sqrt{y_1}}\left[2x_1x_2^2 + 6x_1^5\right]_{x_1=\sqrt{y_1}, x_2=y_2} \\ &= y_2^2 + 3y_1^2, \end{aligned} \tag{20}$$

where to get from the first to the second line we differentiated the inverse of the transformation, $\vec{x} = (\sqrt{y_1}, y_2)$.

1.3.2 Active

Now consider an active transformation. By a simple application of the Chain Rule we have that

$$\begin{aligned} \partial_\mu\left[\phi'(\vec{x})\right] &= \partial_\mu\Big[\phi\big(\vec{f}^{-1}(\vec{x})\big)\Big] \\ &= \frac{\partial\, {f^{-1}}^{\nu}}{\partial x^\mu}(\vec{x})\Big(\partial_\nu\phi\Big)\big(\vec{f}^{-1}(\vec{x})\big). \end{aligned} \tag{21}$$

We see that, as must happen, Eq. (18) and Eq. (21) are the same when we take $\vec{f}^{-1} \leftrightarrow \vec{x}$ and $\vec{x} \leftrightarrow \vec{y}$ (note that even the argument of the second term is the same because $\vec{f}^{-1}(\vec{y}) = \vec{x}(\vec{y})$.

1.4 Jacobians

Ultimately we will use our transformation results for the derivative of a scalar function to derive the transformation rule for vector fields. To do so we will have to know how the inverse of objects like $\partial x^\nu/\partial y^\mu$: we need to understand Jacobians and their inverses.

1.4.1 Passive

The Jacobian of a (Passive) coordinate transformation $\vec{y} = \vec{f}(\vec{x})$ is usually denoted

$$\frac{\partial y^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n}. \end{pmatrix} \tag{22}$$

We may find its inverse from

$$\begin{aligned}
\frac{\partial y^\mu}{\partial y^\nu} &= \frac{\partial}{\partial y^\nu}\left[y^\mu = f^\mu\left(\vec{f}^{-1}(\vec{y})\right)\right] && (23)\\
&= \frac{\left(f^{-1}\right)^\rho}{\partial y^\nu}(\vec{y})\frac{\partial f^\mu}{\partial\left(f^{-1}\right)^\rho}\left(\vec{f}^{-1}(\vec{y})\right) && (24)\\
&= \frac{\partial x^\rho}{\partial y^\nu}(\vec{y})\frac{\partial y^\mu}{\partial x^\rho}\left(\vec{f}^{-1}(\vec{y})\right) && (25)\\
&= \frac{\partial x^\rho}{\partial y^\nu}\left(\vec{f}(\vec{x})\right)\frac{\partial y^\mu}{\partial x^\rho}(\vec{x}) && (26)\\
&= \delta^\mu_\nu. && (27)
\end{aligned}$$

Therefore

$$\left(\frac{\partial y^\mu}{\partial x^\nu}\right)^{-1}\left(\vec{f}^{-1}(\vec{y})\right) = \left(\frac{\partial x^\nu}{\partial y^\nu}\right)(\vec{y}), \tag{28}$$

and the same equation only evaluating both matrices simultaneously at a different point

$$\left(\frac{\partial y^\mu}{\partial x^\rho}\right)^{-1}(\vec{x}) = \left(\frac{\partial x^\rho}{\partial y^\nu}\right)\left(\vec{f}(\vec{x})\right). \tag{29}$$

Expressed as matrix multiplication

$$\mathbf{J}(\vec{x})\mathbf{J}^{-1}\left(\vec{f}(\vec{x})\right) = \frac{\partial y^\mu}{\partial x^\rho}(\vec{x})\frac{\partial x^\rho}{\partial y^\nu}\left(\vec{f}(\vec{x})\right) = \mathbf{I}. \tag{30}$$

We may also derive expressions by differentiating $\vec{x}$:

$$\begin{aligned}
\frac{\partial x^\mu}{\partial x^\nu} &= \frac{\partial}{\partial x^\nu}\left[x^\mu = \left(f^{-1}\right)^\mu(\vec{y})\right]\\
&= \frac{\partial y^\rho}{\partial x^\nu}(\vec{x})\frac{\partial\left(f^{-1}\right)^\mu}{\partial y^\rho}(\vec{y})\\
&= \frac{\partial y^\rho}{\partial x^\nu}(\vec{x})\frac{\partial x^\mu}{\partial y^\rho}\left(\vec{y} = \vec{f}(\vec{x})\right)\\
&= \delta^\mu_\nu.
\end{aligned}$$

Therefore the inverse of the Jacobian (coordinate transformation) matrix is

$$\left(\frac{\partial y^\mu}{\partial x^\nu}\right)^{-1}(\vec{x}) = \frac{\partial x^\mu}{\partial y^\nu}\left(\vec{y} = \vec{f}(\vec{x})\right). \tag{31}$$

Similarly

$$\left(\frac{\partial y^\mu}{\partial x^\nu}\right)^{-1}\left(\vec{x} = \vec{f}^{-1}(\vec{y})\right) = \frac{\partial x^\mu}{\partial y^\nu}(\vec{y}). \tag{32}$$

Because left inverses and right inverses are the same for matrices we have that

$$\frac{\partial y^\mu}{\partial x^\rho}(\vec{x})\frac{\partial x^\rho}{\partial y^\nu}\left(\vec{y} = \vec{f}(\vec{x})\right) = \frac{\partial y^\mu}{\partial x^\rho}\left(\vec{x} = \vec{f}^{-1}(\vec{y})\right)\frac{\partial x^\rho}{\partial y^\nu}(\vec{y}) = \delta^\mu_\rho, \tag{33}$$

where one has to worry about the subtlety of ultimately evaluating the Jacobian and in its inverse in the same set of coordinates.

As an example take the coordinate transformation

$$\begin{array}{c|c} \vec{y} = f(\vec{x}) & \vec{x} = f^{-1}(\vec{y}) \\ \hline y_1 = \sqrt{x_1^2 + x_2^2} & x_1 = y_1 \cos(y_2) \\ y_2 = \tan^{-1}(x_2/x_1) & x_2 = y_1 \sin(y_2) \end{array} \tag{34}$$

Then

$$\begin{aligned} &\frac{\partial y^\mu}{\partial x^\nu}(\vec{x})\frac{\partial x^\nu}{\partial y^\rho}(\vec{y} = \vec{f}(\vec{x})) = \\ &\quad = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} \cos(y_2) & -y_1\sin(y_2) \\ \sin(y_2) & y_1\cos(y_2) \end{pmatrix} \Bigg|_{\substack{y_1 = \sqrt{x_1^2+x_2^2}, \\ y_2 = \tan^{-1}(x_2/x_1)}} \\ &\quad = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & -x_2 \\ \frac{x_2}{x_1^2+x_2^2} & x_1 \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{35}$$

where we have used

$$\begin{aligned} \cos\left(\tan^{-1}(x_2/x_1)\right) &= \frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \sin\left(\tan^{-1}(x_2/x_1)\right) &= \frac{x_2}{\sqrt{x_1^2+x_2^2}}. \end{aligned}$$

We can also see that this works when ultimately evaluating in the $\vec{y}$ coordinates:

$$\begin{aligned} &\frac{\partial y^\mu}{\partial x^\nu}(\vec{x} = \vec{f}^{-1}(\vec{y}))\frac{\partial x^\nu}{\partial y^\rho}(\vec{y}) = \\ &\quad = \begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix} \Bigg|_{\substack{x_1 = y_1\cos(y_2), \\ x_2 = y_1\sin(y_2)}} \begin{pmatrix} \cos(y_2) & -y_1\sin(y_2) \\ \sin(y_2) & y_1\cos(y_2) \end{pmatrix} \\ &\quad = \begin{pmatrix} \cos(y_2) & \sin(y_2) \\ \frac{-\sin(y_2)}{y_1} & \frac{\cos(y_2)}{y_1} \end{pmatrix} \begin{pmatrix} \cos(y_2) & -y_1\sin(y_2) \\ \sin(y_2) & y_1\cos(y_2) \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{36}$$

1.4.2 Active

Since

$$\frac{\partial x^\mu}{\partial x^\nu} = \partial_\nu\left[f^\mu\left(\vec{f}^{-1}(\vec{x})\right)\right] = \frac{\left(f^{-1}\right)^\rho}{\partial x^\nu}(\vec{x})\frac{\partial f^\mu}{\partial x^\rho}\left(\vec{f}^{-1}(\vec{x})\right) = \delta^\mu_\nu, \tag{37}$$

we have that Jacobian ("coordinate transformation") matrix's inverse is

$$\left(\frac{\partial f^\mu}{\partial x^\nu}\left(\vec{f}^{-1}(\vec{x})\right)\right)^{-1} = \frac{\left(f^{-1}\right)^\mu}{\partial x^\nu}(\vec{x}). \tag{38}$$

Similarly

$$\left(\frac{\partial f^\mu}{\partial x^\nu}(\vec{x})\right)^{-1} = \frac{\left(f^{-1}\right)^\mu}{\partial x^\nu}\left(\vec{f}(\vec{x})\right). \tag{39}$$

Note how correctly evaluating the Jacobians and inverses at the same "point" has become a subtlety due to clumsy notation.

As an example take $\vec{f}(\vec{x}) = (x_1^2 x_2, x_2)$. Then $\vec{f}^{-1}(\vec{x}) = (\sqrt{x_1/x_2}, x_2)$. We can now check that

$$\begin{aligned}
&\frac{\partial\left(f^{-1}\right)^\rho}{\partial x^\nu}(\vec{x})\frac{\partial f^\mu}{\partial x^\rho}\left(\vec{f}^{-1}(\vec{x})\right) = \\
&= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{x_1 x_2}} & -\frac{1}{2}\sqrt{\frac{x_1}{x_2^3}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2x_1x_2 & x_1^2 \\ 0 & 1 \end{pmatrix}\Bigg|_{\vec{x}=\vec{f}^{-1}(\vec{x})=(\sqrt{x_1/x_2},x_2)} \\
&= \begin{pmatrix} \frac{1}{2}\frac{1}{\sqrt{x_1 x_2}} & -\frac{1}{2}\sqrt{\frac{x_1}{x_2^3}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\sqrt{x_1x_2} & \frac{x_1}{x_2} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{40}$$

1.5 Vector Transformations

In order to determine the transformation law for Rank 1 tensors we will consider the transformation of the scalar

$$(\partial_\mu \phi)A^\mu. \tag{41}$$

We will follow Carroll's notation and prime indices of transformed vector components.

1.5.1 Passive

From Eq. (18) we have that

$$\left(\partial_{\mu'}\phi'(\vec{x}')\right)A^{\mu'}(\vec{x}') = \frac{\partial x^\nu}{\partial y^\mu}(\vec{y})\Big(\partial_\nu\phi\Big)\left(\vec{f}^{-1}(\vec{y})\right)A^{\mu'}\left(\vec{f}^{-1}(\vec{y})\right). \tag{42}$$

Therefore

$$A^{\mu'}(\vec{x}') = \frac{\partial y^\mu}{\partial x^\nu}\left(\vec{x} = \vec{f}^{-1}(\vec{y})\right)A^\nu\left(\vec{f}^{-1}(\vec{y})\right). \tag{43}$$

This leads to the useful mnemonic

$$A^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}A^\nu, \tag{44}$$

where $x^{\mu'}$ are the new coordinates and x^{μ} are the old; then we see that conservation of indices is extended to transformed coordinates, too.

As a trivial example, let's transform the vector field

$$\vec{A}(\vec{x}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \theta(x_1 - a_1)\theta(b_1 - x_1)\theta(x_2 - a_2)\theta(b_2 - x_2) \tag{45}$$

by taking $x_1 \to -y_2$ and $x_2 \to y_1$, a rotation of the axes by $\pi/2$, Fig. 4.

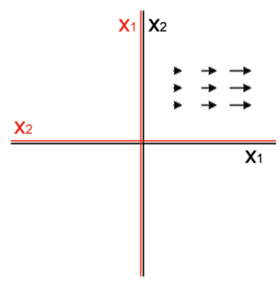

Figure 4: Visualization of a Passive Transformation on a Vector Field.

Using Eq. (43) we find that

$$\begin{aligned} \vec{A}'(\vec{y}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -y_2 \\ 0 \end{pmatrix} \theta(-y_2 - a_1)\theta(b_1 + y_2)\theta(y_1 - a_2)\theta(b_2 - y_1) \\ \vec{A}'(\vec{y}) &= \begin{pmatrix} 0 \\ y_2 \end{pmatrix} \theta(-y_2 - a_1)\theta(b_1 + y_2)\theta(y_1 - a_2)\theta(b_2 - y_1). \end{aligned} \tag{46}$$

We can see by eye that this is the correct answer. The box where the vector field exists is in the proper quadrant ($a_2 \leq y_1 \leq b_2$, and $a_1 \leq -y_2 \leq b_1$), and the vectors are pointing in the correct direction (because $y_2 < 0$ they are pointing in the $-\hat{y}_2$ direction).

Let's do a less trivial example. Take

$$\vec{A}(\vec{x}) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{47}$$

and let's change to polar coordinates. First note that

$$\begin{array}{cc} \vec{y} = f(\vec{x}) & \vec{x} = f^{-1}(\vec{y}) \\ \hline r = \sqrt{x_1^2 + x_2^2} & x_1 = r\cos(\theta) \\ \theta = \tan^{-1}(x_2/x_1) & x_2 = r\sin(\theta) \end{array} \tag{48}$$

Therefore the coordinate transformation matrix

$$M^{\mu}{}_{\nu}(\vec{y}) = \left. \begin{pmatrix} x_1/\sqrt{x_1^2 + x_2^2} & x_2/\sqrt{x_1^2 + x_2^2} \\ -x_2/x_1^2 + x_2^2 & x_1/x_1^2 + x_2^2 \end{pmatrix} \right|_{\vec{x} = f^{-1}(\vec{y})} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix}. \tag{49}$$

Then we have that

$$\begin{aligned}\vec{A}'(r,\theta) &= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta)/r & \cos(\theta)/r \end{pmatrix} \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} \\ &= \begin{pmatrix} r \\ 0 \end{pmatrix}. \end{aligned} \tag{50}$$

We can visualize this (passive) coordinate transformation as in Fig. 5.

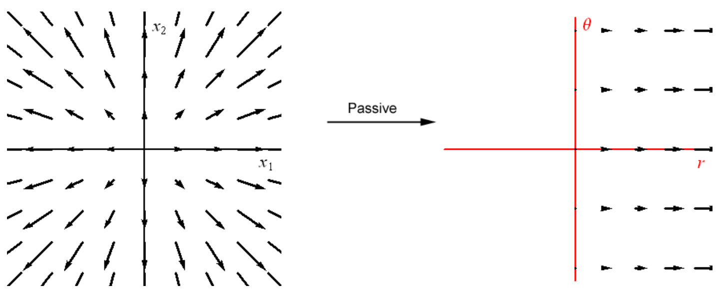

Figure 5: Visualization of a nontrivial Passive Transformation of a Vector Field.

1.5.2 Active

Under an Active Transformation $\vec{f}(\vec{x})$ our vector field transforms according to

$$A'^{\mu}(\vec{x}) = \frac{\partial f^{\mu}}{\partial x^{\nu}}\big(f^{-1}(\vec{x})\big)A^{\nu}\big(f^{-1}(\vec{x})\big). \tag{51}$$

Let's repeat the nontrivial example of above. Then

$$\vec{f}(\vec{x}) = \left(\sqrt{x_1^2+x_2^2}, \tan^{-1}(x_2/x_1)\right) \tag{52}$$

$$\vec{f}^{-1}(\vec{x}) = \big(x_1\cos(x_2), x_1\sin(x_2)\big). \tag{53}$$

And, in admittedly somewhat confusing notation,

$$\begin{aligned}\vec{A}'(x_1,x_2) &= \left.\begin{pmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ \frac{-x_2}{x_1^2+x_2^2} & \frac{x_1}{x_1^2+x_2^2} \end{pmatrix}\right|_{(x_1,x_2)=\left(x_1\cos(x_2),x_1\sin(x_2)\right)} \times \\ &\quad \begin{pmatrix} x_1\cos(x_2) \\ x_1\sin(x_2) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ 0 \end{pmatrix}. \end{aligned} \tag{54}$$

Fig. 6 shows are resulting transformed vector field (which is in the same coordinates as the untransformed field).

Again we see that, as we must, we get that for a trivial relabeling

$$A'^{\mu}_{\text{Passive}}(\vec{y}) = A'^{\mu}_{\text{Active}}(\vec{y}); \qquad A'^{\mu}_{\text{Passive}}(\vec{x}) = A'^{\mu}_{\text{Active}}(\vec{x}). \tag{55}$$

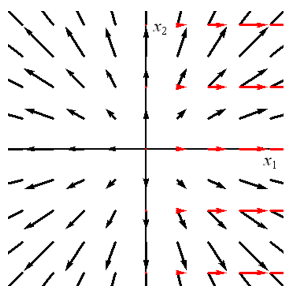

Figure 6: Visualization of a nontrivial Active Transformation of a Vector Field.