1 2D and Periodic in the Sum and Difference

1.1 Change of Variables

Suppose I have the integral

$$I = \int d^2\vec{q} \int d^2\vec{k} f(\vec{k}, \vec{q}) \quad (1)$$

$$= \int qdq \int kdk \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi f(k, q, \theta, \phi), \quad (2)$$

where θ is associated with $\vec{q}$ and ϕ with $\vec{k}$, $q = |\vec{q}|$ and $k = |\vec{k}|$. The region of integration is shown in Fig. 1 (a). It's to my advantage to make a change of integration so that I can do at least one angular integral trivially. Define

$$\begin{cases} \chi = \theta + \phi \\ \psi = \theta - \phi \end{cases} \quad (3)$$

$$\Rightarrow \quad \begin{cases} \theta = (\chi + \psi)/2 \\ \phi = (\chi - \psi)/2. \end{cases} \quad (4)$$

The Jacobian is

$$J(\chi, \psi) = \begin{vmatrix} \frac{\partial \theta}{\partial \chi} & \frac{\partial \theta}{\partial \psi} \\ \frac{\partial \phi}{\partial \chi} & \frac{\partial \phi}{\partial \psi} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}, \quad (5)$$

the new region of integration is shown in Fig. 1 (b), and Eq. (2) becomes

$$\begin{aligned} I &= \frac{1}{2} \int qdqkdk \left\{ \int_0^{2\pi} d\chi \int_{-\chi}^{\chi} d\psi f(k, q, \chi, \psi) \right. \\ &\qquad \left. + \int_{2\pi}^{4\pi} d\chi \int_{\chi - 4\pi}^{4\pi - \chi} d\psi f(k, q, \chi, \psi) \right\} \\ &= \frac{1}{2} \int qdqkdk \left\{ \int_{-2\pi}^{0} d\psi \int_{-\psi}^{\psi + 4\pi} d\chi f(k, q, \chi, \psi) \right. \\ &\qquad \left. + \int_0^{2\pi} d\psi \int_{\psi}^{4\pi - \psi} d\chi f(k, q, \chi, \psi) \right\}. \end{aligned} \quad (6)$$

1.2 Angular Integrals of Trigonometric Functions

The above is all true in general. However if we assume that the integrand is purely a function of trigonometric integrals then we can exploit the periodicity

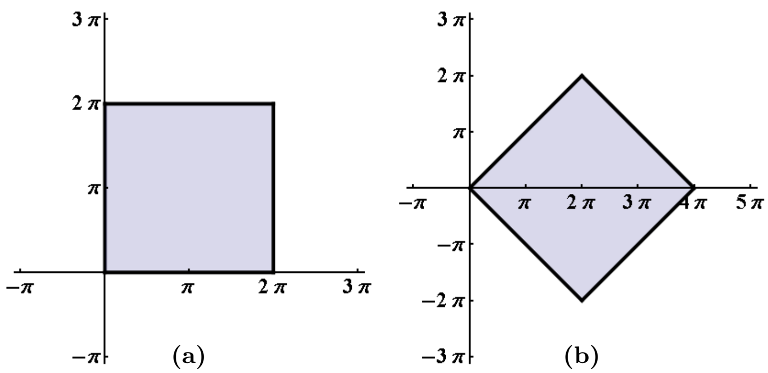

Figure 1: (a) Region of integration for the θ and ϕ variables associated with Eq. (2). (b) Region of integration for the χ and ψ variables associated with Eq. (6), where *chi* is on the "x" axis.

of them in 2π and return our integration region to that of Eq. (2), Fig. 1 (a). Specifically, we can divide the square in Fig. 1 (b) into four triangles (A, B, C, D) by first dividing the region along the $\psi = 0$ axis then along the line of $\chi = 2\pi$; see Fig. 2 (a). The lower right-hand triangle, C, can be moved into the empty triangle space; similarly B and D can be arranged to cover the square a second time. See Fig. 2 (b). We have overcounted twice, so we drop the factor of 1/2 in front of the integral. Therefore Eq. (6) becomes

$$I = \int qdqkdk \int_0^{2\pi} d\chi \int_0^{2\pi} d\psi f(k, q, \chi, \psi). \tag{7}$$

The more physical argument for this rearrangement of the integration region occurs for integrals over dot products of the two vectors $\vec{k}$ and $\vec{q}$. Then one sets, say, $\vec{q}$ as the x direction in the $\vec{k}$ coordinate system. Then ϕ denotes the angle between the two vectors (integrated from 0 to 2π), and θ describes the sweep of the $\vec{q}$ vector (also integrated from 0 to 2π).

Note the importance of the periodicity in 2π of the trigonometric functions in the above rigorous (non-physics-based) argument. One can see explicit failure of the result for, e.g.,

$$\int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \cos\left((\theta - \phi)/4\right) = 32 \neq 2\pi \int_0^{2\pi} d\psi \cos\left(\psi/4\right) = 8\pi \tag{8}$$

2 n-Dimensional and Periodic in Angular Differences

In this section we're interested in functions that depend only on the difference of the angles. This is useful for functions of dot products, where one has to integrate over cosines of the angles between vectors. First we'll examine the 2-D case.

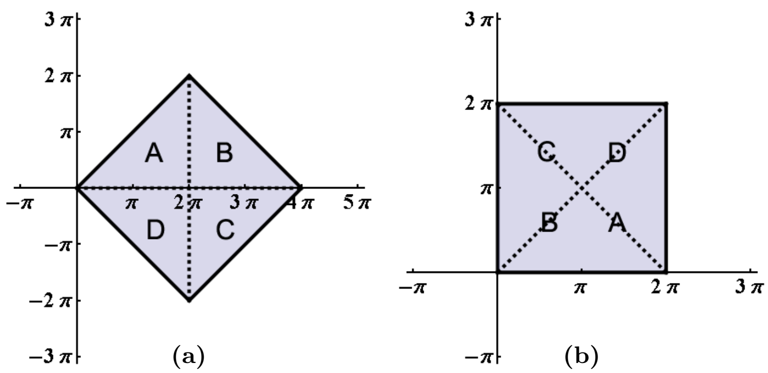

Figure 2: (a) Region of integration for the χ and ψ variables broken up into four triangles. (b) The result of rearranging the triangles back into an upright square region that is the same as in Eq. (2) and Fig. 1 (a).

2.1 2D

Suppose I have the integral

$$\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 f(\theta_2 - \theta_1). \tag{9}$$

A smart change of coordinates is

$$\begin{cases} \psi_1 = \theta_1 \\ \psi_2 = \theta_2 - \theta_1 \end{cases} \tag{10}$$

$$\Rightarrow \quad \begin{cases} \theta_1 = \psi_1 \\ \theta_2 = \psi_2 + \psi_1. \end{cases} \tag{11}$$

The Jacobian is

$$J(\psi_1, \psi_2) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \tag{12}$$

and the new region of integration is shown in Fig. 3.

Therefore Eq. (9) becomes

$$\int_0^{2\pi} d\psi_1 \int_{\psi_1 - 2\pi}^{\psi_1} d\psi_2 f(\psi_2) = \int_0^{2\pi} d\psi_1 \left[\int_{\psi_1 - 2\pi}^{0} d\psi_2 f(\psi_2) + \int_0^{\psi_1} d\psi_2 f(\psi_2) \right] \tag{13}$$

$$= \int_0^{2\pi} d\psi_1 \int_0^{2\pi} d\psi_2 f(\psi_2)$$

$$= 2\pi \int_0^{2\pi} d\psi_2 f(\psi_2) \tag{14}$$

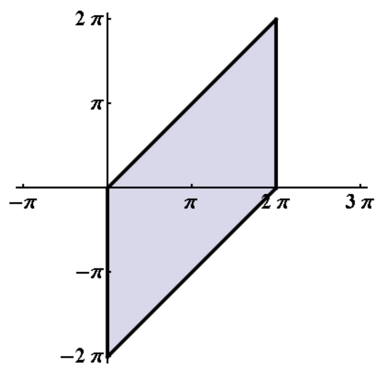

Figure 3: Region of integration in the ψ_1 and ψ_2 coordinates (with ψ_1 on the "x" axis) from Eq. (10).

where in the last line we exploited the 2π periodicity of f in ψ_2. Note that in general f may also depend on ψ_1—and does not have to be periodic in ψ_1—and Eq. (14) still holds. One can even cook up variations on the regions of integration when f has period other than 2π.

2.2 n Dimensions

Suppose now I have a function of n differences of angles (e.g. I want to integrate a function of the dot products of n vectors). Then I have the integral

$$\int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n f(\theta_2 - \theta_1, \ldots, \theta_n - \theta_1, \ldots, \theta_n - \theta_{n-1}). \tag{15}$$

Let's now generalize the previous change of coordinates to

$$\begin{cases} \psi_1 = \theta_1 \\ \psi_2 = \theta_2 - \theta_1 \\ \psi_3 = \theta_3 - \theta_1 \\ \quad \vdots \end{cases} \tag{16}$$

$$\Rightarrow \quad \begin{cases} \theta_1 = \psi_1 \\ \theta_2 = \psi_2 + \psi_1 \\ \theta_3 = \psi_3 + \psi_1 \\ \quad \vdots \end{cases} \tag{17}$$

The Jacobian is

$$J(\psi_1, \ldots, \psi_n) = \begin{vmatrix} 1 & 0 & 0 & \\ 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \\ & \vdots & & \ddots \end{vmatrix} = 1. \tag{18}$$

We can no longer draw the integration region. However we know that each ψ_n, $n > 1$, region must be exactly identical to Eq. (13): after rotating our coordinate system such that the first vector is the "x" axis, the other vectors' rotations are all independent of one another. Therefore we have that our integral, Eq. (15), becomes

$$\int_0^{2\pi} d\psi_1 \int_{\psi_1-2\pi}^{\psi_1} d\psi_2 \cdots \int_{\psi_1-2\pi}^{\psi_1} d\psi_n f(\psi_2,\ldots,\psi_n) \tag{19}$$

$$= \int_0^{2\pi} d\psi_1 \int_0^{2\pi} d\psi_2 \cdots \int_0^{2\pi} d\psi_n f(\psi_2,\ldots,\psi_n)$$

$$= 2\pi \int_0^{2\pi} d\psi_2 \cdots \int_0^{2\pi} d\psi_n f(\psi_2,\ldots,\psi_n). \tag{20}$$

Again we are free to generalize to functions f that also depend on ψ_1 (and are not necessarily periodic in ψ_1), and to functions whose period is not 2π, again so long as we are careful with the integration regions.