1 Rational Functions of Trigonometric Functions

There is a universal substitution that changes integrals of rational functions of trigonometric functions into integrals of rational functions, which are always known to be analytically integrable through the use of partial fractions and basic calculus. Specifically one can go from an integral of $R\big(\sin(\theta), \cos(\theta)\big)$, e.g.

$$\int d\theta \, \frac{1+\cos(\theta)+\sin^3(\theta)}{1-\tan(\theta)}, \tag{1}$$

to an integral of $R\big(u\big)$, e.g.

$$\int du \, \frac{(1-u^2)\big[(1+u^2)^2+8u^3\big]}{(1+u^2)^2(1-2u-u^2)} \tag{2}$$

using the substitution

$$u = \tan(\theta/2). \tag{3}$$

Then we have that

$$\theta = 2\tan^{-1}(u), \tag{4}$$

and

$$d\theta = \frac{2du}{1+u^2} \tag{5}$$

$$\sin\theta = \sin\left(2\tan^{-1}u\right) = 2\big[\sin(\tan^{-1}u)\cos(\tan^{-1}u)\big]$$

$$= \frac{2u}{1+u^2} \tag{6}$$

$$\cos\theta = \cos\left(2\tan^{-1}u\right) = \cos^2\left(\tan^{-1}u\right) - \sin^2\left(\tan^{-1}u\right)$$

$$= \frac{1-u^2}{1+u^2}, \tag{7}$$

where we have made use of several trigonometric identities; see Fig. 1

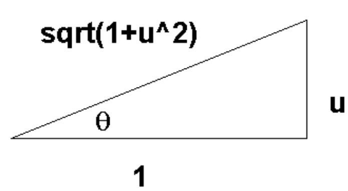

Figure 1: Figure relating u and θ in order to determine the $\sin\left(\tan^{-1}u\right)$, etc.