Traces of Gamma Matrices

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Using Peskin's notation we take

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (1, \vec\sigma), \quad \bar\sigma^\mu = (1, -\vec\sigma), \tag{1}$$

where the σ^i are the usual Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

We are using Peskin's notation that Greek indices denote a Minkowski four-vector and Latin indices denote a usual three-vector. We are also following Peskin and using a mostly minus metric.

First, note that the square of any Pauli matrix yields the identity matrix:

$$(\sigma^\mu)^2 = 1, \tag{3}$$

where index summation is *not* implied in the above equation.

Second, note that

$$(\gamma^0)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{1}, \tag{4}$$

where $\mathbf{1}$ is the 4×4 identity matrix, and

$$(\gamma^i)^2 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\begin{pmatrix} (\sigma^i)^2 & 0 \\ 0 & (\sigma^i)^2 \end{pmatrix} = -\mathbf{1}, \tag{5}$$

where, again, index summation is not implied above. I.e.,

$$(\gamma^\mu)^2 = (\mathbf{1}, -\vec{\mathbf{1}}) = (\mathbf{1}, -\mathbf{1}, -\mathbf{1}, -\mathbf{1}). \tag{6}$$

Since

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ \bar\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} \bar\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \tag{7}$$

we have that

$$(\gamma^0\gamma^i)^2 = \begin{pmatrix} \bar\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}\begin{pmatrix} \bar\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = \mathbf{1}. \tag{8}$$

If we define $C = \gamma^0\gamma^2$, then we know from Eq. (8) that $C^2 = 1$ and can show that $C\gamma^\mu C = -(\gamma^\mu)^T$, where T denotes the transpose. First note that

$$\begin{aligned}
\gamma^0\gamma^2\gamma^\mu\gamma^0\gamma^2 &= \begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sigma^2\sigma^\mu \\ -\sigma^2\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} -\sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\sigma^2\sigma^\mu\sigma^2 \\ \sigma^2\sigma^\mu\sigma^2 & 0 \end{pmatrix}.
\end{aligned} \tag{9}$$

Next, one may readily see from Eq. (3) that

$$\sigma^2\sigma^0\sigma^2 = \sigma^0, \qquad \sigma^2\sigma^2\sigma^2 = \sigma^2, \tag{10}$$

and that

$$\begin{aligned}\sigma^2\sigma^1\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma^1 \qquad (11) \\ \sigma^2\sigma^3\sigma^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma^3. \qquad (12)\end{aligned}$$

Therefore we have that

$$C\gamma^0 C = -\gamma^0 \tag{13}$$
$$C\gamma^1 C = \gamma^0 \tag{14}$$
$$C\gamma^2 C = -\gamma^0 \tag{15}$$
$$C\gamma^3 C = \gamma^0. \tag{16}$$

Since

$$(\gamma^\mu)^T = \begin{pmatrix} 0 & (\bar{\sigma}^\mu)^T \\ (\sigma^\mu)^T & 0 \end{pmatrix} \tag{17}$$

we have that

$$\left(\gamma^0\right)^T = \gamma^0 \tag{18}$$
$$\left(\gamma^1\right)^T = -\gamma^1 \tag{19}$$
$$\left(\gamma^2\right)^T = \gamma^2 \tag{20}$$
$$\left(\gamma^3\right)^T = -\gamma^3. \tag{21}$$

Putting these equations all together we find that

$$C\gamma^\mu C = -\left(\gamma^\mu\right)^T. \tag{22}$$

And then we arrive at the formula we wished to derive,

$$\begin{aligned}\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\right) &= \gamma^{\mu_1}_{i_1,i_2}\gamma^{\mu_2}_{i_2,i_3}\cdots\gamma^{\mu_n}_{i_{n-1},i_1} \\ &= \mathrm{Tr}\left(C\gamma^{\mu_1}CC\gamma^{\mu_2}C\cdots C\gamma^{\mu_n}C\right) \\ &= (-1)^n\,\mathrm{Tr}\left(\left(\gamma^{\mu_1}\right)^T\left(\gamma^{\mu_2}\right)^T\cdots\left(\gamma^{\mu_n}\right)^T\right) \\ &= (-1)^n\gamma^{\mu_1}_{i_2,i_1}\gamma^{\mu_2}_{i_3,i_2}\cdots\gamma^{\mu_n}_{i_1,i_{n-1}}) \\ &= (-1)^n\,\mathrm{Tr}\left(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right) \\ &= \mathrm{Tr}\left(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right), \qquad (23)\end{aligned}$$

where in the last line we dropped the $(-1)^n$ because the trace of any odd number of Dirac matrices is always 0.

1 Trace of 6 Gamma Matrices

Let's calculate

$$\mathrm{Tr}\,(\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f) \equiv \mathrm{Tr}\,(abcdef). \tag{24}$$

From the anticommutation relations of gamma matrices, $\{\gamma^a, \gamma^b\} = 2g^{ab}$,

$$\begin{aligned}\mathrm{Tr}\,(abcdef) &= 2g^{ab}\mathrm{Tr}\,(cdef) - \mathrm{Tr}\,(bacdef) \\ &= 2g^{ab}\mathrm{Tr}\,(cdef) - 2g^{ac}\mathrm{Tr}\,(bdef) + 2g^{ad}\mathrm{Tr}\,(bcef) \\ &\quad - 2g^{ae}\mathrm{Tr}\,(bcdf) + 2g^{af}\mathrm{Tr}\,(bcde) - \mathrm{Tr}\,(bcdefa).\end{aligned} \tag{25}$$

Since traces are cyclic and $\mathrm{Tr}\,(abcd) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{bc})$, we have that

$$\begin{aligned}\mathrm{Tr}\,(abcdef) = 4\Big[&g^{ab}\left(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}\right) \\ &-g^{ac}\left(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}\right) \\ &\ g^{ad}\left(g^{bc}g^{ef} - g^{be}g^{cf} + g^{bf}g^{ce}\right) \\ &-g^{ae}\left(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}\right) \\ &\ g^{af}\left(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd}\right)\Big]\end{aligned} \tag{26}$$