LO pQCD Production

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1 LO pQCD Hadroproduction

I wish to rederive the expression for $d\sigma/dp_T$ given in I. Sarcevic, S. D. Ellis and P. Carruthers, Phys. Rev. **D40** (1989) 1446 for the LO pQCD cross section of $p+p \rightarrow$ back-to-back partons. Start with Peskin's master formula for differential cross sections, Eq. 4.79. We will denote the incoming hadrons as A and B (with momenta p_A and p_B, respectively), and we will work in the CM frame of the hadrons. The incoming interacting partons will be labeled a and b. The outgoing partons will be labeled 1 and 2. Note that our ultimate expression will include integrations over the rapidities of the outgoing partons in the lab frame (the CM frame of the incoming hadrons), y_1 and y_2.

$$d\sigma = \frac{1}{2E_a 2E_b|v_a - v_b|} \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} dx_a dx_b \, x_a f_{a/A} x_b f_{b/B} \, |\mathcal{M}|^2 \\ \times (2\pi)^4 \delta^{(4)}(p_a + p_b - p_1 - p_2) \tag{1}$$

There are several important things to note. First[1],

$$E_a E_b \big| v_a - v_b \big| = \big| p_a^0 p_b^z - p_a^z p_b^0 \big| \tag{2}$$

Second, $x_a f_{a/A}(x_a)$ gives the probability for finding a parton of type a in hadron A with x fraction x_a. These are normalized such that, e.g.,

$$\int_0^1 x f_{u/p}(x) dx \;=\; 2; \qquad\qquad \int_0^1 x f_{s/p}(x) dx \;=\; 0; \tag{3}$$

i.e., there are two valence up quarks in a proton and no valence strange quarks. These can be used from, e.g., MSTW, (Eur.Phys.J.C63:189-285,2009; arXiv:0901.0002).

1.1 Kinematics Expressions

The expressions for $|\mathcal{M}|^2$ involve the Mandelstam variables of the partons; we need to find expressions for $x_{a,b}$, $\hat{s}$, $\hat{t}$, and $\hat{u}$, where hats denote the partonic quantities, in terms of y_1, y_2, p^T, and s. Denote lightcone coordinates with square brackets, $[,]$, and Minkowski coordinates

[1] As an aside, it is worth noting that this quantity is invariant to boosts along the z-direction. This is obvious from the construction

$$|p_a^0 p_b^z - p_a^z p_b^0| = |\epsilon_{\mu x y \nu} p^\mu p^\nu|,$$

but it is worth calculating out in detail. Under a boost,

$$\begin{aligned} p_a^0 p_b^z - p_a^z p_b^0 \;&\rightarrow p_a^{0'} p_b^{z'} - p_a^{z'} p_b^{0'} \\ &\gamma^2 \left[\left(p_a^0 - \beta p_a^z\right)\left(p_b^z - \beta p_b^0\right) - \left(p_a^z - \beta p_a^0\right)\left(p_b^0 - \beta p_b^z\right) \right] \\ &= \gamma^2 \left[p_a^0 p_b^z - \beta\left(p_a^z p_b^z + p_z^0 p_b^0\right) + \beta^2 p_a^z p_b^0 - p_a^z p_b^0 + \beta\left(p_z^0 p_b^0 + p_a^z p_b^z\right) - \beta^2 p_a^0 p_b^z \right] \\ &= p_a^0 p_b^z - p_a^z p_b^0. \end{aligned}$$

So even though the differential cross section has units of Energy^{-2}, it **does not** transform with a γ^{-2} if boosted along the z-direction, it is invariant.

with parentheses, (,). Four vectors will be in plain style, three vectors will have an arrow, and two vectors will be boldface. Note the use of the $p^{\pm} = p^0 \pm p^z$ normalization convention. Then

$$
\begin{aligned}
p_A &= [\sqrt{s}, 0, \mathbf{0}] & p_B &= [0, \sqrt{s}, \mathbf{0}] \\
p_a &= x_a p_A = [x_a\sqrt{s}, 0, \mathbf{0}] & p_b &= x_b p_B = [0, x_b\sqrt{s}, \mathbf{0}] \\
&= \frac{x_a\sqrt{s}}{2}(1, 1, \mathbf{0}) & &= \frac{x_b\sqrt{s}}{2}(1, -1, \mathbf{0}).
\end{aligned} \tag{4}
$$

In order to connect $x_{a,b}$ to $y_{1,2}$, consider the conservation of 4-momentum in the lab frame:

$$p_a + p_b = \frac{\sqrt{s}}{2}(x_a + x_b,\, x_a - x_b,\, \mathbf{0}) = p_1 + p_2 \tag{5}$$

$$\Rightarrow \quad p_1^0 + p_2^0 = \frac{\sqrt{s}}{2}(x_a + x_b), \quad p_1^z + p_2^z = \frac{\sqrt{s}}{2}(x_a - x_b), \quad \mathbf{p}_1^T = -\mathbf{p}_2^T. \tag{6}$$

$$\Rightarrow x_a = \frac{p_1^0 + p_1^z + p_2^0 + p_2^z}{\sqrt{s}} = \frac{p_1^+ + p_2^+}{\sqrt{s}} \tag{7}$$

$$\Rightarrow x_b = \frac{p_1^0 - p_1^z + p_2^0 - p_2^z}{\sqrt{s}} = \frac{p_1^- + p_2^-}{\sqrt{s}} \tag{8}$$

Next consider the definition of rapidity and the outgoing momenta. In general

$$y \equiv \frac{1}{2}\ln\left(\frac{p^0 + p^z}{p^0 - p^z}\right) = \frac{1}{2}\ln\left(\frac{p^+}{p^-}\right). \tag{9}$$

Therefore, with $p^T = |\mathbf{p}^T|$, for massless partons

$$
\begin{aligned}
p_f = [p_f^+, \frac{(p^T)^2}{p_f^+}, \mathbf{p}^T] \qquad p_f = [\frac{(p^T)^2}{p_f^-}, p_f^-, \mathbf{p}^T] \\
\Rightarrow \quad p_f^{\pm} = p^T e^{\pm y_f}.
\end{aligned} \tag{10}
$$

Inserting Eq. (10) in Eqs. (7) and (8) yields

$$x_a = \frac{p^T}{\sqrt{s}}\left(e^{y_1} + e^{y_2}\right) \tag{11}$$

$$x_b = \frac{p^T}{\sqrt{s}}\left(e^{-y_1} + e^{-y_2}\right) \tag{12}$$

Note that Eqs. (11) and (12) are equivalent to the expressions used in Sarcevic.

The partonic Mandelstam variables are

$$\hat{s} \equiv (p_a + p_b)^2 = 2p_a \cdot p_b = p_a^+ p_b^- = x_a x_b s = 2(p^T)^2\big(1 + \cosh(y_2 - y_1)\big) \tag{13}$$

$$
\begin{aligned}
\hat{t} &\equiv (p_1 - p_a)^2 = -2p_1 \cdot p_a = -p_1^- p_a^+ = -x_a\sqrt{s}p^T e^{-y_1} \\
&\equiv (p_2 - p_b)^2 = -2p_2 \cdot p_b = -p_2^+ p_b^- = -x_b\sqrt{s}p^T e^{y_2} \\
&= -(p^T)^2\left(1 + e^{y_2 - y_1}\right)
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\hat{u} &\equiv (p_2 - p_a)^2 = -2p_2 \cdot p_a = -p_2^- p_a^+ = -x_a\sqrt{s}p^T e^{-y_2} \\
&\equiv (p_1 - p_b)^2 = -2p_1 \cdot p_b = -p_1^+ p_b^- = -x_b\sqrt{s}p^T e^{y_1} \\
&= -(p^T)^2\left(1 + e^{y_1 - y_2}\right)
\end{aligned} \tag{15}
$$

1.2 Integrating Over Phase Space

Let's integrate over phase space to the extent possible. First, expand out the four dimensional delta function. Noting that

$$\frac{\sqrt{s}}{2}(x_a + x_b) = p^T\big(\cosh(y_1) + \cosh(y_2)\big) = 2p^T \cosh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} \tag{16}$$

$$\frac{\sqrt{s}}{2}(x_a - x_b) = p^T\big(\sinh(y_1) + \sinh(y_2)\big) = 2p^T \sinh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2}, \tag{17}$$

the energy-momentum conserving delta function becomes

$$\delta^{(4)}(p_a + p_b - p_1 - p_2) = \delta\Big(2p^T\cosh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - p_1^0 - p_2^0\Big)$$
$$\times\,\delta\Big(2p^T\sinh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - p_1^z - p_2^z\Big)\delta^{(2)}\big(\mathbf{p}_1^T - \mathbf{p}_2^T\big) \tag{18}$$

Use the 3 integrals over $\vec{p}_2$ to set $\mathbf{p}_2^T = -\mathbf{p}_1^T$ and $p_2^z = 2p^T\sinh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - p_1^z$. Using on-shellness of the outgoing partons 1 and 2 we can rewrite the argument of the remaining delta function (which we will ultimately integrate over p_1^z) as

$$f(p_1^z) \equiv 2p^T\cosh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - \sqrt{(p^T)^2 + (p_1^z)^2}$$
$$- \sqrt{(p^T)^2 + (2p^T\sinh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - p_1^z)^2}. \tag{19}$$

After some slightly tedious algebra, we find the zeros of $f(p_1^z)$ are

$$p_{1,\pm}^z = \frac{p^T}{2}\left(\sinh y_1 + \sinh y_2 \pm |\sinh y_1 - \sinh y_2|\right). \tag{20}$$

Hence the two solutions are

$$p_{1,+}^z = \begin{cases} p^T\sinh y_1, & y_1 \geq y_2 \\ p^T\sinh y_2, & y_1 < y_2 \end{cases} \tag{21}$$

$$p_{1,-}^z = \begin{cases} p^T\sinh y_2, & y_1 \geq y_2 \\ p^T\sinh y_1, & y_1 < y_2 \end{cases} \tag{22}$$

More algebra shows that

$$\left|1\Big/\frac{\partial f}{\partial p_1^z}\right|_{p_1^z = p^T\sinh y_{1,2}} = \left|\frac{\cosh y_{1,2}\cosh y_{2,1}}{\sinh y_{1,2}\cosh y_{2,1} - \sinh y_{2,1}\cosh y_{1,2}}\right| = \frac{\cosh y_1\cosh y_2}{|\sinh(y_2 - y_1)|} \tag{23}$$

Putting the above pieces together we find that

$$\delta\big(f(p_1^z)\big) = \frac{\cosh y_1\cosh y_2}{|\sinh(y_2-y_1)|}\big[\theta(y_1-y_2)\delta(p_1^z - p^T\sinh y_1) + \theta(y_2-y_1)\delta(p_1^z - p^T\sinh y_2)$$
$$+\theta(y_1-y_2)\delta(p_1^z - p^T\sinh y_2) + \theta(y_2-y_1)\delta(p_1^z - p^T\sinh y_1)\big]$$
$$= \frac{\cosh y_1\cosh y_2}{|\sinh(y_2-y_1)|}\big[\delta(p_1^z - p^T\sinh y_1) + \delta(p_1^z - p^T\sinh y_2)\big]. \tag{24}$$

Note that since $p_2^z = 2p^T\sinh\frac{y_1+y_2}{2}\cosh\frac{y_1-y_2}{2} - p_1^z = p^T(\sinh y_1 + \sinh y_2) - p_1^z$,

$$p_1^z = p^T\sinh y_{1,2} \quad\Rightarrow\quad p_2^z = p^T\sinh y_{2,1}. \tag{25}$$

That the outgoing particles are on mass-shell implies that their momenta are constrained to be

$$p_{1,2} = \big(p^T\cosh y_{1,2},\, p^T\sinh y_{1,2},\, \mathbf{p}^T\big). \tag{26}$$

It seems that the E-M conserving delta function above leads to over counting. E-M conservation only means that one particle emerges with y_1 and the other emerges with y_2. However, we can't have particle 1 emerging with particle 2's rapidity! Therefore we take the above delta function

$$\delta\big(f(p_1^z)\big) \to \frac{\cosh y_1\cosh y_2}{|\sinh(y_2-y_1)|}\delta(p_1^z - p^T\sinh y_1). \tag{27}$$

1.3 Relative Velocity, Change of Variables, and the Cross Section

We need to find an expression in terms of the pertinent variables for the relative velocity denominator of the cross section, Eq. (1). Using Eqs. (2) and (10) we find that

$$\left|p_a^0 p_b^z - p_a^z p_b^0\right| E_1 E_2 = (x_a x_b s/2)\, p^T \cosh y_1\, p^T \cosh y_2 = (\hat{s}/2)(p^T)^2 \cosh y_1 \cosh y_2. \tag{28}$$

Therefore the cross section becomes, after integrating over p_1^z,

$$d\sigma = \frac{1}{2^3 (2\pi)^2 \hat{s} (p^T)^2 \cosh y_1 \cosh y_2} p^T dp^T dp_1^\phi \frac{\cosh y_1 \cosh y_2}{\left|\sinh(y_2 - y_1)\right|}\, dx_a dx_b\, x_a f_{a/A} x_b f_{b/B} \left|\mathcal{M}\right|^2 \tag{29}$$

Now we just have to make a change of integration variables from x_a, x_b, and p^T to y_1, y_2, and $\tilde{p}^T$, where we will temporarily use a tilde sign on the latter p^T in order to distinguish it from the old variable. From Eqs. (11) and (12), which give the old coordinates $x_{a,b}$ in terms of the new coordinates $y_{1,2}$ and $\tilde{p}^T$, we have that

$$\left|J(y_1,\, y_2,\, \tilde{p}^T;\, x_a,\, x_b,\, p^T)\right| = \begin{vmatrix} \frac{\partial x_a}{\partial y_1} & \frac{\partial x_a}{\partial y_2} & \frac{\partial x_a}{\partial \tilde{p}^T} \\ \frac{\partial x_b}{\partial y_1} & \cdots & \cdots \\ \frac{\partial p^T}{\partial y_1} & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} \frac{\tilde{p}^T}{\sqrt{s}} e^{y_1} & \frac{\tilde{p}^T}{\sqrt{s}} e^{y_2} & . \\ -\frac{\tilde{p}^T}{\sqrt{s}} e^{-y_1} & -\frac{\tilde{p}^T}{\sqrt{s}} e^{-y_2} & . \\ 0 & 0 & 1 \end{vmatrix} \tag{30}$$

$$= \frac{\left(\tilde{p}^T\right)^2}{s} \left|e^{y_2 - y_1} - e^{y_1 - y_2}\right| = 2\frac{\left(\tilde{p}^T\right)^2}{s} \left|\sinh(y_2 - y_1)\right|. \tag{31}$$

Using this result, canceling factors, removing the dummy tilde and dropping the T to a subscript on the p's, and integrating over p_1^ϕ yields

$$\frac{d\sigma}{dp_T dy_1 dy_2} = \frac{p_T}{8\pi \hat{s} s}\, x_a f_{a/A} x_b f_{b/B} \left|\mathcal{M}\right|^2, \tag{32}$$

where $x_a f_{a/A} x_b f_{b/B}$ is understood to be evaluated in terms of $x_{a,b}(y_{1,2}, p_T)$.

1.4 Simplest Amplitude

In order to check our result against Sarcevic, let's evaluate the simplest amplitude, $q_i g_j' \to q_i q_j'$. If one factors out $g^4 = (4\pi\alpha_s)^2$ from the amplitude, then

$$\frac{d\sigma}{dp_T dy_1 dy_2} = \frac{2\pi \alpha_s^2 p_T}{\hat{s} s}\, x_a f_{a/A} x_b f_{b/B} \left|\mathcal{M}\right|^2, \tag{33}$$

where for, e.g., $q_i q_j' \to q_i q_j'$ scattering properly summed and averaged over both spins and colors

$$\left|\mathcal{M}\{q_i q_j' \to q_i q_j'\}\right|^2 = \frac{4}{9}\left(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}\right). \tag{34}$$

The above equation comes from using Eq. (5.71) of Peskin for $e^- + \mu^- \to e^- + \mu^-$ scattering amplitude,

$$\frac{1}{4}\sum_{\text{spins}} \left|\mathcal{M}\{e^-\mu^- \to e^-\mu^-\}\right|^2 = 2\left(\frac{s^2 + u^2}{t^2}\right), \tag{35}$$

combined with the correct color averaging:

$$\begin{aligned} \frac{1}{N_c^2} \sum_{a_c, b_c, 1_c, 2_c} t^a_{1_c,a_c} t^a_{2_c,b_c} \left(t^b_{1_c,a_c}\right)^* \left(t^b_{2_c,b_c}\right)^* &= \frac{1}{N_c^2} \sum_{a_c, b_c, 1_c, 2_c} t^a_{1_c,a_c} t^b_{a_c,1_c} t^a_{2_c,b_c} t^b_{b_c,2_c} \\ &= \frac{1}{N_c^2} \mathrm{Tr}\, t^a t^b \mathrm{Tr}\, t^a t^b = \frac{1}{N_c^2} \frac{1}{2}\delta^{ab} \frac{1}{2}\delta^{ab} \\ &= \frac{N_c^2 - 1}{4N_c^2} \to \frac{2}{9} \end{aligned} \tag{36}$$

The firt equality is due to the Hermiticity of the generators, the rest to color algebra. In the final expression N_c is taken to be 3.