Eikonal Approximation in QCD Scattering

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1 2-2 Scattering

1.1 Exact 2-2 Scattering

We wish to compute the exact summed and averaged over matrix element squared for the 2-2 scattering process of 2 quarks going to 2 quarks as displayed in Fig. 1. The exact, turgid formula

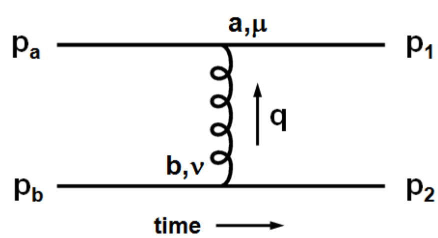

Figure 1: Feynman diagram for the (t-channel) 2-2 scattering process we are interested in calculating. Each incoming and outgoing quark has an associated spin and color; i.e. particle a has spin s_a and color c_a. These will ultimately be summed and averaged over.

we have is

$$i\mathcal{M}^{s_1,c_1,s_2,c_2,s_a,c_a,s_b,c_b} = \bar{u}^{s_1,c_1}_{i_{s_1},i_{c_1}}(p_1) ig\gamma^\mu_{i_{s_1},j_{s_a}} t^a_{i_{c_1},j_{c_a}} u^{s_a,c_a}_{i_{s_a},i_{c_a}}(p_a) \frac{-ig_{\mu\nu}\delta_{ab}}{q^2} \bar{u}^{s_2,c_2}_{i_{s_2},i_{c_2}}(p_2) ig\gamma^\nu_{i_{s_2},j_{s_b}} t^b_{i_{c_2},j_{c_b}} u^{s_b,c_b}_{i_{s_b},i_{c_b}}(p_b), \tag{1}$$

where the a and b superscripts on the t color matrices are color indices, implicitly summed over (there are N_c^2-1 gluons, each of which contributes to $\mathcal{M}$), not to be confused with the initial a and b particles, and the matrix indices (such as i_{s_1}, the index associated with the spin of particle 1) have been kept explicit. The spin and color summed and averaged result is (dropping the indices on $\mathcal{M}$)

$$\frac{1}{4}\frac{1}{N_c^2}\sum_{\substack{s_1,c_1,s_2,c_2\\ s_a,c_a,s_b,c_b}} |\mathcal{M}|^2 = \frac{1}{4}\frac{1}{N_c^2} g^4 \frac{g_{\mu\nu}g_{\alpha\beta}}{q^4} \sum \bar{u}^{s_1,c_1}_{i_{s_1},i_{c_1}} \gamma^\mu t^a u^{s_a,c_a}_j \bar{u}^{s_2,c_2}_i \gamma^\nu t^a u^{s_b,c_b}_j \bar{u}^{s_b,c_b}_i \gamma^\alpha t^b u^{s_2,c_2}_j \bar{u}^{s_a,c_a}_i \gamma^\beta t^b u^{s_1,c_1}_j, \tag{2}$$

where I've suppressed some of the indices for brevity. In gory detail this is then equal to, noting that the u and $\bar{u}$ form a complete basis in color space:

$$\frac{1}{4}\frac{1}{N_c^2}g^4\frac{g_{\mu\nu}g_{\alpha\beta}}{q^4}\left[(\not{p}_1)_{j_{s_1},i_{s_1}}\gamma^{\mu}_{i_{s_1},j_{s_a}}(\not{p}_a)_{j_{s_a},i_{s_a}}\gamma^{\beta}_{i_{s_a},j_{s_1}}\right]\left[(\not{p}_2)_{j_{s_2},i_{s_2}}\gamma^{\nu}_{i_{s_2},j_{s_b}}(\not{p}_b)_{j_{s_b},i_{s_b}}\gamma^{\alpha}_{i_{s_b},j_{s_2}}\right]$$
$$\left[\delta_{j_{c_1},i_{c_1}}t^a_{i_{c_1},j_{c_a}}\delta_{j_{c_a},i_{c_a}}t^b_{i_{c_a},j_{c_1}}\right]\left[\delta_{j_{c_2},i_{c_2}}t^a_{i_{c_2},j_{c_b}}\delta_{j_{c_b},i_{c_b}}t^b_{i_{c_b},j_{c_2}}\right] \tag{3}$$

$$= \frac{1}{4}\frac{1}{N_c^2}g^4\frac{g_{\mu\nu}g_{\alpha\beta}}{q^4}\,\mathrm{Tr}\,(t^a t^b)\mathrm{Tr}\,(t^a t^b)\,\mathrm{Tr}\,(\not{p}_1\gamma^{\mu}\not{p}_a\gamma^{\beta})\mathrm{Tr}\,(\not{p}_2\gamma^{\nu}\not{p}_b\gamma^{\alpha}) \tag{4}$$

$$= \frac{1}{4}\frac{1}{N_c^2}g^4\frac{g_{\mu\nu}g_{\alpha\beta}}{q^4}\,C(r)\delta^{ab}C(r)\delta^{ab}\,p_{1_\gamma}p_{a_\delta}p_{2_\rho}p_{b_\sigma}\,(4)^2\left[g^{\gamma\mu}g^{\delta\beta}-g^{\gamma\delta}g^{\mu\beta}+g^{\gamma\beta}g^{\mu\delta}\right]$$
$$\left[g^{\rho\nu}g^{\sigma\alpha}-g^{\rho\sigma}g^{\nu\alpha}+g^{\rho\alpha}g^{\nu\sigma}\right] \tag{5}$$

$$= \frac{1}{4}\frac{1}{N_c^2}\frac{g^4}{q^4}\left(C(r)\right)^2N_c\,p_{1_\gamma}p_{a_\delta}p_{2_\rho}p_{b_\sigma}\,(4)^2\left[g^{\gamma}_{\nu}g^{\delta}_{\alpha}-g^{\gamma\delta}g_{\nu\alpha}+g^{\gamma}_{\alpha}g^{\delta}_{\nu}\right]\left[g^{\rho\nu}g^{\sigma\alpha}-g^{\rho\sigma}g^{\nu\alpha}+g^{\rho\alpha}g^{\nu\sigma}\right] \tag{6}$$

$$= \frac{1}{4}\frac{1}{N_c^2}\frac{g^4}{q^4}\frac{N_c}{4}\,p_{1_\gamma}p_{a_\delta}p_{2_\rho}p_{b_\sigma}\,(4)^2 2\left[g^{\rho\gamma}g^{\sigma\delta}+g^{\rho\delta}g^{\gamma\sigma}\right] \tag{7}$$

$$= \frac{1}{4}\frac{1}{N_c^2}\frac{g^4}{q^4}\frac{N_c}{4}\,(32)\left[p_1\cdot p_2\,p_a\cdot p_b+p_2\cdot p_a\,p_1\cdot p_b\right] \tag{8}$$

From Fig. 1 we can see that

$$p_b = q + p_2 \quad\Rightarrow\quad q = p_b - p_2 \quad\Rightarrow\quad q^4 = t^2 \tag{9}$$

and that

$$s = (p_1+p_2)^2 = 2p_1\cdot p_2 = 2p_a\cdot p_b \tag{10}$$
$$u = (p_2-p_a)^2 = -2p_2\cdot p_a = -2p_1\cdot p_b. \tag{11}$$

Therefore we have that

$$\frac{1}{4}\frac{1}{N_c^2}\sum|\mathcal{M}|^2 = \frac{1}{4N_c}2\frac{g^4}{t^2}\left(s^2+u^2\right) \tag{12}$$
$$= \frac{1}{4N_c}\frac{32\pi^2\alpha_s^2}{t^2}\left(s^2+u^2\right), \tag{13}$$

as

$$g = \sqrt{4\pi\alpha_s}. \tag{14}$$

1.2 Lightcone Convention

We will take as convention that the four momentum p is, in Minkowski and lightcone coordinates,

$$p = (p^0, p^z, \boldsymbol{p}) = [p^+, p^-, \boldsymbol{p}], \tag{15}$$
$$\boldsymbol{p} = p^i,\ \text{for } i = 1{,}2, \tag{16}$$

with the normalization such that

$$p^{\pm} = \frac{1}{\sqrt{2}}\left(p^0 \pm p^z\right) \tag{17}$$
$$p^{0,z} = \frac{1}{\sqrt{2}}\left(p^+ \pm p^z\right). \tag{18}$$

In this case the metric becomes

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{19}$$

Then

$$p \cdot q = p^+ q^- + p^- q^+ - \boldsymbol{p} \cdot \boldsymbol{q}. \tag{20}$$

In particular

$$p^2 = 2p^+ p^- - \boldsymbol{p}^2, \tag{21}$$

and if p is on-shell and corresponds to a massless particle, then

$$p = \left[p^+, \frac{\boldsymbol{p}^2}{2p^+}, \boldsymbol{p}\right] = \left[\frac{\boldsymbol{p}^2}{2p^-}, p^-, \boldsymbol{p}\right]. \tag{22}$$

1.3 2-2 Scattering with Large Lightcone Momenta

Now consider the process depicted in Fig. 1 assuming that p_a has a very large p_a^+ component while p_b has a very large p_b^- component (note that "large" here is with respect to their momenta in the perpendicular direction). We can use the formulae for the Mandelstam variables, Eqs. (10) and (11), to immediately arrive at the result using the exact result from above, Eq. (12). Specifically, we take

$$p_a = [p_a^+, 0, \mathbf{0}] \qquad \text{and} \qquad p_b = [0, p_b^-, \mathbf{0}]. \tag{23}$$

Then, to lowest order in perp momenta over large momenta,

$$q^2 = \boldsymbol{q}^2 \qquad \text{and} \qquad s = 2p_a^+ p_b^- \qquad \text{and} \qquad u \approx -2p_a^+ p_b^-. \tag{24}$$

Therefore Eq. (12) becomes

$$\frac{1}{4N_c} \frac{g^4}{t^2} 2 \left(s^2 + u^2\right) \approx \frac{1}{4N_c} \frac{g^4}{\boldsymbol{q}^4} 16 \left(p_a^+ p_b^-\right)^2. \tag{25}$$

We should also be able to derive this using the eikonal approximation,

$$\bar{u}_r \gamma^\mu u_\sigma \approx 2p^\mu \delta_{\sigma r}, \tag{26}$$

where σ and r denote the chirality of the incoming and outgoing particles, respectively; ultimately, when computing a summed and averaged matrix element squared, these chiralities will have to be summed over (chirality is either $+$ or $-$ depending on whether the spin is aligned or anti-aligned with the particle's motion). Taking care with chiralities and color indices, Fig. 1 yields

$$\begin{aligned} i\mathcal{M}^{r_1,r_2,\sigma_a,\sigma_b,c_1,c_2,c_a,c_b} &= \bar{u}_{i_{c_1}}^{r_1,c_1}(p_1) ig\gamma^\mu t^a_{i_{c_1},j_{c_a}} u_{j_{c_a}}^{\sigma_a,c_a}(p_a) \frac{-ig_{\mu\nu}}{\boldsymbol{q}^2} \bar{u}_{i_{c_2}}^{r_2,c_2}(p_2) ig\gamma^\nu t^a_{i_{c_2},j_{c_b}} u_{j_{c_b}}^{\sigma_b,c_b}(p_b) \\ &= \frac{ig_{\mu\nu}}{\boldsymbol{q}^2} 2p_a^\mu \delta_{r_1,\sigma_a} t^a_{i_{c_1},j_{c_a}} 2p_b^\nu \delta_{r_2,\sigma_b} t^a_{i_{c_2},j_{c_b}} \\ &= \frac{4ig_{\mu\nu}}{\boldsymbol{q}^2} p_a^+ p_b^- \, t^a_{i_{c_1},j_{c_a}} t^a_{i_{c_2},j_{c_b}} \end{aligned} \tag{27}$$

Hence

$$\begin{aligned} \frac{1}{4}\frac{1}{N_c^2} \sum_{\substack{s_1,c_1,s_2,c_2 \\ s_a,c_a,s_b,c_b}} |\mathcal{M}|^2 &= \frac{1}{4}\frac{1}{N_c^2} \frac{4^2 g^4}{\boldsymbol{q}^4} \left(p_a^+ p_b^-\right)^2 \delta_{r_1,\sigma_a}\delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b}\delta_{r_2,\sigma_b} t^a_{i_{c_1},j_{c_a}} t^a_{i_{c_2},j_{c_b}} t^b_{j_{c_a},i_{c_1}} t^b_{j_{c_b},i_{c_2}} \\ &= \frac{1}{4}\frac{1}{N_c^2} \frac{4^2 g^4}{\boldsymbol{q}^4} \left(p_a^+ p_b^-\right)^2 4 \, \frac{N_c}{4} \\ &= \frac{1}{4N_c} \frac{g^4}{\boldsymbol{q}^4} 16 \left(p_a^+ p_b^-\right)^2, \end{aligned} \tag{28}$$

where, again, the color indices a and b are implicitly summed over. We see that the final result of Eq. (28) here is the same as the expansion of the full result above, Eq. (25).

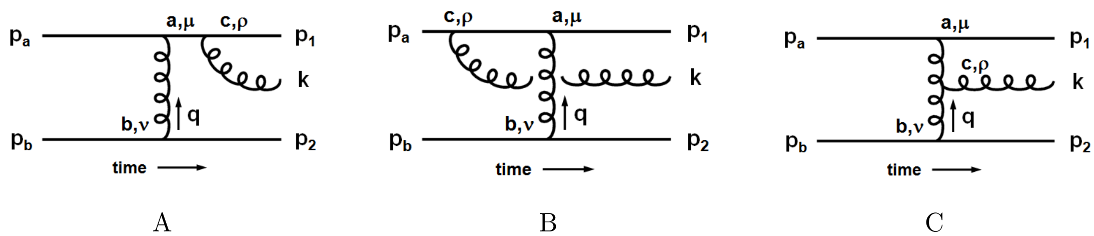

Figure 2: Dominant Feynman diagrams contributing to the 2-3 gluon production cross section in high energy quark-quark scattering. The diagrams with gluons connected to the lower quark line are suppressed by inverse powers of large momenta due to the $A^+ = 0$ light-cone gauge choice.

2 2-3 Scattering

We now wish to calculate in gory detail the leading order contribution to 2-3 scattering in the eikonal approximation. (One can also refer to Horowitz and Kovchegov, arXiv:1009.0545.) The relevant diagrams are shown in Fig. 2. Let's first compute the relevant momenta to leading order. We will have five unknown momenta left undetermined that we will integrate over for the cross section; we will take these five as k^+, $\boldsymbol{k}$, and $\boldsymbol{q}$. Then using on-shell-ness for the final state particles and 4-momentum conservation we immediately have that

$$\begin{aligned} p_a + p_b = p_1 + p_2 + k \qquad &\Rightarrow \qquad p_a + q = p_1 + k \\ p_a = [p_a^+, 0, \mathbf{0}] \qquad & \qquad p_b = [0, p_b^-, \mathbf{0}] \qquad (29) \\ p_1 = [p_1^+, \frac{(\boldsymbol{q}-\boldsymbol{k})^2}{2p_1^+}, \boldsymbol{q}-\boldsymbol{k}] \qquad & \qquad p_2 = p_b - q = [-q^+, p_b^- - q^-, -\boldsymbol{q}] \qquad (30) \\ k = [k^+, \frac{\boldsymbol{k}^2}{2k^+}, \boldsymbol{k}] \qquad & \qquad = [\frac{\boldsymbol{q}^2}{2(p_b^- - q^-)}, pb^- - q^-, -\boldsymbol{q}]. \qquad (31) \end{aligned}$$

EM conservation can be used again to find the only remaining unknown, q^-:

$$p_a + q = p_1 + k \Rightarrow \begin{cases} p_1^+ = p_a^+ + q^+ - k^+ \\ q^- = p_1^- + k^- = \dfrac{(\boldsymbol{q}-\boldsymbol{k})^2}{2p_1^+} + \dfrac{\boldsymbol{k}^2}{2k^+} \end{cases} \qquad (32)$$

Consistently solving for q^- yields a quadratic equation. The solution of interest is found unambiguously by requiring that to lowest order $q^- \sim \boldsymbol{k}^2/2k^+$; the incorrect solution for our set of approximations yields $q^- \sim p_b^-$. To leading order, then, the momenta are

$$\begin{aligned} p_a &= [p_a^, 0, \mathbf{0}] & p_1 &\simeq [p_a^+, \frac{(\boldsymbol{q}-\boldsymbol{k})^2}{2p_a^+}, \boldsymbol{q}-\boldsymbol{k}] \\ p_b &= [0, p_b^+, \mathbf{0}] & p_2 &\simeq [\frac{\boldsymbol{q}^2}{2p_b^-}, p_b^-, -\boldsymbol{q}] \qquad (33) \\ k &= [k^+, \frac{\boldsymbol{k}^2}{2k^+}, \boldsymbol{k}] & q &\simeq [-\frac{\boldsymbol{q}^2}{2p_b^-}, \frac{\boldsymbol{k}^2}{2k^+}, \boldsymbol{q}]. \end{aligned}$$

Note that $q^2 \simeq -\boldsymbol{q}^2 < 0$. We will do this computation in lightcone gauge such that $\eta \cdot A = A^+ = 0$; i.e.,

$$\eta_\mu = [1, 0, \mathbf{0}] \qquad \Rightarrow \qquad \eta^\mu = [0, 1, \mathbf{0}]. \qquad (34)$$

In lightcone gauge the propagator for a gluon of four-momentum q with Lorentz indices μ and ν is

$$\frac{-i}{q^2}\left(g_{\mu\nu} - \frac{\eta_\mu q_\nu + \eta_\nu q_\mu}{\eta \cdot q}\right). \qquad (35)$$

We also know that the polarization vectors ϵ^λ obey $\eta \cdot \epsilon = \epsilon^+ = 0$ and $k \cdot \epsilon = 0$ for $\lambda = 1, 2$. Therefore

$$\epsilon^\mu = \epsilon^{*\mu} = [0, \frac{\boldsymbol{k} \cdot \boldsymbol{\epsilon}}{k^+}, \boldsymbol{\epsilon}]. \qquad (36)$$

We can now start evaluating matrix elements. First, let's examine $\mathcal{M}_A$:

$$
\begin{aligned}
i\mathcal{M}_A^{r_1,\sigma_a,r_2,\sigma_b,c_1,c_2,c_a,c_b,\lambda} &= \bar{u}_{i_{c_1}}^{r_1,c_1}(p_1) ig\gamma^\rho t^c_{i_{c_1},j_c} \frac{i\not{p}_A}{p_A^2} ig\gamma^\mu t^a_{j_c,j_{c_a}} u^{\sigma_a,c_a}_{j_{c_a}}(p_a) \\
&\quad \times \bar{u}_{i_{c_2}}^{r_2,c_2}(p_2) ig\gamma^\nu t^a_{i_{c_2},j_{c_b}} u^{\sigma_b,c_b}_{j_{c_b}}(p_b) \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{\eta_\mu q_\nu + \eta_\nu q_\mu}{\eta\cdot q}\right)\epsilon^{*\lambda}_\rho
\end{aligned} \tag{37}
$$

$$
\simeq \frac{-8(i)^5 g^3\, p_a^\mu p_{b\mu}\, p_a\cdot\epsilon^{*\lambda}}{p_A^2 q^2}\, t^c_{i_{c_1},j_c} t^a_{j_c,j_{c_a}} t^a_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \tag{38}
$$

$$
\simeq \frac{-8ig^3 p_a^+ p_b^-}{-\boldsymbol{q}^2} \frac{p_a^+ \boldsymbol{k}\cdot\boldsymbol{\epsilon}^\lambda}{k^+} \frac{k^+}{p_a^+\boldsymbol{k}^2}\, t^c_{i_{c_1},j_c} t^a_{j_c,j_{c_a}} t^a_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \tag{39}
$$

$$
= \frac{8ig^3\, p_a^+ p_b^-}{\boldsymbol{q}^2\,\boldsymbol{k}^2}\, \boldsymbol{k}\cdot\boldsymbol{\epsilon}^\lambda\, t^c_{i_{c_1},j_c} t^a_{j_c,j_{c_a}} t^a_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b}. \tag{40}
$$

Let's explain the derivation. We exploited a number of tricks in going from Eq. (37) to Eq. (38). Specifically, we used the $\bar{u}$ version of "Peskin's" trick:

$$
\bar{u}(p)\not{\epsilon}\not{p} = 2p\cdot\epsilon\,\bar{u}(p) \tag{41}
$$

$$
\not{p}\not{\epsilon}u(p) = 2p\cdot\epsilon\,u(p). \tag{42}
$$

Since $q = p_b - p_2$ we are able to drop the $\eta_\mu q_\nu$ term in the gluon propagator by noticing that it is proportional to

$$
\bar{u}(p_2)\gamma^\nu u(p_b) q_\nu = \bar{u}(p_2)\not{q}u(p_b) = \bar{u}(p_2)\big(\not{p}_b - \not{p}_2\big)u(p_b) = 0 \tag{43}
$$

by the Dirac equation. Eikonality was used twice. And since the dominant contribution from the p^ν that comes from exploiting eikonality is its minus component, we are able to drop the $\eta_\nu q_\mu$ term in the gluon propagator as $\eta_- = 0$.

To get from Eq. (38) to Eq. (39) we need p_A^2. From Fig. 2 we have that

$$
p_A = p_a + q \quad\Rightarrow\quad p_A^2 \simeq 2p_a^+\, q^- \simeq \frac{p_a^+\boldsymbol{k}^2}{k^+}. \tag{44}
$$

Let's now evaluate $\mathcal{M}_B$. We can see from Fig. 2 that there are two differences between A and B: the order of the t^c and one t^a color matrix is switched, and the fermion propagator is evaluated at momentum $p_B = p_a - k$ instead of p_A. Noting that

$$
p_B^2 = -2p_a\cdot k \simeq -\frac{p_a^+\boldsymbol{k}}{k^+} \tag{45}
$$

we can readily write down

$$
i\mathcal{M}_B^{r_1,\sigma_a,r_2,\sigma_b,c_1,c_2,c_a,c_b,\lambda} \simeq -\frac{8ig^3\, p_a^+ p_b^-}{\boldsymbol{q}^2\,\boldsymbol{k}^2}\, \boldsymbol{k}\cdot\boldsymbol{\epsilon}^\lambda\, t^a_{i_{c_1},j_c} t^c_{j_c,i_{c_a}} t^a_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b}. \tag{46}
$$

Now let's compute $\mathcal{M}_C$. From Fig. 2 we can immediately write down using the usual Feynman rules (and noting that the outgoing gluon's momentum is in the opposite direction of that usually taken when using the triple-gluon vertex formula)

$$
\begin{aligned}
i\mathcal{M}_C^{r_1,\sigma_a,r_2,\sigma_b,c_1,c_2,c_a,c_b,\lambda} &= \bar{u}_{i_{c_1}}^{r_1,c_1}(p_1) ig\gamma^\mu t^a_{i_{c_1},j_{c_a}} u^{\sigma_a,c_a}_{j_{c_a}}(p_a)\, \bar{u}_{i_{c_2}}^{r_2,c_2}(p_2) ig\gamma^\nu t^b_{i_{c_2},j_{c_b}} u^{\sigma_b,c_b}_{j_{c_b}}(p_b) \\
&\quad \times \frac{-i}{(k-q)^2}\left[g_{\mu\alpha} - \frac{\eta_\mu(k-q)_\alpha - \eta_\alpha(k-q)_\mu}{(k-q)^+}\right] \frac{-i}{q^2}\left[g_{\nu\beta} - \frac{\eta_\nu q_\beta + \eta_\beta q_\nu}{q^+}\right] \\
&\quad \times g f^{abc}\left[g^{\alpha\beta}(k-2q)^\rho + g^{\beta\rho}(q+k)^\alpha + g^{\rho\alpha}(q-2k)^\beta\right]\epsilon^{*\lambda}_\rho
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
&\simeq \frac{4g^3\, p_a^+ p_b^-}{(\boldsymbol{k}-\boldsymbol{q})^2\,\boldsymbol{q}^2}\left[g_{+\alpha} - \frac{(k-q)_\alpha}{(k-q)^+}\right] f^{abc}\, t^a_{i_{c_1},j_{c_a}} t^b_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \\
&\quad \times \left[g^{\alpha+}(k-2q)^\rho + g^{+\rho}(q+k)^\alpha + g^{\rho\alpha}(q-2k)^+\right]\epsilon^{*\lambda}_\rho
\end{aligned} \tag{48}
$$

$$
\simeq -\frac{8g^3\, p_a^+ p_b^-}{(\boldsymbol{k}-\boldsymbol{q})^2\,\boldsymbol{q}^2}\,(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{\epsilon}^\lambda\, f^{abc}\, t^a_{i_{c_1},j_{c_a}} t^b_{i_{c_2},j_{c_b}} \delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \tag{49}
$$

In order to get from Eq. (47) to Eq. (48) we used, as usual, eikonality twice. We were able to drop the $\eta_\alpha(k-q)_\mu$ term in the first gluon propagator as $k-q=p_a-p_1$, and $\bar{u}(p_1)(\not{k}-\not{q})u(p_a)=0$ by the Dirac equation. The $\eta_\nu q_\beta$ and $\eta_\beta q_\nu$ terms in the second gluon propagator can also be dropped for the same reasons they were in A: the Dirac equation and suppression by the large p_b^-. Since the dominant contribution from $\bar{u}(p_2)\gamma^\nu u(p_b)$ is proportional to p_b^- we can set $\nu=-$. The second gluon propagator, with its latter two terms dropped, then forces $\beta=+$. In the same way, since the dominant contribution from $\bar{u}(p_1)\gamma^\mu u(p_a)$ is proportional to p_a^+ we can set $\mu=+$.

To get from Eq. (48) to Eq. (49), first notice that the gluon propagator is zero unless $\alpha=i$; i.e., unless α is in one of the perpendicular directions. This is because $(k-q)_+=(k-q)^-\simeq 0$ and

$$g_{+-}-\frac{(k-q)_-}{(k-q)^+}=1-\frac{(k-q)^+}{(k-q)^+}=0. \tag{50}$$

Now consider the three terms from the triple gluon vertex in turn. The first term yields nothing as $\alpha=i$ and $g^{i+}=0$. The second term also yields nothing as the $g^{+\rho}$ is zero unless $\rho=-$; however, $\epsilon_-=\epsilon^+=0$. And so the only remaining term is the last one, for which the matrix element receives the contribution $(k-q)\cdot\epsilon^*=(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{\epsilon}$.

To make progress the easy way notice that

$$i(\mathcal{M}_A+\mathcal{M}_B)^{r_1,\sigma_a,r_2,\sigma_b,c_1,c_2,c_a,c_b,\lambda}=\frac{8ig^3\,p_a^+p_b^-}{\boldsymbol{q}^2\,\boldsymbol{k}^2}\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}^\lambda\,[t^c,t^a]_{i_{c_1},j_{c_a}}t^a_{i_{c_2},j_{c_b}}\delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \tag{51}$$

and

$$[t^c,t^a]_{i_{c_1},j_{c_a}}t^a_{i_{c_2},j_{c_b}}=if^{cab}t^b_{i_{c_1},j_{c_a}}t^a_{i_{c_2},j_{c_b}}=if^{cba}t^a_{i_{c_1},j_{c_a}}t^b_{i_{c_2},j_{c_b}}=-if^{abc}t^a_{i_{c_1},j_{c_a}}t^b_{i_{c_2},j_{c_b}}. \tag{52}$$

Therefore

$$i(\mathcal{M}_A+\mathcal{M}_B)^{r_1,\sigma_a,r_2,\sigma_b,c_1,c_2,c_a,c_b,\lambda}=\frac{8g^3\,p_a^+p_b^-}{\boldsymbol{q}^2\,\boldsymbol{k}^2}\,\boldsymbol{k}\cdot\boldsymbol{\epsilon}^\lambda\,f^{abc}\,t^a_{i_{c_1},j_{c_a}}t^b_{i_{c_2},j_{c_b}}\delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b}, \tag{53}$$

and

$$i(\mathcal{M}_A+\mathcal{M}_B+\mathcal{M}_C)^{\cdots}=\frac{8g^3\,p_a^+p_b^-}{\boldsymbol{q}^2}\left(\frac{\boldsymbol{k}}{\boldsymbol{k}^2}-\frac{(\boldsymbol{k}-\boldsymbol{q})}{(\boldsymbol{k}-\boldsymbol{q})^2}\right)\cdot\boldsymbol{\epsilon}^\lambda\,f^{abc}\,t^a_{i_{c_1},j_{c_a}}t^b_{i_{c_2},j_{c_b}}\delta_{r_1,\sigma_a}\delta_{r_2,\sigma_b} \tag{54}$$

We now want to sum and average over the result. First, let's examine the color factor.

$$\begin{aligned} f^{abc}\,t^a_{i_{c_1},j_{c_a}}t^b_{i_{c_2},j_{c_b}} \quad\rightarrow\quad & f^{abc}f^{dec}\,\mathrm{Tr}\,\left(t^at^d\right)\mathrm{Tr}\,\left(t^bt^e\right)=C^2(r)f^{abc}f^{abc}=C^2(r)C_2(G)\delta^{cc}\\ &=\frac{N_c(N_c^2-1)}{4}. \end{aligned} \tag{55}$$

Summing and averaging the entire matrix element, noting that $\sum_\lambda \epsilon^{i,\lambda}\epsilon^{j,\lambda}=g^{ij}$ for $i,j=1,2$ as the polarization vectors form an orthonormal basis in the perpendicular direction, we have that

$$\frac{1}{4}\frac{1}{N_c^2}\sum|\mathcal{M}|^2=\frac{1}{4}\frac{1}{N_c^2}\frac{(2g)^6\left(p_a^+p_b^-\right)^2}{\boldsymbol{q}^4}\left(\frac{\boldsymbol{k}}{\boldsymbol{k}^2}-\frac{\boldsymbol{k}-\boldsymbol{q}}{(\boldsymbol{k}-\boldsymbol{q})^2}\right)^2\frac{N_c(N_c^2-1)}{4}4 \tag{56}$$

Noting that

$$\begin{aligned}\left(\frac{\boldsymbol{k}}{\boldsymbol{k}^2}-\frac{\boldsymbol{k}-\boldsymbol{q}}{(\boldsymbol{k}-\boldsymbol{q})^2}\right)^2&=\frac{1}{\boldsymbol{k}^2}-\frac{2\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{q})}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2}+\frac{1}{(\boldsymbol{k}-\boldsymbol{q})^2}\\ &=\frac{(\boldsymbol{k}-\boldsymbol{q})^2-2\boldsymbol{k}^2+2\boldsymbol{k}\cdot\boldsymbol{q}+\boldsymbol{k}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2}\\ &=\frac{\boldsymbol{q}^2}{\boldsymbol{k}^2(\boldsymbol{k}-\boldsymbol{q})^2},\end{aligned} \tag{57}$$

we have that

$$\frac{1}{4}\frac{1}{N_c^2}\sum|\mathcal{M}|^2=\frac{16\,(N_c^2-1)\,g^6\left(p_a^+p_b^-\right)^2}{N_c}\frac{1}{\boldsymbol{k}^2}\frac{1}{\boldsymbol{q}^2(\boldsymbol{k}-\boldsymbol{q})^2} \tag{58}$$

$$=\frac{2^{10}\pi^3\,(N_c^2-1)\,\alpha_s^3\left(p_a^+p_b^-\right)^2}{N_c}\frac{1}{\boldsymbol{k}^2}\frac{1}{\boldsymbol{q}^2(\boldsymbol{k}-\boldsymbol{q})^2}. \tag{59}$$

3 Trace of 6 Gamma Matrices

Let's calculate

$$\text{Tr}\,(\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f) \equiv \text{Tr}\,(abcdef). \tag{60}$$

From the anticommutation relations of gamma matrices, $\{\gamma^a, \gamma^b\} = 2g^{ab}$,

$$\begin{aligned}\text{Tr}\,(abcdef) &= 2g^{ab}\text{Tr}\,(cdef) - \text{Tr}\,(bacdef)\\ &= 2g^{ab}\text{Tr}\,(cdef) - 2g^{ac}\text{Tr}\,(bdef) + 2g^{ad}\text{Tr}\,(bcef)\\ &\quad - 2g^{ae}\text{Tr}\,(bcdf) + 2g^{af}\text{Tr}\,(bcde) - \text{Tr}\,(bcdefa).\end{aligned} \tag{61}$$

Since traces are cyclic and $\text{Tr}\,(abcd) = 4(g^{ab}g^{cd} - g^{ac}g^{bd} + g^{ad}g^{bc})$, we have that

$$\begin{aligned}\text{Tr}\,(abcdef) = 4\Big[&g^{ab}\left(g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}\right)\\ &-g^{ac}\left(g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}\right)\\ &g^{ad}\left(g^{bc}g^{ef} - g^{be}g^{cf} + g^{bf}g^{ce}\right) \qquad (62)\\ &-g^{ae}\left(g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}\right)\\ &g^{af}\left(g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd}\right)\Big]\end{aligned} \tag{63}$$