From Action to Answers: Feynman Rules

In the canonical formulation of quantum mechanics the lagrangian is only used to set up the hamiltonian, then discarded along with the global space-time viewpoint implicit in the action. The advantage of retaining the lagrangian view can be appreciated by noting that the 4-momentum may be written $p^\mu = m U^\mu$, $U^\mu = dx^\mu / d\tau$ with τ the 'proper time' ie the time measured by comoving clock: $d\tau = \sqrt{dt^2 - d\vec{x}^2}$. In these terms the equation of motion for, and 4-current associated to, a charge are

$$m \frac{dU^\mu}{d\tau} = e F^\mu{}_\nu U^\nu \tag{7.1a}$$

$$J^\mu = e \int d\tau\, U^\mu \delta^{(4)}(x - x(\tau)) \tag{7.1b}$$

Now, because time is treated on the same footing as space one includes trajectories such as in Figure 7.1; on the segment (B) to (C) $U^0 < 0$ so $E < 0$ but according to (7.1b) this is equivalent to $e \to -e$, ie the negative energy particle propagated backwards in coordinate time is equivalent to the positive energy antiparticle propagated forward in coordinate time. From the hamiltonian view, looking at successive time slices: a particle enters at (A), at (C) a particle-antiparticle pair is created, the three propagate, then at (B) the antiparticle annihilates the original particle, the new particle exiting at (D).

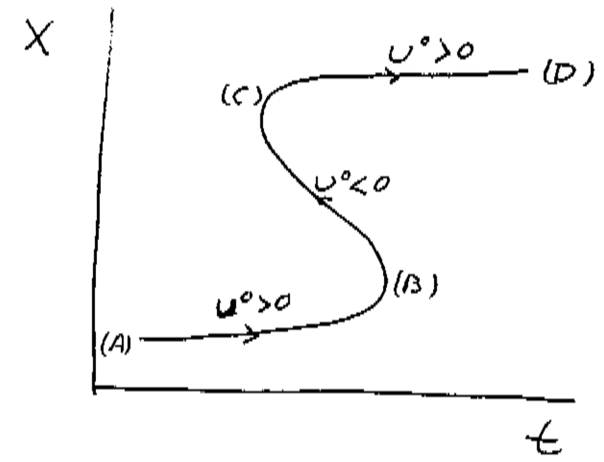


Figure 7.1 : A space - time path parameterized by τ

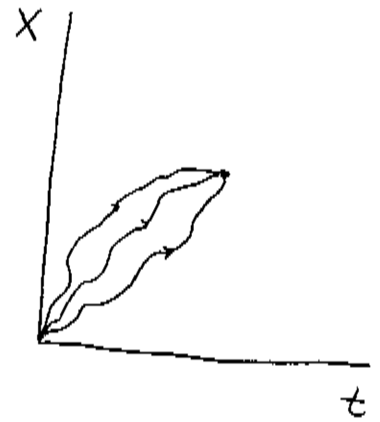


Figure 7.2 : $P(\vec{X},t)$ as a sum of random walks

To reformulate quantum mechanics from the lagrangian perspective consider the free particle Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = -\frac{\vec{\nabla}^2}{2m}\psi(\vec{x},t) \tag{7.2}$$

and regard it as equivalent to a Huygens principle

$$\psi(\vec{x},t) = \int d^3x'\, K_F(\vec{x}-\vec{x}';t)\,\psi(\vec{x}';0) \tag{7.3}$$

For this to hold the Greens function or 'propagator' must satisfy (i)

$$i\frac{\partial}{\partial t}K_F(\vec{x},t) = -\frac{\vec{\nabla}^2}{2m}K_F(\vec{x},t)\ ,\quad t>0, \tag{7.4}$$

(ii) $\lim_{t\to 0^+} K_F(\vec{x},t) = \delta(\vec{x})$, and (iii) $K_F(\vec{x},t<0) = 0$. This last is causality, but note that $\psi^*(x,t)$ is "determined" by the future!

Now, (7.4) is reminiscent of the diffusion equation (D the diffusion constant)

$$\frac{\partial}{\partial t}P(\vec{x},t) = D\,\vec{\nabla}^2 P(\vec{x},t)\quad ,\ t>0; \tag{7.5}$$

$P(\vec{x},t)\,d^3x$ gives the probability to find a particle undergoing Brownian motion at $\vec{x}$ at $t>0$ given it was at $\vec{x}=t=0$. The solution of (7.5) is well known

$$P(\vec{x},t) = (4\pi D t)^{-3/2}\exp\left(-\frac{\vec{x}^2}{4Dt}\right) \tag{7.6}$$

as is that it represents the 'sum of random walks from $\vec{x} = t = 0$ to $\vec{x}, t$, Figure 7.2:

$$\int d^3y \; P(\vec{x}-\vec{y}, \Delta t)\, P(\vec{y}, t-\Delta t) = P(\vec{x}, t) \tag{7.7}$$

so one can break $[0,t]$ into $n+1$ steps of size Δt and associate to $t_j = j\Delta t$ a $\vec{y}_j$, $\vec{y}_0 = 0$, $\vec{y}_{n+1} = \vec{x}$ forming a path.

Since (7.5) becomes (7.4) if we set $D = 1/2m$ and change $t \to it$

$$K_F(\vec{x}, t>0) = \left[\frac{m}{2\pi i t}\right]^{3/2} \exp\left(i\,\frac{m\vec{x}^2}{2t}\right) \tag{7.8}$$

The propagator is the <u>amplitude</u> to find the particle at $\vec{x}, t$ given it was at $\vec{x} = t = 0$ - that is the essential difference between quantum and classical mechanics. Note $m\vec{x}^2/2t$ is just the classical action and, similar to P, K_F is 'the sum over all paths weighted by a phase equal to the classical action' - Feynman's 'path integral'.

When the particle isn't free translational invariance is lost and one can only write the analogue of (7.8) on a sufficiently short interval that the potential is constant; the finite time propagator is built from an infinite number of infintesimal steps. If the action is large (compared to $\hbar = 1$) paths which differ from the extremal destructively interfere due to their large random phases leaving $K_F \sim \exp(iS)$, $S \gg 1$, $\delta S = 0$.

Turning to the relativistic case, the free Klein-Gordon equation may be separated into a Schrödinger-like equation in proper time together with an eigenvalue condition on τ

$$i \frac{\partial}{\partial \tau} \varphi(x,\tau) = \Box \varphi(x,\tau) \qquad (7.9a)$$

$$\varphi(x,\tau) = \Phi(x)\, e^{im^2 \tau} \qquad (7.9b)$$

The immediate analogues of (7.3) and (7.8) are

$$\varphi(x,\tau) = \int d^4x' \, K_F(x-x',\tau)\, \varphi(x',0) \qquad (7.10a)$$

$$K_F(x,\tau) = (4\pi i \tau)^{-2} \exp\left(-i \frac{x\cdot x}{4\tau}\right) \qquad (7.10b)$$

The new φ is made consistent with (7.9b) by performing a Laplace transform on τ with conjugate $im^2+\epsilon$, $\epsilon \to 0+$ for the integral to exist. This is facilitated by first doing a Fourier transform on x^μ yielding the 'Feynman propagator in momentum space'

$$\int_0^\infty d\tau \int d^4x \, \exp(ik\cdot x - i\tau(m^2 - i\epsilon))\, K_F(x,\tau) =$$

$$\int_0^\infty d\tau \, \exp(i\tau(k^2 - m^2 + i\epsilon)) = \frac{i}{k^2-m^2+i\epsilon} \equiv i\Delta_F(k) \qquad (7.11)$$

Inverting the Fourier transform gives the 'Feynman propagator in configuration space

$$i\Delta_F(x-x') = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-x')}\, i\Delta_F(k) \qquad (7.12)$$

which satisfies

$$\Theta\, i(\Box + m^2)\, i\Delta_F(X-X') = \Theta\, i(\Box'^2+m^2)\, i\Delta_F(X-X') = \delta^{(4)}(X-X') \quad . \qquad (7.13)$$

As a function of complex k^0, $\Delta_F(k)$ has poles at $k^0 = \pm\sqrt{\vec{k}^2+m^2} \mp i\epsilon$, Figure 7.3, and if $t > t'$ ($< t'$) in $i\Delta_F(X-X')$ the contour is closed at infinity in the lower (upper) half complex plane in applying the residue theorem

$$i\Delta_F(X-X') = \int \frac{d^3P}{(2\pi)^3}\, \frac{1}{2E}\left[e^{-iP\cdot(X-X')}\theta(t-t') + e^{iP(X-X')}\theta(t'-t)\right], \qquad (7.14)$$

$P^0 = E = \sqrt{\vec{P}^2+m^2}$. One sees positive (negative) energy solutions of the Klein-Gordon equation $(\Box+m^2)\Phi(X)$ propagated forward (backward) in time. The 'density of states' is $d^3P/(2\pi)^3 2E$ because probability density must be the zeroth component of a 4-vector and the only such thing is $\Phi^* i\overleftrightarrow{\partial}^\mu \Phi$. Multiplying the second form in (7.13) by $\Phi(X')$, subtracting $i\Delta_F(X-X')\, i(\Box'^2+m^2)\Phi(X')=0$ and integrating over a 4-volume containing X^μ Gauss' theorem, Figure 7.4, gives

$$\Phi(X) = \begin{cases} \int d^3X'\, i\Delta_F(X-X')\, i\overleftrightarrow{\partial_0'}\, \Phi(X') & t > t' \\ -\int d^3X'\, i\Delta_F(X-X')\, i\overleftrightarrow{\partial_0'}\, \Phi(X') & t' > t \end{cases} \qquad (7.15)$$

That is to say Φ is determined from the past <u>and</u> future.

The Dirac equation may be subjected to a proper time treatment, however the γ-matrices require a 'path ordered exponential', making it simpler

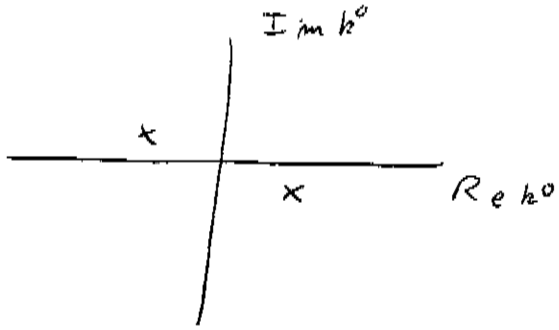


Figure 7.3 : Poles of $\Delta_F(k)$ as a function of complex k^0

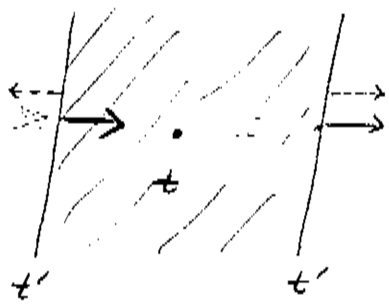


Figure 7.4 : Gauss theorem for the determination of $\Phi(x)$, (7.15). Note the outwards normal is opposite to increasing time for $t' < t$.

to look for the analogue of (7.13). Noting $(i\not\partial - m)(i\not\partial + m) = -\Box - m^2$ one sees that

$$i S_F(x-x') = (i\not\partial + m)\, i \Delta_F(x-x') \tag{7.16a}$$

$$S_F(k) = \frac{\not k + m}{k^2 - m^2 + i\epsilon} = \frac{1}{\not k - m + i\epsilon} \tag{7.16b}$$

satisfies

$$-i\,(i\not\partial - m)\, i S_F(x-x') = i S_F(x-x')(-i)(i\overleftarrow{\not\partial} - m) = \delta^{(4)}(x-x') \tag{7.17}$$

and using (3.26, 31)

$$i S_F(x-x') = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}\left[(\not p + m)e^{-ip\cdot(x-x')}\theta(t-t') + (-\not p + m)e^{ip\cdot(x-x')}\theta(t'-t)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{2m}{2E}\sum_{pol}\left[u(p)\bar u(p)e^{-ip\cdot(x-x')}\theta(t-t') - v(p)\bar v(p)e^{ip\cdot(x-x')}\theta(t'-t)\right] \tag{7.18}$$

The density of states reflects that for the free-particle solutions (3.24, 29) $\bar u \gamma^\mu u = \bar v \gamma^\mu v = p^\mu/m$. By steps similar to those leading to (7.15)

$$\psi(x) = \begin{cases} \int d^3x'\, i S_F(x-x')\gamma_0\, \psi(x') & t > t' \\ -\int d^3x'\, i S_F(x-x')\gamma_0\, \psi(x') & t < t' \end{cases} \tag{7.19}$$

In the presence of the electromagnetic field the Greens function $S_F(x, x'; A]$ for the Dirac equation (3.33) obeys

$$-i\,(i\not\partial - e\not A - m)\, i S_F(x, x'; A] = \delta^{(4)}(x-x') \tag{7.20}$$

which may be converted to an integral equation that is solved by iteration (meaning substitute the LHS in the RHS ad nauseum)

$$i S_F(X,X';A] = i S_F(X-X') + \int d^4Y\, i S_F(X-Y)\,[-ie\not{A}(Y)]\, i S_F(Y;X';A]$$

$$= i S_F(X-X') + \int d^4Y_1\, i S_F(X-Y_1)\,[-ie\not{A}(Y_1)]\, i S_F(Y_1-X') +$$

$$+ \int d^4Y_2 \int d^4Y_1\, i S_F(X-Y_2)[-ie\not{A}(Y_2)]\, i S_F(Y_2-Y_1)[-ie\not{A}(Y_1)]\, i S_F(Y_1-X') + \cdots \quad (7.21)$$

The various terms correspond to 'Feynman diagrams', Figure 7.5, and vice versa. A typical potential scattering problem involves evaluating the transition amplitude

$$i T_{fi} = \int d^3X \int d^3X'\, \bar{\psi}_0(X)\, \gamma_0\, [\, i S_F(X,X';A] - i S_F(X-X')]\, \gamma_0\, \psi(X') \quad (7.22)$$

assuming $A^\mu(Y)$ nonzero only between initial and final times t_i, t_f (eg (A) and (D) in Figure 7.1) with ψ_0, $\bar{\psi}_0$ free particle solutions. Different from the nonrelativistic case there are four possabilities (the minus signs account for the normals in Figure 7.4)

$$\psi_0(X') = \begin{cases} u(P_1)\, e^{-iP_1\cdot X'}, & t' = t_i, \text{ initial particle} \\ -v(P_2')\, e^{iP_2'\cdot X'}, & t' = t_f, \text{ final antiparticle} \end{cases} \quad (7.23a)$$

$$\bar{\psi}_0(X) = \begin{cases} \bar{u}(P_1')\, e^{iP_1'\cdot X}, & t = t_f, \text{ final particle} \\ -\bar{v}(P_2)\, e^{-iP_2\cdot X}, & t = t_i, \text{ initial antiparticle} \end{cases} \quad (7.23b)$$

These get propagated to the interaction point(s) Y by the outer iS_F; to first order in A one gets $iT_{fi}^{(1)}$

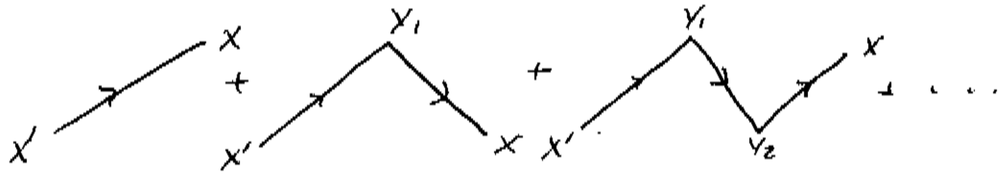
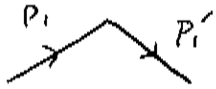
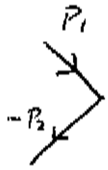


Figure 7.5 : Diagramatic representation of the Series (7.21)

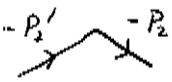


$\bar{u}(p_1')[-ie\not{A}(p_1'-p_1)]u(p_1)$ particle scattering



$\bar{v}(p_2)[-ie\not{A}(-p_2-p_1)]u(p_1)$ pair annihilation

$\bar{u}(p_1')[-ie\not{A}(p_1'+p_2')]v(p_2')$ pair production



$\bar{v}(p_2)[-ie\not{A}(-p_2+p_2')]v(p_2')$ antiparticle scattering

Figure 7.6 : First order contributions to iT_{fi} in (7.22)

Note reading right to left follow the arrow.

in Figure 7.6. Using the Fourier representation of $iS_F(Y_2 - Y_1)$ to carry out the Y integrals the second order contribution to particle scattering is

$$i T_{fi}^{(2)} = \int \frac{d^4k}{(2\pi)^4} \bar{u}(P_1')[-ie \not{A}(P_1' - k)]\, iS_F(k)[-ie\not{A}(k - P_1)]u(P_1) \qquad (7.24)$$

A few words are in order about the difference between 'real particles' with 'on-shell 4-momenta' $P^2 = m^2$ and 'virtual particles' with 'off-shell 4-momenta' $k^2 \neq m^2$. A truly free particle would never interact, making it irrelevant. The "free-particle" solutions (7.23) we put in iT_{fi} just mean that the interactions going into preparing (detecting) the initial (final) state occur outside the scattering space-time volume. Inside the "particle" may be scattered forward or backward in time - everything not forbidden is compulsory as a partial amplitude - propagating on-shell between, (7.18); $iS_F(k)$ in (7.24) takes both time orderings into account.

With two lines threading the space-time region one needs a two particle propagator $iS_F(X_1, X_2; X_1', X_2'; A] = iS_F^{(a)}(X_1, X_1'; A] \otimes iS_F^{(b)}(X_2, X_2'; A]$ if they are distinct (say electron and proton) else $iS_F(X_1, X_1'; A] \otimes iS_F(X_2, X_2'; A] - iS_F(X_2, X_1'; A] \otimes iS_F(X_1, X_2'; A]$ the sign due to the exclusion principle. Having admitted that pairs may be produced and annihilated, clearly this may happen inside, Figure 7.7, contributing a 'loop' amplitude

$$L[A] = -\frac{1}{2} \mathrm{tr} \int d^4Y_1 \int d^4Y_2 [-ie\not{A}(Y_2)]\, iS_F(Y_2 - Y_1)[-ie\not{A}(Y_1)]\, iS_F(Y_1 - Y_2) + \cdots \qquad (7.25)$$

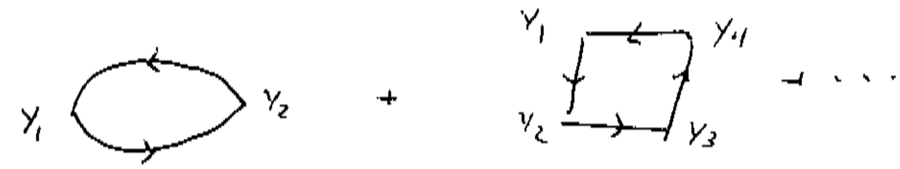


Figure 7.7 Vacuum diagrams giving $L[A]$, (7.25)

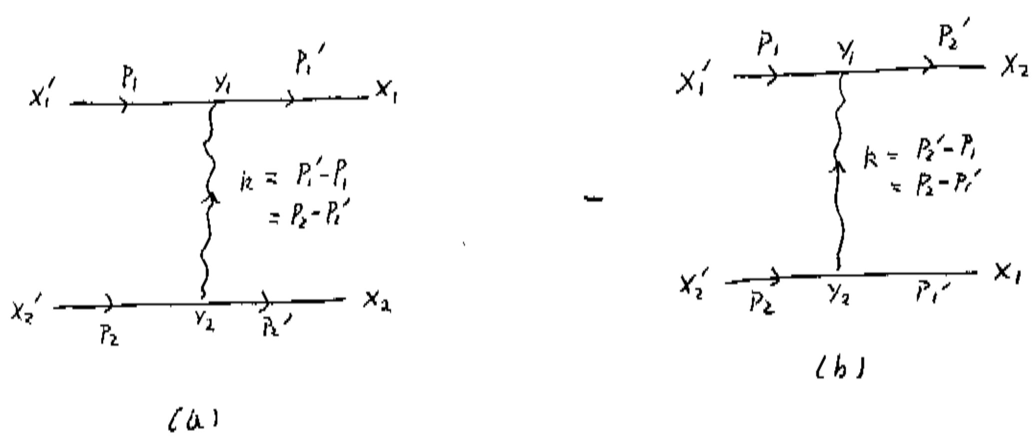


Figure 7.8 Lowest order diagrams for the scattering of two particles; (b) exists if they are identical types. Note 4-momentum conservation at vertices

- terms odd in A^μ vanish by charge conjugation which reverses the sense of the arrow thereby changing $e^- \to e$ (Furry's theorem). The '$-$tr' again reflects the exclusion principle for identical fermions, originating in the sign difference between the two terms in (7.18).

Instead of potential scattering, consider two particles $\underset{\sim}{a}$ and $\underset{\sim}{b}$. Particle $\underset{\sim}{b}$ provides a current $e\bar{\Psi}_{(b)}\gamma^\mu\Psi_{(b)} = J^\mu_{(b)}$ which via (2.30) generates A^μ. Denote $\Delta_F(x)|_{m^2=0} = D_F(x)$ so $\Box D_F(x) = -\delta^{(4)}(x)$, hence

$$A^\mu(y_1) = \int d^4y_2\,[-g^{\mu\nu} i D_F(y_1-y_2)]\,\bar{\Psi}_{(b)}(y_2)[-ie\gamma_\nu]\Psi_{(b)}(y_2) \qquad (7.26)$$

from which particle $\underset{\sim}{a}$ scatters to first order, Figure 7.8(a) wherein the "wavey" line corresponds to

$$i D_F^{\mu\nu}(y_1-y_2) = \int\frac{d^4k}{(2\pi)^4} e^{-ik\cdot(y_1-y_2)}\,\frac{-ig^{\mu\nu}}{k^2+i\epsilon}\,. \qquad (7.27)$$

and eg $\Psi_{(b)}(y_2)$ is found via (7.19). As before the wavefunctions (7.23) are propagated to the ys by the iS_F. The y integrals are carried out yielding[+] $(2\pi)^4\delta^{(4)}(P_1'-P_1-k)$ and $(2\pi)^4\delta^{(4)}(P_2'-P_2+k)$, then the $\int d^4k/(2\pi)^4$ fixes k^μ and produces an overall $(2\pi)^4\delta^{(4)}(P_1'+P_2'-P_1-P_2)$ which is factored from iT_{fi}, calling the remainder $-i\mathcal{M}_{fi}$. For identical particle types there is the additional diagram of Figure 7.8(b). We have

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[+] Albeit these are restricted to the scattering space-time volume L^3T, L and T are so large compared to the wavelengths concerned that one approaches $\lim_{a\to\infty} \sin(ax)/\pi x = \delta(x)$

$$-i\mathcal{M}_{fi} = \bar{u}(P_1')[-ie\gamma_\mu]u(P_1)\frac{-ig^{\mu\nu}}{(P_1'-P_1)^2}\bar{u}(P_2')[-ie\gamma_\nu]u(P_2) \qquad (7.28a)$$

$$-\bar{u}(P_2')[-ie\gamma_\mu]u(P_1)\frac{-ig^{\mu\nu}}{(P_2'-P_1)^2}\bar{u}(P_1')[-ie\gamma_\nu]u(P_2)\,. \qquad (7.28b)$$

Let us dissect (7.27): for fixed k^μ, $\eta^\mu = (1,\vec{0})$, $\hat{k}^\mu = (0, \vec{k}/|\vec{k}|)$ and $\epsilon^\mu(k,\lambda=1,2) = (0, \vec{\epsilon}(\vec{k},\lambda))$ with $\vec{\epsilon}(\vec{k},1)$, $\vec{\epsilon}(\vec{k},2)$ unit vectors orthogonal to each other and to $\vec{k}$, form a basis such that

$$-g^{\mu\nu} = -\eta^\mu\eta^\nu + \hat{k}^\mu\hat{k}^\nu + \sum_{\lambda=1}^{2}\epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda)$$

$$= \sum_{\lambda=1}^{2}\epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda) + \frac{k^2}{|\vec{k}|^2}\eta^\mu\eta^\nu + \frac{k^\mu k^\nu - k^0(k^\mu\eta^\nu + \eta^\mu k^\nu)}{|\vec{k}|^2} \qquad (7.29)$$

The k^μ and/or k^ν pieces are 'gauge terms' that can be discarded between conserved currents - eg $\bar{u}(P_1')(\not{P}_1'-\not{P}_1)u(P_1) = \bar{u}(P_1')(m-m)u(P_1)$ by virtue of the free Dirac equation. The $\eta^\mu\eta^\nu$-term gives the denominator $|\vec{k}|^2$ in place of k^2 producing the instantaneous Coulomb interaction. Thus

$$iD_F^{\mu\nu}(y_1-y_2) = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega}\sum_{\lambda=1}^{2}\epsilon^\mu(k,\lambda)\epsilon^\nu(k,\lambda)\left[e^{-ik\cdot(y_1-y_2)}\theta(t_1-t_2)+e^{ik\cdot(y_2-y_1)}\theta(t_2-t_1)\right]$$

$$+ \text{ Coulomb } + \text{ gauge }, \qquad (7.30)$$

$k^0 = \omega = |\vec{k}|$; it is conventional to use $k^\mu = (\omega, \vec{k})$ for the 'real photon' also.

Suppose we have one particle threading the scattering space-time volume responding to currents acting before and after t_i and t_f. The latter are registered as real photons and the process is 'Compton scattering, Figure 7.9, with $iT_{fi} = (2\pi)^4 \delta^{(4)}(P'+k'-P-k)$ times

$$-i\mathcal{M}_{fi} = \bar{u}(P')\{[-ie\not{\epsilon}'(k')]\, iS_F(P+k)[-ie\not{\epsilon}(k)] + [-ie\not{\epsilon}(k)]\, iS_F(P'-k)[-ie\not{\epsilon}'(k')]\}u(P) \quad (7.31)$$

'Gauge invariance' is readily verified replacing $\epsilon^\mu(k)$ with k^μ:

$$\bar{u}(P')\{[-ie\not{\epsilon}'(k')]\, iS_F(P+k)[-ie(\not{P}+\not{k}-m)] + [ie(\not{P}'-\not{k}-m)]\, iS_F(P'-k)[-ie\not{\epsilon}'(k')]\}u(P) =$$

$$\bar{u}(P')\{[-ie\not{\epsilon}'(k')]e - e[-ie\not{\epsilon}'(k')]\}u(P) = 0 \,. \quad (7.32)$$

'Vacuum currents' also produce fields, and a charge can absorb the photon it emits, ie interact with itself (in classical electrodynamics this accounts for 'radiation resistance'). Thus there are 'higher order corrections' to the processes above, Figure 7.10. We shall return to these and the issues they raise later.

Evidently, expressions like (7.28) and (7.31) are built up from 'elementary processes' of photon emission / absorbtion linked by propagators. Let us split the action associated to (7.27) in the Lorenz gauge $\partial \cdot A = 0$ into a 'free part' S_0 and an interaction part S_I; using the Fourier representations

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} A^\mu(k)\,, \quad \psi(x) = \int \frac{d^4P}{(2\pi)^4} e^{-iP\cdot x}\, \psi(P) \quad (7.33)$$

$$S_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} A^\mu(-k)\,[-k^2 g_{\mu\nu}]\,A^\nu(k) + \int \frac{d^4P}{(2\pi)^4} \bar{\psi}(P)\,[\not{P}-m]\,\psi(P) \quad (7.34a)$$

$$S_I = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4P'}{(2\pi)^4} \int \frac{d^4P}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(P'-P-k)\, A^\mu(k)\, \bar{\psi}(P')\,[-e\gamma_\mu]\,\psi(P) \quad (7.34b)$$

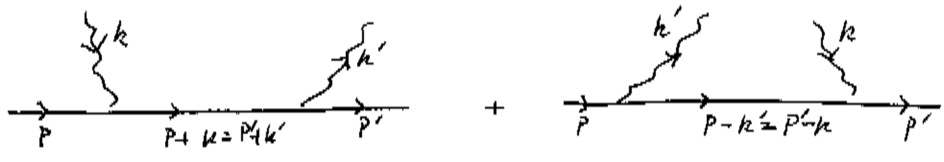


Figure 7.9 Lowest order diagrams for Compton scattering

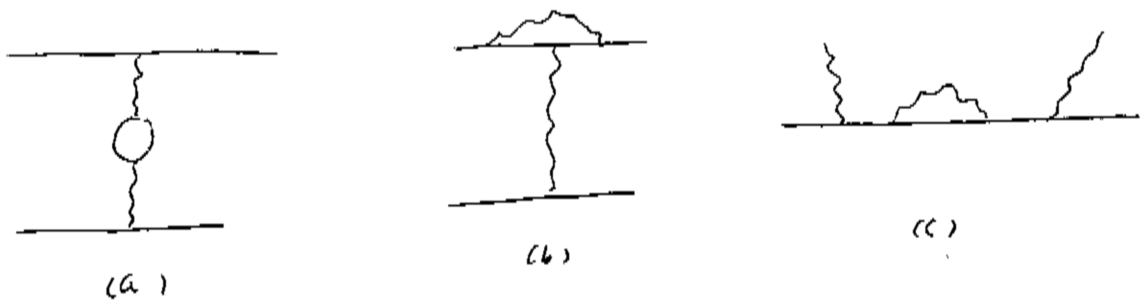


Figure 7.10 : Examples of higher order corrections: (a) vacuum polarization, (b) vertex correction, (c) self energy.

Hence, striping away the As and Ψs, i times the inverse of the free part is the propagator, i times the interaction part is the vertex.

This generalizes to eg 'scalar electrodynamics' consisting of (6.19) with $m^2 > 0$ in (6.4)

$$S_0 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{2} A^\mu(-k)\left[-k^2 g_{\mu\nu}\right]A^\nu(k) + \int \frac{d^4P}{(2\pi)^4} \Phi^*(P)\left[P^2 - m^2\right]\Phi(P) \tag{7.35a}$$

$$S_I = \int\frac{d^4k}{(2\pi)^4}\int\frac{d^4P'}{(2\pi)^4}\int\frac{d^4P}{(2\pi)^4}(2\pi)^4\delta^{(4)}(P'-P-k)A^\mu(k)\,\Phi^*(P')\left[-e(P'+P)_\mu\right]\Phi(P)$$

$$+\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4P'}{(2\pi)^4}\int\frac{d^4P}{(2\pi)^4}(2\pi)^4\delta^{(4)}(P'+P-k_1-k_2)\,\frac{1}{2}A^\mu(k_1)A^\nu(k_2)\Phi^*(P')\left[2e^2 g_{\mu\nu}\right]\Phi(P)$$

$$+\int\frac{d^4P_1'}{(2\pi)^4}\int\frac{d^4P_2'}{(2\pi)^4}\int\frac{d^4P_1}{(2\pi)^4}\int\frac{d^4P_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}(P_1'+P_2'-P_1-P_2)\,\frac{1}{2}\Phi^*(P_1')\Phi^*(P_2')\left[-4\lambda\right]\frac{1}{2}\Phi(P_1)\Phi(P_2) \tag{7.35b}$$

The 'Feynman rules' for the two cases are listed in Figure 7.11. Note the numerical factors in the 'contact verticies' come from the observation that linking may be made with either of the As, Φ^*s or Φ. As the reader will verify there are <u>three</u> graphs for Compton scattering in lowest order in the scalar case and the contact term $2ie^2\, \epsilon'(k')\cdot\epsilon(k)$ is necessary for gauge invariance. Also, the $-\lambda|\Phi|^4$ term in $\mathcal{L}$ can be rewritten $\sigma|\Phi|^2 + \sigma^2/4\lambda$ using a auxiliar field σ whose "propagator" is $2i\lambda$ – the $\sigma\Phi^*\Phi$ vertex is i and $2\cdot i(2i\lambda)i = -4i\lambda$.

It is straightforward to extend this construction to arbitrary nongauge or abelian gauge models. In the Yang-Mills case a complication arises because the currents are <u>covariantly</u> conserved

$\Rightarrow \quad \dfrac{-i g^{\mu\nu}}{k^2 + i\epsilon}$

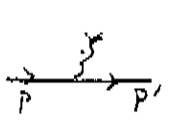
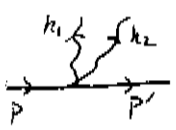
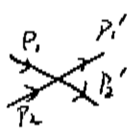
Spinor		Scalar
	$\Rightarrow \dfrac{i}{\not p - m + i\epsilon}$	$\dfrac{i}{p^2 - m^2 + i\epsilon}$
	$= -ie\gamma_\mu$	$-ie(p'+p)_\mu$
	N.A.	$2ie^2 g_{\mu\nu}$
	N.A.	$-4i\lambda$

Figure 7.11 Feynman rules for spinor and scalar electrodynamics

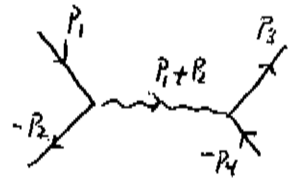


Figure 7.12 Lowest order diagram for pair annihilation creating a pair of a different type

and one cannot glibly dismiss the "gauge terms" in (7.29). If the nonabelian gauge group is spontaniously broken to no more than an abelian gauge subgroup (as is the case in the Standard Electroweak Model) the unitary gauge rules may easily be written down but are specific to the symmetry breaking pattern; the Standard Model rules will be given in the next section. The 'massive vector propagator' follows from its physical (U-gauge) contents, using (2.31)

$$i D_F^{\mu\nu}(x;m) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \sum_{\lambda=1}^{3} \epsilon^{\mu}(p,\lambda)\epsilon^{\nu}(p,\lambda) \left[e^{-ip\cdot x}\theta(t) + e^{ip\cdot x}\theta(-t) \right] \tag{7.36a}$$

$$D_F^{\mu\nu}(k;m) = \frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}}{k^2 - m^2 + i\epsilon} . \tag{7.36b}$$

There remains the question of how to relate the transition amplitude to an observable number. In general $i T_{fi} = -i \mathcal{M}_{fi} \cdot (2\pi)^4 \delta^{(4)}(\sum_f p_f - \sum_i p_i)$ expressing conservation of total initial and final 4 momenta. Squaring, we remember the approximation made by $|T_{fi}|^2 = |\mathcal{M}_{fi}|^2 \cdot (2\pi)^4 \delta^{(4)}(\sum_f p_f - \sum_i p_i) \cdot (T L^3)$. Dividing by $T L^3$ and multiplying by the density of final states gives the relativistic generalization of Fermi's golden rule:

$$W = \frac{|\mathcal{M}_{fi}|^2}{S} (2\pi)^4 \delta^{(4)}(\sum_f p_f - \sum_i p_i) \prod_f \frac{d^3p_f}{(2\pi)^3} \frac{2m_f}{2E_f} \tag{7.37}$$

is the transition probability per unit time per unit volume, $S = \prod_f n_f!$ if the final state contains n_f identical particles so as to not double count. This equals for $a + b \to 1 + 2 + \cdots$ in the b rest frame the flux $v_a E_a / m_a = |\vec{p}_a| / m_a$ times the target density m_b/m_b times the differential cross-section $d\sigma$, or since $|\vec{P}_a| m_b = \sqrt{(P_a\cdot P_b)^2 - m_a^2 m_b^2}$

$$d\sigma = \frac{1}{4\sqrt{(P_a\cdot P_b)^2 - m_a^2 m_b^2}} \; \frac{|M_{fi}|^2 (\prod 2m_F)}{S} (2\pi)^4 \delta^{(4)}(P_a + P_b - \sum_f P_f) \prod_f \frac{d^3 P_f}{(2\pi)^3} \frac{1}{2E_f} \qquad (7.38)$$

Thus written, (7.38) applies also to scalars and vectors. When the initial state is a single particle W equals the density E_a/m_a times the differential inverse lifetime or 'decay width $d\tau^{-1} = d\Gamma$:

$$d\Gamma = \frac{1}{2E_a} \; \frac{|M_{fi}|^2 (\prod 2m_F)}{S} (2\pi)^4 \delta^{(4)}(P_a - \sum_f P_f) \prod_f \frac{d^3 P_f}{(2\pi)^3 2E_f} \qquad (7.39)$$

Often the helicities are not observed so one sums over final and averages over initial polarizations(?) using (3.26, 31) for spinors, (7.29) with $k^2 = 0$ for massless vector and (2.31) for massive vectors; $\overline{|M_{fi}|^2}$ is a function of kinematic invariants only, eg for $a + b \to c + d$ $\quad s = (P_a + P_b)^2$ and $t = (P_a - P_c)^2$

($u = (P_a - P_d)^2 = m_a^2 + m_b^2 + m_c^2 + m_d^2 - s - t$ is not independant)

Using the property

$$\delta(f(x)) = \sum_i |f'(x_i)|^{-1} \delta(x - x_i), \quad f(x_i) = 0 \qquad (7.40)$$

one has $\int d^3P / 2E = \int d^4P \, \delta(P^2 - m^2)\, \theta(E)$ so (7.38, 39) are covariant (even time dilation for decay in flight is included in (7.39)). The 'two-body phase-space' for scattering, (7.38), is reduced easily in the 'centre of momentum' frame $\vec{P}_a + \vec{P}_b = 0$:

$$\int \frac{d^3 P_c}{(2\pi)^3 2E_c} \int \frac{d^4 P_d}{(2\pi)^3} \delta(P_d^2 - m_d^2)\, (2\pi)^4 \delta^{(4)}(P_a + P_b - P_c - P_d) =$$

$$\frac{1}{2(2\pi)^2} \int \frac{2\pi \, d\cos\theta_{ac} \, |\vec{P}_c|^2 d|\vec{P}_c|}{E_c} \; \delta((P_a + P_b - P_c)^2 - m_d^2) =$$

$$\frac{1}{4\pi}\int |\vec{P}_c|\, d\cos\theta_{ac}\, dE_c\, \delta(s - 2\sqrt{s}\,E_c + m_c^2 - m_d^2) = \frac{1}{8\pi}\int \frac{dt}{2\sqrt{s}\,|\vec{P}_a|} \qquad (7.38)$$

and $4s|\vec{P}_a|^2 = 4[(P_a\cdot P_b)^2 - m_a^2 m_b^2] = (s - m_a^2 - m_b^2)^2 - 4m_a^2 m_b^2 \equiv \lambda(s, m_a^2, m_b^2)$.

Denoting $\overline{|\mathcal{M}_{cd\leftarrow ab}|^2}\,(\prod_F 2m_F) = F(s,t)$

$$\frac{d\bar{\sigma}(a+b\to c+d)}{dt} = \frac{F(s,t)}{16\pi\,\lambda(s, m_a^2, m_b^2)\,S} \qquad (7.39)$$

in any frame. In the centre of momentum frame

$$\frac{d\bar{\sigma}(a+b\to c+d)}{d\cos\theta} = \frac{F(s,t)}{32\pi\, s\, S}\;\frac{\lambda^{1/2}(s, m_c^2, m_d^2)}{\lambda^{1/2}(s, m_a^2, m_b^2)} \qquad (\text{C.M}) \qquad (7.40)$$

A similar reduction for $a\to b+c$ in the rest frame gives

$$\bar{\Gamma}(a\to b+c) = \frac{\lambda^{1/2}(m_a^2, m_b^2, m_c^2)}{16\pi\, m_a^3}\;\frac{\overline{|\mathcal{M}_{fi}|^2}\,(\prod_F 2m_F)}{S}\,. \qquad (7.41)$$

As a example of the above we take electron-positron annihilation to a scalar particle - antiparticle pair to lowest order, Figure 7.12. From the rules

$$-i\mathcal{M}_{fi} = \bar{v}(P_2)[-ie\gamma_\mu]u(P_1)\,\frac{-ig^{\mu\nu}}{(P_1+P_2)^2}\,[-ie(P_3-P_4)_\nu] \qquad (7.42a)$$

$$\mathcal{M}_{fi} = -\frac{e^2}{s}[\bar{v}(P_2)\gamma_\mu u(P_1)](P_3-P_4)^\mu \qquad (7.42b)$$

$$\mathcal{M}_{fi}^* = -\frac{e^2}{s}[\bar{u}(P_1)\bar{\gamma}_\mu v(P_2)](P_3-P_4)^\mu \qquad (7.42c)$$

where $\bar{\gamma}_\mu = \gamma^0\gamma_\mu^+\gamma^0 = \gamma_\mu$. Then by (3.26, 31)

$$F = \left(\tfrac{1}{2}\right)^2 \sum_{pol} \mathcal{M}_{fi}^* \mathcal{M}_{fi}\,(2m_e)^2$$

$$= \frac{e^4}{4s^2}\, tr\,\left((\not{P}_1 + m_e)\gamma_\mu(\not{P}_2 - m_e)\gamma_\nu \right)(P_3-P_4)^\mu (P_3-P_4)^\nu \qquad (7.43)$$

References

The classical interpretation of the positron as a time reversed electron is found in: R. P. Feynman, Phys. Rev. 74, 939 (1948).

Path integrals were introduced in: R. P. Feynman, Rev. Mod. Phys. 20, 267 (1948)*, required reading for any student of quantum mechanics. More on the analogies between the Schrödinger equation/path integral and diffusion equation/random walk can be obtained in: E. P. Raposo, S. M. de Oliveira, A. M. Nemirovsky and M. D. Coutinho-Filho, Am. J. Phys. 59, 633 (1991).

We have followed the "intuitive" space-time approach of: R. P. Feynman, Phys. Rev 76, 749, 769 (1949)*. Formal derivations are given in: R. P. Feynman, Phys. Rev 80, 440 (1950)*, Phys. Rev. 84, 108 (1951).

Problems

7.1. Verify (7.6) is a solution of (7.5). Show that

$$\int_{-\infty}^{\infty} dx\, \exp\left(-\frac{(x-b_1)^2}{a_1}-\frac{(x-b_2)^2}{a_2}\right) = \sqrt{\frac{\pi a_1 a_2}{a_1+a_2}}\exp\left(-\frac{(b_1-b_2)^2}{a_1+a_2}\right), \quad (7.49)$$

hence (7.7) holds.

7.2. Show that (7.25) is consistent with pair production and annihilation, Figure 7.6.

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* Reprinted in J. Schwinger, editor "Quantum Electrodynamics, Dover, New York, 1958.

7.3. Draw the diagrams and write the matrix element M_{fi} for Compton scattering from a charged scalar; verify it is gauge invariant. Obtain $d\bar{\sigma}/d\Omega$ in the rest frame of the initial charge (hint: $\sum_{pol} (\vec{\varepsilon}\cdot\vec{\varepsilon}')^2 = 1 + \cos^2\theta$)

7.4. Write the Feynman rules for the model (6.7), and evaluate the decay width for $\rho \to 2\theta$.