Symmetry Breaking for the Masses

Gravity aside, the only long range force known is electromagnetism which is aptly described by an abelian gauge theory. Since the Yang-Mills construction leaves no room for introducing a gauge mass term like $m^2 A^{a\mu} A^a_\mu / 2$ which is not invariant under gauge tranformations, nonabelian phenomenology would appear problematic. This did not prevent some early papers speculating that the lagrangian density of the world consists of a part which is (gauge) symmetric $\mathcal{L}_{sym}$ and a part $\mathcal{L}_{SB}$ which explicitly breaks the (gauge) symmetry (the gauge mass term). To paraphrase Weinberg: if gauge symmetry is an expression of the simplicity of nature, what does it mean to say nature is partially simple?

The answer came with the assimilation into relativistic field theory of ideas born in condensed matter physics, by people who had worked in both. Consider the Ising model for the humble ferromagnet: spins $S_i = \pm 1$ are placed at lattice sites "i" with a hamiltonian consisting of nearest neighbours

$$H = -\mu \sum_{\langle i,j \rangle} S_i S_j \tag{6.1}$$

Clearly H is unchanged if all the spins are flipped $S_i \to -S_i$ so there is a discrete symmetry isomorphic to S_2. It is equally clear that the lowest energy state is one where all the spins are up (+1) or down (-1) which is not invariant. At high temperature the spins are uncorrelated with average $\langle S \rangle = 0$ but as the system is cooled one of the possabilities $\langle S \rangle = +1$ or $\langle S \rangle = -1$ will be choosen spontaniously. Hence one says the ground state spontaniously breaks the symmetry.

In practice one does not look at individual spins but rather the average over small domains containing many spins; the average can take on a range of values so if we label the centre of the domain by "$\vec{x}$" one can deal with a (approximately) continuous field $\phi(\vec{x})$. As noted by Ginzburg and Landau, at least near the Curie, or critical, temperature T_c the magnetization field ϕ is small and it makes sense to functionally expand the free energy consistent with $\phi \to -\phi$ symmetry

$$F_{GL}[\phi] = \int d^3x \left\{ \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{a}{2}\phi^2 + \frac{b}{4}\phi^4 \right\}. \qquad (6.2)$$

Take $a \propto T - T_c$, $b > 0$ independant of T. For any T F_{G-L} is a minimum for $\vec{\nabla}\phi = 0$, ie ϕ constant, so the free energy density $\mathcal{F}_{GL} = F_{GL}/V$ is just the remaining terms in $\{\ \}$ in (6.2). When $T > T_c$, "a" is positive and the minimum $\mathcal{F}_{G-L}$ obtains at $\phi = 0$:

$$\frac{\partial}{\partial\phi} F_{GL} = a\phi + b\phi^3 \tag{6.3a}$$

$$\frac{\partial^2}{\partial\phi^2} F_{GL} = a + 3b\phi^2 \tag{6.3b}$$

Conversely, when $T < T_c$ "a" is negative and (6.3) says F_{GL} is a minimum for $\phi = \pm\sqrt{-a/b}$ spontaniouly breaking the reflection symmetry. All this is illustrated in Figure 6.1.

With the spins confined to a plane instead of an axis one needs two Ginzburg-Landau fields ϕ^1, ϕ^2 to describe the magnetization; these can be combined in a complex field to produce a free energy that looks very much like (4.17). Take the potential to be

$$V(|\Phi|^2) = m^2|\Phi|^2 + \frac{1}{2}\lambda|\Phi|^4 \tag{6.4}$$

so the minimum of $E = P^0$ is for $\partial_\mu\Phi = 0$, ie Φ a constant minimizing V

$$\frac{\partial V}{\partial\Phi^*} = (m^2 + 2\lambda|\Phi|^2)\Phi \tag{6.5a}$$

$$\frac{\partial^2 V}{\partial\Phi\,\partial\Phi^*} = m^2 + 4\lambda|\Phi|^2 \tag{6.5b}$$

If $m^2 > 0$ the minimum is $\Phi = 0$ preserving global phase symmetry and as $\lambda \to 0$ (4.13) becomes the Klein-Gordon equation for two real degrees of freedom of mass m.

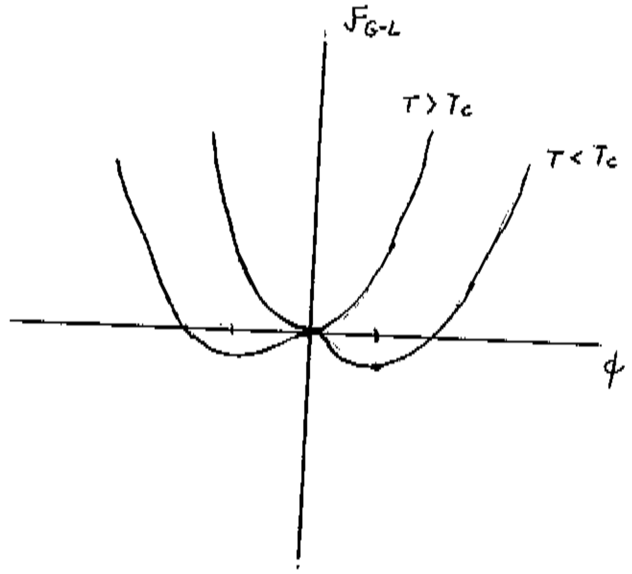

Figure 6.1 : Ginzburg - Landau free enery density

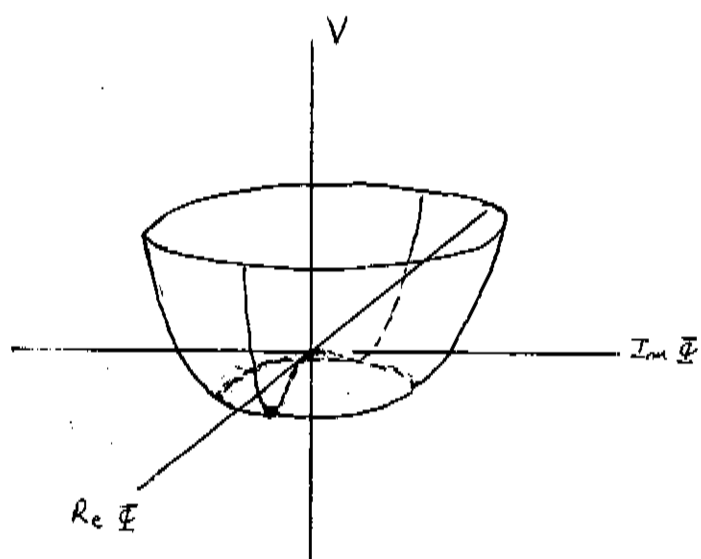

Figure 6.2 : Potential (6.4) for $m^2 < 0$

Things are very different if $m^2 < 0$: plane wave solutions of the Klein-Gordon equation give $\phi \sim \exp(\sqrt{-m^2}\, t)$ for small k indicating a 'tachyonic' instability. Ah, but (6.5) says $\Phi = 0$ is not the stable point which is at $\sqrt{2}|\Phi| = v = \sqrt{-m^2/\lambda}$ - note this fixes the modulus of Φ but not the phase: the possible ground states form a circle in the $\phi^1 - \phi^2$ plane. The system chooses one of them and since they are all equivalent anyway we can take it to be $\Phi = \Phi^* = v/\sqrt{2}$. This is illustrated in Figure 6.2. Now write

$$\Phi = \exp(i\theta/\sqrt{2}\,v) \cdot (v + \tfrac{\rho}{\sqrt{2}})/\sqrt{2} \qquad (6.6)$$

and get after some algebra (!)

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \rho)^2 + \tfrac{1}{2}\left(1 + \tfrac{2\rho}{v} + \tfrac{\rho^2}{v^2}\right)(\partial_\mu \theta)^2 +$$

$$+ \tfrac{\lambda}{4} v^4 - \tfrac{1}{2}(2\lambda v^2)\rho^2 - \lambda v\, \rho^3 - \tfrac{\lambda}{4}\rho^4 \qquad (6.7)$$

which says the ρ field has a real mass $\sqrt{2\lambda}\,v$. The only remnant of the original phase symmetry is that nothing changes if θ is shifted by a constant because the θ field is massless. Goldstone first noted this phenomenon: when a global symmetry is spontaneously broken a massless boson appears. In tribute they are termed 'Goldstone bosons'. Looking at Figure 6.2 it is apparent that changing θ costs no energy but changing ρ does.

Next suppose the "spins" are unrestricted so there are three fields ϕ^1, ϕ^2, ϕ^3 which can be conviently assembled as a vector $\vec{\phi}$, and

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2 - V(\vec{\phi}^2) \qquad (6.8a)$$

$$V(\vec{\phi}^2) = \frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2 \qquad (6.8b)$$

The global symmetry group is $SO(3)$ wich is nonabelian. The ground state or 'vacuum' will be $\vec{\Phi}$ constant minimizing V:

$$\frac{\partial V}{\partial \phi^i} = (m^2 + \lambda\, \vec{\Phi}^2)\, \phi^i \qquad (6.9a)$$

$$\frac{\partial^2 V}{\partial \phi^i \partial \phi^j} = (m^2 + \lambda \vec{\phi}^2)\, \delta^{ij} + 2\lambda\, \phi^i \phi^j \qquad (6.9b)$$

If $m^2 < 0$ the possible vacua are $|\vec{\phi}|^2 = -m^2/\lambda$, ie a sphere. There are (one) two tangents to the (circle) sphere at a point. The vacuua are equivalent so take $\vec{v} = (0, 0, v)$ which is unchanged by rotations around ϕ^3 but not ϕ^1 or ϕ^2:

$$\underset{\sim}{T_1}\, \vec{v} \neq 0, \quad \underset{\sim}{T_2}\, \vec{v} \neq 0, \quad \underset{\sim}{T_3}\, \vec{v} = 0 \qquad (6.10)$$

Hence we expect two goldstone bosons because $SO(3)$ is broken spontaniously down to a $SO(2)$ subgroup, and indeed shifting $\vec{\phi} = \vec{v} + \vec{\varphi}$ (6.8) becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \vec{\Phi})^2 + \tfrac{\lambda}{4}(\vec{v}^2)^2 - \tfrac{1}{2}(m_s^2)_{ij}\,\varphi^i\varphi^j - \mathcal{O}(\varphi^3) \qquad (6.11)$$

with positive semidefinite scalar mass (squared) matrix

$$(m_s^2)_{ij} = \frac{\partial^2 V}{\partial\phi^i\partial\phi^j}\Big|_{\phi^k=v^k} = 2\lambda v^2 \delta_i^3 \delta_j^3 . \qquad (6.12)$$

The analogue of (6.6) is here

$$\vec{\Phi} = \underset{\sim}{R}(\theta)\,\hat{v}\,(v+\rho)\ , \quad \underset{\sim}{R}(\theta) = \exp(i\underset{\sim}{T_1}\theta_1/v + i\underset{\sim}{T_2}\theta_2/v) \qquad (6.13)$$

which is locally the same as $\vec{\Phi} = \vec{v} + \vec{\varphi}$. The general lesson is: when a continuous global symmetry G is spontaneously broken to a continuous subgroup G′ there are as many Goldstone bosons as generators which do not "annihilate" the "vacuum", and they parameterize the coset G/G′.

Goldstone bosons would seem bad news, leading to long ranged r^{-1} potentials. In fact because by nature they are derivatively coupled the associated potential is more like r^{-3}. Also by applying a small "magnetic field" through $\mathcal{L}_{SB} = h\phi^3$ to (6.8) they can be reduced to pseudo-Goldstone bosons with mass squared h/v. Something like this occurs for pions which are anomalously light compared to other hadrons. There is however a better way to get rid of Goldstone bosons — one which resolves the Yang-Mills mass problem.

Back to condensed matter: the Ginzburg-Landau theory for a superfluid involves a complex 'order parameter field' $\psi(\vec{x})$ which is really the wavefunction for the macroscopically occupied ground state, ie. $|\psi|^2$ is the number density of bosons and the phase is the velocity potential. In the case of a superconductor the current is charged and gauge invariance dictates[+]

$$F_{GL} = \int d^3x \left[\frac{\vec{B}^2}{2} + \frac{1}{2\mu} |(\vec{\nabla}+iq\vec{A})\psi|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4 \right] \qquad (6.14)$$

Below T_c, $|\psi|^2 = -a/b = v^2$, ignoring spatial variations of the density, and $\psi = e^{i\chi} v$ so

$$F_{G\text{-}L} = \int d^3x \left[\frac{\vec{B}^2}{2} + \frac{(q\vec{A}+\vec{\nabla}\chi)^2}{2\mu} + \text{const} \right] \qquad (6.15)$$

Making a gauge transformation $\vec{A} \to \vec{A}' - \vec{\nabla}\chi/q$

$$F_{G\text{-}L} = \int d^3x \left[\frac{\vec{B}'^2}{2} + \frac{m_A^2}{2} \vec{A}'^2 + \text{const} \right], \quad m_A \doteq \frac{(qv)^2}{\mu} \qquad (6.16)$$

resulting in the London equation

$$(\vec{\nabla}^2 + m_A^2)\vec{B}' = 0 \qquad (6.17)$$

The solution for a semi-infinite slab is

[+] As shown by Gorkov this can be derived from the microscopic Bardeen-Cooper-Schrieffer (BCS) theory and $q = 2e$

with it manifest gauge invariance. The gauge field in (6.20) is massive: 'the gauge field has eaten the would-be Goldstone boson and become heavy from the meal'. The reality is that the Goldstone boson degree of freedom manifests itself as the zero helicity polarization (2.30); gauge invariance is not broken but hidden by 'unitary gauge' transformation. Note that in the debris of (6.20) is a left over 'Higgs scalar' ρ.

The 'Higgs[+] mechanism' easily generalizes to the nonabelian model (6.8); first gauge it

$$\mathcal{L} = \tfrac{1}{2}(\underset{\sim}{D}_\mu \vec{\phi})^2 - V(\vec{\phi}^2) - \tfrac{1}{4} F_{a\mu\nu} F_a^{\mu\nu} \qquad (6.21a)$$

$$\underset{\sim}{D}_\mu = \partial_\mu + i g A^a_\mu \underset{\sim}{T}_a \qquad (6.21b)$$

Then, with $m^2<0$ take the parameterization (6.13) and make the gauge transformation (5.9) with $\underset{\sim}{U} = \underset{\sim}{R}$ so

$$\mathcal{L} = \tfrac{1}{2}\left((\partial_\mu + i g \underset{\sim}{A})\hat{v}(v+\rho)\right)^2 - V((v+\rho)^2) - \tfrac{1}{4} F_a^{\mu\nu} F^{a}{}_{\mu\nu}$$

$$= \tfrac{1}{2}(\partial_\mu \rho)^2 + \tfrac{1}{2}\left(1+\tfrac{\rho}{v}\right)^2 (m_A^2)_{ab} A^a_\mu A^{b\mu} - \tfrac{1}{4} F_a^{\mu\nu} F^a{}_{\mu\nu} +$$

$$+ \tfrac{\lambda}{4} v^4 - \tfrac{m_s^2}{2}\rho^2 - \lambda v \rho^3 - \tfrac{\lambda}{4}\rho^4 , \qquad (6.22a)$$

$$(m_A^2)_{ab} = g^2 v^T \underset{\sim}{T}_a \underset{\sim}{T}_b v = (gv)^2(\delta_a{}^1\delta_b{}^1 + \delta_a{}^2\delta_b{}^2) \qquad (6.22b)$$

[+] Sometimes Higgs-Kibble-...

$$m_s^2 = 2\lambda v^2 \tag{6.22c}$$

The two would-be Goldstone bosons have been consumed to give mass to two of the gauge fields, A_μ^1, A_μ^2, but A_μ^3 remains massless corresponding to the 'unbroken' SO(2) subgroup. The lesson becomes: when a gauge symmetry G is spontaneously broken to G′ the gauge fields belonging to generators which do not annihilate the vacuum become massive, the would-be Goldstone bosons manifesting as the 'longitudinal' component of the massive gauge fields.

Now SO(3) and SU(2) are homomorphic, and we could view the preceeding exercise as a SU(2) gauge theory with the scalars in the adjoint representation. If the scalars transform as the fundamental representation

$$\mathcal{L} = |D_\mu \Phi|^2 - V - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{6.23a}$$

$$V = m^2 |\Phi|^2 + \lambda |\Phi|^4 \tag{6.23b}$$

the vacuum $\langle \Phi \rangle = \frac{1}{\sqrt{2}}\binom{0}{v}$ is not annihilated by any of the generators so all three gauge components get equal mass (the details are left as an exercise). Thus the gauge mass pattern depends upon the scalar sector.

The Higgs field(s) can also be used to give spinors masses. Indeed this is mandatory if the gauge group is 'chiral', ie ψ_L and ψ_R transform differently, because the mass term $\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L$ is clearly not invariant. As a case in point, if ψ_L and Φ transform in the same way but ψ_R is neutral the most general thing we can write down which is gauge invariant is schematically

$$\mathcal{L}_F = \psi_L^\dagger i \tilde{\sigma}^\mu D_\mu \psi_L + \psi_R^\dagger i \sigma^\mu \partial_\mu \psi_R +$$

$$- h\, \psi_L^\dagger \Phi \psi_R - h\, \psi_R^\dagger \Phi^\dagger \psi_L \qquad (6.24)$$

Then, when the symmetry is spontaniously broken the fermion picks up a mass $m_F = h\langle\Phi\rangle$ plus 'Yukawa' interactions with the Higgs boson.

The general recipee is clear: choose the gauge group and matter content, with the 'Higgs' sector' arranged to give the desired mass pattern of vectors and fermions when the potential spontaniously breaks the symmetry. What we lack as yet is the means to quantize the theory, ie to get out numbers.

References

The seminal papers of J. Goldstone (Nuovo Cimento 19, 154 (1961)) and P.W Higgs (Phys. Rev.

145, 1156 (1966)), plus many others important contributions are reprinted in C. H. Lai, editor, "Gauge Theory of Weak and Electromagnetic Interactions", World Scientific, Singapore 1981.

For the Ginzburg-Landau theory see: D. R. Tilley and J. Tilly, "Superfluidity and Superconductivity", Van Nostrand Reinhold LTD, London, 1974.

Problems

6.1. If $\mathcal{L} = \mathcal{L}_{SYM} + \mathcal{L}_{SB}$ where $\delta\mathcal{L}_{SYM} = 0$ but $\delta\mathcal{L}_{SB} \neq 0$ under $\phi^i \to \phi^i + \delta\phi^i$, $\delta x^\mu = 0$,
(a) assuming the field equations hold relate the 4-divergence of the Nöether current to $\delta\mathcal{L}_{SB}$;
(b) illustrate with $\mathcal{L}_{SYM} = |\partial\Phi|^2$, $\mathcal{L}_{SB} = -C(\Phi^* + \Phi)$; (c) if $|\Phi|^2 = v^2/2$ in that model what is the mass of the 'pseudoGoldstone boson θ?

6.2. Verify (6.7) follows from (6.6)

6.3 Writing for $m^2 < 0$

$$\Phi = \frac{1}{\sqrt{2}} U(\vec{\theta}) \begin{pmatrix} 0 \\ v+e \end{pmatrix} \tag{6.25}$$

analyze the model (6.23) in the U-gauge. Show that if $\lambda \to \infty$ with v fixed, before the U gauge transformation/rescaling

$$\mathcal{L} \to \mathcal{L}_{MYM} = -\frac{1}{2g^2} \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{v^2}{4} \mathrm{tr}((U^\dagger D_\mu U) U^\dagger D^\mu U) \tag{6.26}$$

(hint: add a constant to make $V=0$ the minimum)