Fun with Yang & Mills

Like Frankenstein's monster (4.27) is sewn together from bits and pieces. An omnipotent creator might say "in the begining there was matter described by the free Dirac equation (3.12), and that was good. Lo, it is invariant under global U(1) phase transformations

$$\psi \to \psi' = e^{i\alpha}\,\psi \qquad (5.1)$$

α = constant, but not local ones $\alpha = \alpha(x)$."

"Behold; replace the derivative ∂_μ by the 'gauge covariant derivative'

$$D_\mu = \partial_\mu + i A_\mu \qquad (5.2)$$

transforming as

$$D_\mu \to D'_\mu = e^{i\alpha} D_\mu e^{-i\alpha} \qquad (5.3)$$

so

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\,\psi \qquad (5.4)$$

is thus, and it was good. Yea, the 'gauge field A^μ is only a Lagrange multiplier for $J^\mu = \bar{\psi}\gamma^\mu\psi = 0$."

"Behold: define

$$iF^{\mu\nu} = [D^\mu, D^\nu]_- \tag{5.5}$$

and

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi \tag{5.6}$$

which is gauge invariant and nontrivial. And it was good. Then he rescaled $A^\mu \to eA^\mu$ and rested" Note (5.3) gives $A'_\mu = A_\mu - \partial_\mu \alpha$.

Phase transformations form an abelian Lie group, U(1). The ingenuity of Yang and Mills was to see the steps could be extended to nonabelian Lie groups - more specifically SU(2). The inspiration for this was that the strong interactions have an approximate global 'isospin' symmetry, with protons and neutrons in a isodoublet $\Psi = \binom{p}{n}$. Ignoring electromagnetism (and the small proton-neutron mass difference) what one calls a proton and what a neutron is a matter of convention, being mixed by elements of SU(2); but why must the same convention hold at all space-time points, particularly those out of causal contact ($s^2 < 0$). Enforcing the convention or 'gauge' to be chosen locally necessitates introducing a 'gauge field' and produces a 'gauge field theory'.

The general case is just as easy as the special one of SU(2): start with matter described by ψ carrying both spinor and 'internal symmetry indicies. Under some Lie group (algebra) G ($\mathcal{G}$) with generators $\underset{\sim}{t}_a$ it transforms as the basis for the (usualy fundamental) representation

$$\psi \to \psi' = e^{i\underset{\sim}{\alpha}}\psi \ , \quad \underset{\sim}{\alpha} = \alpha^a \underset{\sim}{t}_a = \vec{\alpha}\cdot\vec{t} \qquad (5.7)$$

The free lagrangian density (4.24) is not invariant under (5.7) for $\vec{\alpha} = \vec{\alpha}(x)$, but will become so if we replace ∂_μ by

$$\underset{\sim}{D}^\mu = \partial_\mu + i\underset{\sim}{A}_\mu \ , \quad \underset{\sim}{A}^\mu = A^{\mu a} t_a = \vec{A}^\mu\cdot\vec{t} \qquad (5.8)$$

with the rule

$$\underset{\sim}{D}_\mu \to \underset{\sim}{D}'_\mu = \underset{\sim}{U}\,\underset{\sim}{D}_\mu \underset{\sim}{U}^{-1} ; \quad \underset{\sim}{U} = e^{i\underset{\sim}{\alpha}} \qquad (5.9)$$

The transformed gauge field follows as

$$\underset{\sim}{A}'_\mu = -i\,\underset{\sim}{U}(\underset{\sim}{D}_\mu \underset{\sim}{U}^{-1}) \ ; \qquad (5.10)$$

for α_a infinitesimal this gives

$$\underset{\sim}{A}'_\mu = \underset{\sim}{A}_\mu - [\underset{\sim}{D}_\mu, \underset{\sim}{\alpha}] \qquad (5.11)$$

or using the Lie algebra (1.8), (total) antisymmetry of the structure constants and (1.9)

$$A_a^{\mu\prime} = A_a^{\mu} - D_{ab}^{\mu}\,\alpha_b \,, \tag{5.12}$$

$$D_{ab}^{\mu} = \delta_{ab}\,\partial^{\mu} + i\,(T_c)_{ab}\,A_c^{\mu} \tag{5.13}$$

being the 'gauge covariant derivative in the adjoint representation'.

To complete the Yang-Mills construction define the 'nonabelian field strength'

$$\underset{\sim}{F}^{\mu\nu} \equiv -i\,[\underset{\sim}{D}^{\mu}, \underset{\sim}{D}^{\nu}]_- = \partial^{\mu}\underset{\sim}{A}^{\nu} - \partial^{\nu}\underset{\sim}{A}^{\mu} + i\,[\underset{\sim}{A}^{\mu}, \underset{\sim}{A}^{\nu}]$$

$$= (\partial^{\mu}A_a^{\nu} - \partial^{\nu}A_a^{\mu} - C_{abc}A_b^{\mu}A_c^{\nu})\,t_a = F_a^{\mu\nu}\,t_a \tag{5.14}$$

which evidently transforms as

$$\underset{\sim}{F}^{\mu\nu\prime} = \underset{\sim}{U}\,\underset{\sim}{F}^{\mu\nu}\,\underset{\sim}{U}^{-1} \tag{5.15}$$

so $tr\,(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu})$ is gauge invariant. With the standard normalization $tr\,(\underset{\sim}{t}_a\underset{\sim}{t}_b) = \delta_{ab}/2$

$$\mathcal{L}_{YM} = -\frac{1}{2g^2}\,tr\,(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}) = -\frac{1}{4g^2}\,F_{a\mu\nu}F_a^{\mu\nu} \tag{5.16}$$

is the Yang-Mills Lagrangian density. The total is $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_F$

$$\mathcal{L}_F = \bar{\Psi}\,(i\,\not{D} - m\,\underset{\sim}{I})\,\Psi \tag{5.17}$$

for spinor matter. In the case of scalar (5.17') matter described by a multicomponent Φ transforming like Ψ one has

$$\mathcal{L}_S = (D_\mu \Phi)^+ (D^\mu \Phi) - V(\Phi^+\Phi) \tag{5.18}$$

Again one can rescale $A^\mu_a \to g A^\mu_a$ to put the Yang-Mills coupling constant g in the interaction terms.

There are important differences between the abelian and nonabelian theories. In the former one could have a multicomponent Ψ (or Φ) transforming as $\Psi' = \exp(i\alpha Q)\Psi$ if $D_\mu = \partial_\mu + iAQ$ and Q is diagonal - in other words different fields can have arbitrarily different charges. Nonlinearity forbids this in the nonabelian case; so long as G is semisimple charges are fixed by the group theory representations. Associated to this is the nonlinear terms in (5.14) which say that the Yang-Mills field carries the Yang-Mills charges. Thus even the pure Yang-Mills theory (5.16) is nontrivial:

$$\frac{\partial \mathcal{L}_{YM}}{\partial A^a_{\nu,\mu}} = -\frac{1}{g^2} F_a^{\mu\nu} \quad , \quad \frac{\partial \mathcal{L}_{YM}}{\partial A^a_\nu} = \frac{1}{g^2} C_{abc} A_\mu{}^c F_b{}^{\mu\nu} \tag{5.19}$$

so

$$D_{\mu ab} F_b{}^{\mu\nu} = 0 \tag{5.20}$$

contains quadratic and cubic terms!

Consequent of (5.15)

$$D_{\nu\, ab}\, D_{\mu\, bc}\, F_c^{\mu\nu} = 0 \tag{5.21}$$

holds as an identity. This implies that with matter included

$$g^{-2} D_{\mu\, ab}\, F_b^{\mu\nu} = J_a^{\nu} \tag{5.22}$$

the current is covariantly conserved.

$$D_{\mu\, ab}\, J_b^{\mu} = 0 \tag{5.23}$$

References

The original work, which still makes valuable reading, is: C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954). Similar lines were developed in the PhD thesis of R. Shaw, Cambridge University 1955 (unpublished)

Problems

5.1. Verify (5.21) follows from (5.15). (Hint: invariance implies $\delta \mathcal{L}_{YM}$ under an infinitesimal gauge transformation.)

5.2. Obtain the ψ, $\bar{\psi}$ field equations and current J_a^{μ} from (5.17); show the latter satisfies (5.23).