Where the Action is

Hamilton's principle provides a powerful tool in the classical dynamics of particles and even more so in 'classical' field theory. In the former one has generalized coordinates $q^i(t)$, the index "i" labeling different components and/or particles; in the latter relativistic considerations require the generalized coordinates to be $\phi^i(x)$, functions of x^μ carrying an index "i" to distinguish different components and/or fields. For particles the generalized velocities are $\dot{q}^i = dq^i/dt$; time is the 0-th component of a 4-vector so for relativisic fields the analogue is $\partial_\mu \phi^i$. Particle mechanics says: take a Lagrangian $L(q, \dot{q})$ and integrate it over time to get the action S, a real number. Field theory says: form a Lagrangian density $\mathcal{L}(\phi, \partial\phi)$ which is a Lorentz scalar and integrate it over the 4-volume $d^4x = dt\,dx\,dy\,dz$ to form

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial\phi) \qquad (4.1)$$

Notationally: S is a 'functional' of ϕ, a mapping from functions to numbers - in other words its value depends on ϕ^i at every space-time point.

Hamiltons principle for particles deals with infantesimal variations $q^i(t) \to q'^i(t) = q^i(t) + \delta q^i(t)$. Field theory allows us to change both the generalized coordinates and the space time ones

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu, \quad \phi^i(x) \to \phi'^i(x') = \phi^i(x) + \delta\phi^i(x) \qquad (4.2)$$

It is useful to have a 'local variation' which compares things with the same coordinates - compare (2.28') - so

$$\delta\phi^i(x) = \phi'^i(x') - \phi^i(x) = \Delta\phi^i(x) + \delta x^\mu \partial_\mu \phi^i(x) \qquad (4.3)$$

to first order. The difference of $\delta\phi^i$ and $\Delta\phi^i$ is important for

$$\delta(\partial_\mu \phi^i(x)) \equiv \partial'_\mu \phi'^i(x') - \partial_\mu \phi^i(x)$$

$$= \partial_\mu \Delta\phi^i(x) + \partial_\mu(\delta x^\nu \partial_\nu \phi^i(x)) - (\partial_\mu \delta x^\nu)\partial_\nu \phi^i(x) \qquad (4.4)$$

Since

$$d^4x' = \det\left(\frac{\partial x'}{\partial x}\right) d^4x = (1 + \partial_\mu \delta x^\mu)\, d^4x \qquad (4.5)$$

one finds

$$\delta S[\phi] = \int d^4x \left[(\partial_\mu \delta x^\mu)\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\phi^i}(\Delta\phi^i + \delta x^\mu \partial_\mu \phi^i) + \right.$$

$$\left. \frac{\partial\mathcal{L}}{\partial(\phi^i_{,\mu})}(\partial_\mu \Delta\phi^i + \delta x^\nu \partial_\mu \partial_\nu \phi^i) \right]$$

$$= \int d^4x \left[\left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\phi^i_{,\mu})} \right)\Delta\phi^i - \partial_\mu J^\mu \right], \qquad (4.6)$$

$$J^\mu \equiv - \frac{\partial \mathcal{L}}{\partial(\phi^i{}_{,\mu})} \Delta\phi^i - \delta X^\mu \mathcal{L} \tag{4.7}$$

Thus $\delta S = 0$ provides us with the Euler - Lagrange field equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\phi^i{}_{,\mu})} - \frac{\partial \mathcal{L}}{\partial \phi^i} = 0 \tag{4.8}$$

and a conserved current (Noether current)

$$\partial_\mu J^\mu = 0 \tag{4.9}$$

plus a conserved (Noether) "charge"

$$Q = \int d^3x\, J^0 \quad , \quad \frac{d}{dt} Q = 0 \tag{4.10}$$

if the fields fall off sufficiently fast at spatial infinity.

Lets see how this works for a pertinent case, that of a complex (Lorentz) scalar field Φ. One could work with the real and imaginary parts, $\Phi = (\phi^1 + i\phi^2)/\sqrt{2}$ but we'll stick with Φ and let "i" include the independent complex conjugate Φ^*. We want $\mathcal{L}$ real and so take

$$\mathcal{L} = g^{\mu\nu}(\partial_\mu\Phi^*)(\partial_\nu\Phi) - V(\Phi^*\Phi) = (\partial_\mu\Phi^*)(\partial^\mu\Phi) - V(|\Phi|^2)$$

$$= |\partial\Phi|^2 - V \tag{4.11}$$

indicating the range of notations. Then

$$\frac{\partial \mathcal{L}}{\partial \Phi^*} = -V'\Phi, \qquad \frac{\partial \mathcal{L}}{\partial \Phi} = -\Phi^* V',$$

$$\frac{\partial \mathcal{L}}{\partial (\Phi^*_{,\mu})} = \partial^\mu \Phi \quad , \frac{\partial \mathcal{L}}{\partial \Phi_{,\mu}} = \partial^\mu \Phi^*, \qquad (4.12)$$

$V' = V'(|\Phi|^2) = \partial V(|\Phi|^2)/\partial |\Phi|^2$, and we get two field equations

$$\Box \Phi + V'\Phi = 0 \quad , \quad \Box \Phi^* + \Phi^* V' = 0 \qquad (4.13)$$

If $V = m^2 |\Phi|^2$ (4.13) are just Klein-Gordon equations. Under space-time translations $\delta x^\mu = \xi^\mu$, a constant 4-vector $\delta \Phi$ vanishes but

$$\Delta \Phi = -\xi^\mu \partial_\mu \Phi \qquad (4.14)$$

does not. The conserved current reads

$$J^\mu = (\partial^\mu \Phi^*)\xi^\nu(\partial_\nu \Phi) + (\xi^\nu \partial_\nu \Phi^*)(\partial^\mu \Phi) - \xi^\mu \mathcal{L}$$

$$= T^\mu{}_\nu \xi^\nu, \qquad (4.15)$$

whence four associated charges written

$$P^\mu = \int d^3x \; T^{0\mu} \qquad (4.16)$$

Examining P^0,

$$P^0 = \int d^3x \left[|\partial_0 \Phi|^2 + |\vec\nabla \Phi|^2 + V(|\Phi|^2) \right]$$

$$= \int d^3x \; \mathcal{P}^0, \qquad (4.17)$$

we see the analogue of the energy or hamiltonian in particle mechanics - indeed P^i are the 'three momenta' of the field(s). Note the 'stress energy tensor' $T^{\mu\nu}$ is symmetric.

Lorentz invariance of (4.11) gives six more generalized charges; we leave this as an execise. There is yet another symmetry of $\mathcal{L}$: it is unchanged by 'global phase transformations' $\Phi \to e^{i\alpha}\Phi$, $\Phi^* \to \Phi^* e^{-i\alpha}$ with α constant. In that case

$$\delta\Phi = \Delta\Phi = i\alpha\Phi \;, \qquad (4.18)$$

$\delta x^\mu = 0$ and the conserved current is

$$J^\mu = i\alpha(\Phi^*(\partial^\mu\Phi) - (\partial^\mu\Phi^*)\Phi) \equiv i\alpha\Phi^* \overleftrightarrow{\partial^\mu}\Phi \,. \qquad (4.19)$$

Now, it must be possible to design an $\mathcal{L}$ whose Euler-Lagrange field equations are those of electrodynamics*. The basic building block is $F^{\mu\nu}$ in (2.19) and Lorentz invariance says we need $F_{\mu\nu}F^{\mu\nu}$ and $A^\mu J_\mu$. Thus try

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu \qquad (4.20)$$

with the "μ" on A^μ like the "i" on ϕ^i. Since

$$\frac{\partial\mathcal{L}}{\partial A_{\nu,\mu}} = -\frac{1}{2}F^{\mu\nu} + \frac{1}{2}F^{\nu\mu} = -F^{\mu\nu} \;, \quad \frac{\partial\mathcal{L}}{\partial A_\nu} = -J^\nu \qquad (4.21)$$

* If not you wouldn't be reading this!

the relative sign is correct but what about the absolute sign? Uner translations $\Delta A^\mu = -\xi^\nu \partial_\nu A^\mu$ and our recipee gives the 'canonical' stress-energy tensor ($J^\mu = 0$)

$$T^{\mu\nu}_{can} = F^{\mu\alpha}\partial^\nu A_\alpha - g^{\mu\nu}\mathcal{L} \qquad (4.22)$$

This is a bit of an embarrasment: $T^{\mu\nu}_{can}$ isnt symmetric or even gauge invariant! Still, the field equations are unaltered if we modify $\mathcal{L}$ by a 4-divergence but the Noether current is changed; by a judicious choice we can arrange this so

$$T^{\mu\nu} = F^{\mu\alpha}F^{\nu}{}_{\alpha} - g^{\mu\nu}\mathcal{L}\ . \qquad (4.23)$$

Using the result of Problem 2.1, $T^{00} = (\vec{E}^2 + \vec{B}^2)/2 \geq 0$ so the overall sign is right.

Spinor fields can be treated similarly:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi\ , \qquad (4.24)$$

$$\not{\partial} \equiv \gamma^\mu \partial_\mu \qquad (4.25)$$

yields up the free Dirac equation (3.12) as $\partial\mathcal{L}/\partial\bar{\Psi}_{,\mu} = 0$. Integration by parts gives $\partial\mathcal{L}/\partial\Psi_{,\mu} = 0$ and the conjugate

$$\bar{\Psi}(i\overleftarrow{\not{\partial}} + m) = i\partial_\mu\bar{\Psi}\gamma^\mu + \bar{\Psi}m = 0\ . \qquad (4.26)$$

Taking

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{\partial} - e\not{A} - m)\,\psi \tag{4.27}$$

produces the Dirac equation in an electromagnetic field, (3.33), and the electrodynamic field equations with the source current

$$J^\mu = e\,\bar{\psi}\,\gamma^\mu \psi\,. \tag{4.28}$$

This raises an important point about the Dirac field: because the charges are determining the field in which they move in (4.27) one expects $\mathcal{L}$ to be the same if we replace all the fields by their charge conjugates, including $A^{\mu c} = -A^\mu$

$$\mathcal{L}^c = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi^c}\,(i\not{\partial} + e\not{A} - m)\,\psi^c \tag{4.29}$$

but of course $J^{\mu c} = -J^\mu$. Note that in the Weyl representation

$$\begin{aligned} &\gamma^0 = \gamma^{0T} = \gamma^{0*}, \quad \gamma^1 = -\gamma^{1T} = \gamma^{1*}, \\ &\gamma^2 = \gamma^{2T} = -\gamma^{2*}, \quad \gamma^3 = -\gamma^{3T} = \gamma^{3*} \end{aligned} \tag{4.30}$$

and (3.38) can be written

$$\psi^c = -i\gamma^2\gamma^0\,\bar{\psi}^T \tag{4.31}$$

so

$$\overline{\psi^c} = \psi\, i\gamma^{2+}\gamma^0 = \psi^T i\gamma^0\gamma^2 \tag{4.32}$$

Since

$$(i\gamma^0\gamma^2)(\gamma^\mu, 1)(-i\gamma^2\gamma^0) = (-1, \gamma^{\mu T}) \qquad (4.33)$$

in fact

$$\mathcal{L}_c^c = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \psi^T\gamma^{\mu T}(i\partial_\mu + eA_\mu)\bar{\psi}^T + m\psi^T\bar{\psi}^T, \qquad (4.34a)$$

$$J^{\mu c} = e\psi^T\gamma^{\mu T}\bar{\psi}^T \qquad (4.34b)$$

The resolution of this paradox is that the Dirac field does not commute like an ordinary c-number, even classically, but anticommutes; eg putting in the spinor indicies

$$J^{\mu c} = e\psi_j(\gamma^\mu)_{ij}\bar{\psi}_i = -e\bar{\psi}_i(\gamma^\mu)_{ij}\psi_j \qquad (4.35)$$

Thus the Dirac fields are 'anticommuting c-numbers' or elements of a 'Grassmann algebra'. This is the origin of Fermi-Dirac statistics and the exclusion principle!

References

Classical field theory is discussed in eg H. Goldstein "Classical Mechanics", Addison-Wesley, Reading, Massachusetts, 1980; beware though the ict (pronounced yuch) metric. Our treatment follows Sections 6,4,5 of Cushing, ibid. . The pecularities of spinor fields is discussed in Ramond, ibid.

Problems

4.1. Verify eqs (4.4) and (4.5).

4.2. Show (4.10) follows from (4.9).

4.3. For the model (4.11) obtain the Noether current associated to infinitesimal Lorentz transformations (2.7) as

$$J^{\mu} = \frac{1}{2} M^{\mu\alpha\beta} \omega_{\alpha b}$$

Show the space-space components of

$$Q^{\alpha\beta} = \int d^3x \; M^{0\alpha b}$$

give the angular momentum of the field

4.4. Determine the 4-divergence $\partial_\mu f^\mu$ which must be added to (4.20) to yield (4.23).