Spinors : Matter

In view of the similarities of the rotation and Lorentz group, given the homomorphism between SO(3) and SU(2) one expects something similar. Indeed, remembering that electrons belong to $J = 1/2$ this will be what describes matter.

A convient (but nonhistorical) way to proceed is to mock up what we know for rotations. Since there are only three Pauli matricies but time must enter try

$$\sigma^\mu = (\sigma^0, \vec{\sigma}) \quad , \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \qquad (3.1)$$

and

$$\underset{\sim}{X} = \sigma^\mu X_\mu = \begin{pmatrix} t - z & -(x - iy) \\ -(x+iy) & t+z \end{pmatrix} , \qquad (3.2)$$

so $\det(\underset{\sim}{X}) = s^2$, with the transformation rule

$$\underset{\sim}{X}' = \underset{\sim}{M}^{+} \underset{\sim}{X} \, \underset{\sim}{M} \qquad (3.3)$$

To have $s'^2 = \det(X') = s^2$ we need $\underset{\sim}{M}$ to be 'unimodular', $\det(\underset{\sim}{M}) = 1$, but not unitary, $M^{-1} \neq M^{+}$, else we just get the SU(2) subgroup.* The group so described is SL(2,C) (special, linear, 2 dimensions, complex)

* Nonunitarity is connected to noncompactness, and why we did not put an "i" in (2.7)

For a boost along the z axis (2.8) gives $(t' \mp z') = (t \mp z)\, e^{\pm\theta}$ and we see

$$\underset{\sim}{M} = \begin{pmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{pmatrix} \tag{3.4}$$

reproduces that. In general

$$\underset{\sim}{M} = \exp\left(\tfrac{\theta}{2}\, \vec{\sigma}\cdot\hat{v} \right) \tag{3.5}$$

and

$$\underset{\sim}{M}^{+} \sigma^{\mu} \underset{\sim}{M} = \Lambda^{\mu}{}_{\nu}\, \sigma^{\nu} \tag{3.6}$$

We could just as well have taken

$$\tilde{\sigma}^{\mu} = (\sigma^{0}, -\vec{\sigma})$$

and $\tilde{\underset{\sim}{X}} = \tilde{\sigma}^{\mu} X_{\mu}$. This is equivalent to a spatial inversion - parity - and so a little though demonstrates* $\tilde{\underset{\sim}{X}}' = (\underset{\sim}{M}^{-1})^{+} \tilde{\underset{\sim}{X}} M^{-1}$ or

$$(\underset{\sim}{M}^{-1})^{+}\, \tilde{\sigma}^{\mu} M^{-1} = \Lambda^{\mu}{}_{\nu}\, \tilde{\sigma}^{\nu} \tag{3.7}$$

Again we have a homomorphism: two elements of $SL(2,C)$ are mapped to one of the proper Lorentz group.

Now, we can introduce two sorts of 2 component spinors, ψ_L and ψ_R according according to how they transform

$$\psi_L'(x') = M\, \psi_L(x) \tag{3.8a}$$

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* For boosts only!

$$\psi_R'(x') = M^{-1}\psi_R(x) \tag{3.8b}$$

Care must be exercised when acting with derivatives : $i\tilde{\sigma}^\mu \partial_\mu \psi_L$ transforms as ψ_R and $i\sigma^\mu\partial_\mu \psi_R$ as ψ_L. Thus we end up with a pair of coupled equations

$$i\tilde{\sigma}^\mu \partial_\mu \psi_L = m\,\psi_R \tag{3.9a}$$

$$i\sigma^\mu \partial_\mu \psi_R = m\,\psi_L \tag{3.9b}$$

Recalling the Pauli matrices satisfy the anticommutation relation

$$[\sigma^i, \sigma^j]_+ = 2\delta^{ij} \tag{3.10}$$

applying $i\sigma^\mu\partial_\mu$ to (3.9a) and $i\tilde{\sigma}^\mu\partial_\mu$ to (3.9b) one readily finds $(\Box + m^2)\psi_{L/R} = 0$, ie they satisfy the 'Klein-Gordon equation' for mass m.

We can assemble ψ_L and ψ_R into a 4-component spinor

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \tag{3.11}$$

and write (3.9) as the 'Dirac equation'

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0 \tag{3.12}$$

where in block form

$$\gamma^\mu = \begin{pmatrix} 0 & \tilde{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \tag{3.13}$$

and recover the 2-spinors a la

$$\psi_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\psi, \quad \gamma_5 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \equiv \gamma^5 \tag{3.14}$$

It is left as a problem to check that

$$\tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu = 2 g^{\mu\nu} \mathbb{1} \tag{3.15}$$

$$[\gamma^\mu, \gamma^\nu]_+ = 2 g^{\mu\nu} \mathbb{1} \tag{3.16}$$

- often by convention identity matrices are left implicit! Also,

$$[\gamma_5, \gamma^\mu]_+ = 0 \tag{3.17}.$$

Eq (3.16) defines the 'Dirac (a Clifford) algebra'. There is nothing sancrosanct about the 'Weyl representation' we are using; by a block similarity transform

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} = U^{-1} \tag{3.18}$$

one obtains the 'Dirac representation

$$\gamma'^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \vec{\gamma}' = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma_5' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.19}$$

but then the representations $\psi_{L/R}$ are mixed in ψ' and must in any case be disentangled for constructing the standard model.

As noted above parity connects the two representations, and the Dirac equation is unchanged if $\vec{x} \to -\vec{x}$ and $\psi \to \psi^P$

$$\psi^P = \gamma^0 \psi \equiv (\bar{\psi})^+ \qquad (3.20)$$

which interchanges ψ_L and ψ_R. Albeit $\psi_L^+ \tilde{\sigma}^\mu \psi_L$, $\psi_R \sigma^\mu \psi_R$, $\psi^+_L \psi_R$ and $\psi^+_R \psi_L$ have definite transformations under $SL(2,C)$, they do not under the improper Lorentz transformation of parity. We can, however, form such objects:

$$\bar{\psi}\psi = \psi_L^+ \psi_R + \psi_R^+ \psi_L \quad , \quad \text{scalar} \qquad (3.21a)$$

$$\bar{\psi}\gamma_5\psi = \psi^+_L \psi_R - \psi_R^+ \psi_L \quad , \quad \text{pseudoscalar} \qquad (3.21b)$$

$$\bar{\psi}\gamma^\mu\psi = \psi^+_L \tilde{\sigma}^\mu \psi_L + \psi_R^+ \sigma^\mu \psi_R \quad , \quad \text{vector} \qquad (3.21c)$$

$$\bar{\psi}\gamma_5\gamma^\mu\psi = \psi^+_L \tilde{\sigma}^\mu \psi_L - \psi^+_R \sigma^\mu \psi_R \quad , \quad \text{pseudovector}. \qquad (3.21d)$$

Next, consider plane wave (to become free particle) solutions of (3.9). Writing

$$\psi_{L/R} = e^{-iP\cdot X} u_{L/R} \quad , \quad P^0 = E > 0 \qquad (3.22)$$

we have

$$(E + \vec{\sigma}\cdot\vec{P})u_L = m\, u_R, \quad (E - \vec{\sigma}\cdot\vec{P})u_R = m\, u_L \qquad (3.23)$$

when $\vec{p} \to 0$ these say $u_L = u_R = \varphi/\sqrt{2}$, φ a two component spinor normalised to $\varphi^+ \varphi = 1$. As is easily verified the solution is

$$u_R(p, \varphi') = \frac{E + m + \vec{\sigma}\cdot\vec{p}}{2\sqrt{m(E+m)}}\, \varphi \tag{3.24a}$$

$$u_L(p, \varphi) = \frac{E + m - \vec{\sigma}\cdot\vec{p}}{2\sqrt{m(E+m)}}\, \varphi \tag{3.24b}$$

such that

$$\bar{u}(p,\varphi)\, u(p,\varphi') = \varphi^+ \varphi' \tag{3.25}$$

Now one sees what "L" and "R" mean: the solutions can be chosen as eigenstates of helicity $\vec{\sigma}\cdot\vec{p}/|\vec{p}|$ but as $E \to \infty$ u_R (u_L) is purely 'right (left) handed' with eigenvalue +1 (−1). Chirality, unlike helicity is Lorentz invariant. Summing over 'polarizations' $\varphi = \binom{1}{0}, \binom{0}{1}$

$$\sum_{pol} u(p,\varphi) \otimes \bar{u}(p,\varphi) = \frac{1}{2m}\left(\gamma^\mu p_\mu + m\right) \tag{3.26}$$

Because the Klein-Gordon equation has $p^2 = (p^0)^2 - \vec{p}^2 = m^2$ there must be solutions with $p^0 = -E < 0$. It is more convenient to write these instead as

$$\psi_{L/R} = e^{ip\cdot x}\, v_{L/R}, \qquad p^0 = E > 0 \tag{3.27}$$

so

$$(E + \vec{\sigma}\cdot\vec{p})\, v_L = -m\, v_R, \qquad (E - \vec{\sigma}\cdot\vec{p})\, v_R = -m\, v_L \tag{3.28}$$

For these $v_L = v_R = -\chi/\sqrt{2}$ as $\vec{p} \to 0$, χ a two component spinor, and one finds

$$v_R(P,\chi) = \frac{\vec{\sigma}\cdot\vec{P} + E + m}{2\sqrt{m(E+m)}}\,\chi \tag{3.29a}$$

$$v_L(P,\chi) = \frac{\vec{\sigma}\cdot\vec{P} - E - m}{2\sqrt{m(E+m)}}\,\chi \tag{3.29b}$$

with normalization

$$\bar{v}(P,\chi)\,v(P,\chi') = -\chi^{+}\chi' \tag{3.30}$$

while

$$\sum_{pol} v(P,\chi)\otimes\bar{v}(P,\chi) = \frac{1}{2m}(\gamma^{\mu}P_{\mu} - m) \tag{3.31}$$

Note that $\bar{v}(P,\chi)\,u(P,\varphi) = \bar{u}(P,\varphi)\,v(P,\chi) = 0$.

Historically the 'negative energy (frequency)' solutions caused some consternation; later we will see this concern is misplaced. More importantly, they lead to 'antimatter', the prototype being the positron*. To see this aspect (without splashing holes in the Dirac sea) recall from electrodynamics that the introduction of electromagnetic potentials follows the 'minimal substitution' rule $E \to E - e\phi$, $\vec{P} \to \vec{P} - e\vec{A}$; quantum mechanics says $E \sim i\partial^0$, $\vec{P} \sim -i\vec{\nabla}$ so this becomes

$$i\partial^{\mu} \to i\partial^{\mu} - eA^{\mu} \tag{3.32}$$

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* Initially peer pressure against new particles led Dirac to identify this as the proton!

and the Dirac equation reads

$$[\,\gamma^\mu(i\partial_\mu - eA_\mu) - m\,]\,\psi = 0 \tag{3.33}$$

Observe this is invariant under simultaneous gauge transformations (2.22) and $\psi \to e^{-iex}\psi$. 'Charge conjugation' is the replacement $e \to -e$, $\psi \to \psi^c$,

$$[\,\gamma^\mu(i\partial_\mu + eA_\mu) - m\,]\,\psi^c = 0 \tag{3.34}$$

Comparing (3.22) and (3.27) we expect $\psi^c \sim \psi^*$ but the complex conjugate of (3.33) is

$$[\,\gamma^{\mu *}(-i\partial_\mu + eA_\mu) - m\,]\,\psi^* \tag{3.35}$$

so that is not the whole story; γ^0, γ^1 and γ^3 are real while $\gamma^{2*} = -\gamma^2$ which gives the way out as

$$(\gamma^\mu)^c = i\gamma^2\gamma^{\mu *}i\gamma^2 = -\gamma^\mu \quad , \tag{3.36}$$

$$[\,\gamma^\mu(i\partial_\mu + eA_\mu) - m\,](i\gamma^2\psi^*) = 0 \tag{3.37}$$

Taking (advantage of a phase choice)

$$\psi^c = -i\gamma^2\psi^* \tag{3.38}$$

indeed turns (3.24) into (3.29) - just be cautious in that $(\psi^c)_{R/L} = (\psi_{L/R})^c$.

In the everyday world matter predominates, relativistic effects are small and (3.9) can be usefully rewritten in term of linear combinations

$$\psi_L + \psi_R = e^{-imt}\varphi\ , \quad \psi_L - \psi_R = e^{-imt}\chi \qquad (3.39)$$

yielding

$$(i\partial^0 - e\phi)\varphi = \vec{\sigma}\cdot(i\vec{\nabla} + e\vec{A})\chi \qquad (3.40a)$$

$$[2m + (i\partial^0 - e\phi)]\chi = \vec{\sigma}\cdot(i\vec{\nabla} + e\vec{A})\varphi \qquad (3.40b)$$

Approximately solving (3.40b), $\chi \simeq \vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A})\varphi/2m$, reduces (3.40a) to the Pauli-equation

$$i\partial^0\varphi = \left[\frac{1}{2m}(\vec{\sigma}\cdot(i\vec{\nabla}+e\vec{A}))^2 + e\,\phi\right]\varphi \qquad (3.41)$$

or after a little manipulation

$$i\frac{\partial}{\partial t}\varphi = \left[\frac{1}{2m}(i\vec{\nabla}+e\vec{A})^2 - \frac{e}{2m}\vec{\sigma}\cdot\vec{B} + e\phi\right]\varphi \qquad (3.42)$$

Thus the Dirac equation <u>predicts</u> the electron has a magnetic moment $\vec{\mu} = 2(e/2m)(\vec{\sigma}/2)$, ie a gyromagnetic ratio $g_0 = 2$.

Finally, when $m=0$ eqs (3.9) decouple. The distinct free solutions can be taken as

$$u_L = \tfrac{1}{2}(\mathbb{1} - \vec{\sigma}\cdot\hat{p})\varphi_L\ , \quad v_R = \tfrac{1}{2}(\mathbb{1} + \vec{\sigma}\cdot\hat{p})\chi_R \qquad (3.43)$$

which are related by charge conjugation

References:

Insights into the history of the Dirac equation can be found in: S. Schweber, "QED and the men who made it", Princeton U. Press, New Jersey, 1994. The ahistorical approach, based on the Lorentz group may be found in: P. Ramond, "Field Theory: A Modern Primer" Addison-Wesley, Reading, Massachusetts, 2nd edition 1989.

Problems:

3.1. Verify eqs (3.15, 16).

3.2. Verify eqs (3.26, 31)

3.3. Fill in the steps between eqs (3.41) and (3.42).

3.4. Why for $m=0$ do only two solutions (3.43) obtain? (Hint: take $\hat{p} = \hat{z}$).