Electrodynamics and the Lorentz Group

Looking at Maxwell's equations

$$\vec\nabla\cdot\vec E = \rho \tag{2.1a}$$

$$\vec\nabla\cdot\vec B = 0 \tag{2.1b}$$

$$\vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} = 0 \tag{2.1c}$$

$$\vec\nabla\times\vec B - \frac{\partial\vec E}{\partial t} = \vec J \tag{2.1d}$$

one would scarcely guess invariance^{+} under Lorentz transformations

$$t' = \gamma(t - \vec v\cdot\vec x)\ , \quad \vec x' = \gamma(\vec x - \vec v t)\ , \quad \gamma = (1-\vec v^2)^{-1/2} \tag{2.2}$$

- in this sense Lorentz invariance may be the mother of all hidden symmetries! In dealing, as we shall, with relativistic theories much agony can be avoided by employing an efficient notation which Lorentz (and other) symmetry apparent:

Recall Lorentz transformations can be defined, similar to SO(3), as those leaving $s^2 = t^2 - \vec x^2$ invariant (so the measured speed of light is the same to observers having constant relative velocity). To deal with the (-) sign in s^2 introduce 'contravariant' x^μ and covariant x_μ 4-vectors where

$^{+}$ more properly 'covariant'; see below

$$X^0 = X_0 = t\ ,\quad X^1 = -X_1 = X,\quad X^2 = -X_2 = Y\ ,\quad X^3 = -X_3 = Z \quad * \tag{2.3}$$

so Greek indicies run from 0 to 3, and with the summation convention $s^2 = X^\mu X_\mu$. The co- and contra-variant 4-vectors are connected by the Minkowski covariant metric

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1\ , \tag{2.4}$$

a la $X_\mu = g_{\mu\nu} X^\nu$, and the contravariant metric $g^{\mu\nu}$, $g^{\mu\alpha} g_{\alpha\nu} = g^\mu{}_\nu = \delta^\mu{}_\nu$ (= 1 if μ and ν same, zero otherwise), $X^\mu = g^{\mu\nu} X_\nu$. A Lorentz transformation takes the form

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu \tag{2.5}$$

and if $s^{2\prime} = s^2$

$$g_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu = g_{\mu\nu} \tag{2.6}$$

For an infintesimal Lorentz transformation[+]

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \tag{2.7}$$

and putting (2.7) in (2.6) one readily finds the infintesimal parameters $\omega_{\mu\nu} = g_{\mu\alpha}\omega^\alpha{}_\nu$ form an antisymmetric 'tensor' $\omega_{\nu\mu} = -\omega_{\mu\nu}$. For a boost along the z axis $\omega_{03} = -\omega_{30} = -\theta$ are the nonzero elements so $\omega^0{}_3 = \omega^3{}_0 = -\theta$; using the latter in matrix notation the

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[+] Contemplate: why dont we include an "i" here?

* More compactly: $X^\mu = (t, \vec{x})$, $X_\mu = (t, -\vec{x})$

finite transformation is

$$\Lambda^\mu{}_\nu = (e^{\omega})^\mu{}_\nu = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix} \tag{2.8}$$

and (2.2) recovers with $\tanh\theta = v_z$.

Evidently the transformation of x_μ is $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ where $\Lambda_\mu{}^\nu = g_{\mu\alpha} g^{\nu\beta} \Lambda_\alpha{}^\beta$. Generally we can define tensors as contravariant, covariant or mixed if they transform like $x^\mu x^\nu \cdots$, $x_\mu x_\nu \cdots$, or $x^\mu x_\nu \cdots$. Replacing the usual 3-gradient $\vec{\nabla}$ is

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{2.9}$$

which transforms as a covariant 4-vector. Note

$$\partial^\mu = g^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \tag{2.10}$$

allows us to form the Lorentz invariant d'alembertian

$$\Box = \partial^\mu \partial_\mu = \left(\frac{\partial}{\partial t}\right)^2 - \vec{\nabla}^2 .$$

Occasionally one denotes $f_{,\mu} \equiv \partial_\mu f$, $f_,{}^\mu = \partial^\mu f$.

Back to electrodynamics: the charge density ρ and 3-current $\vec{J}$ have the right number to form a 4-vector current

$$J^\mu = (\rho, \vec{J}) \tag{2.11}$$

but not the six electric and magnetic fields.

The number of fields is, however, just right for an antisymmetric tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} \qquad (2.12)$$

Introducing the the Levi-Civita tensor

$$\epsilon^{\mu\nu\alpha\beta} \equiv \begin{cases} +1 & \mu\nu\alpha\beta \text{ even permutation } 0123 \\ -1 & \text{" " odd " "} \\ 0 & \text{otherwise} \end{cases} \qquad (2.13)$$

the 'dual' field strength

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \qquad (2.14)$$

is obtained from $F^{\mu\nu}$ by $E \to B$, $B \to -E$. Maxwells equations can thus be expressed

$$\partial_\mu F^{\mu\nu} = J^\mu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \qquad (2.15)$$

Strictly, these are not invariant like $x^2 \equiv x^\mu x_\mu$ but 'covariant': they take the same form in every Lorentz frame and both sides transform identically. Since $F^{\mu\nu}$ is antisymmetric but $\partial_\mu\partial_\nu$ is symmetric it follows from the first of (2.15) that the 4-current J^μ is conserved:

$$\partial_\mu J^\mu = 0 \qquad (2.16)$$

A more profitable approach is to recall that eqs (2.1b, c) can be solved by introducing 3-vector ($\vec{A}$) and 3-scalar (ϕ) potentials

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad , \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \tag{2.17}$$

Using (2.17) in the remaining Maxwell equations and recalling $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$ it is straightforward to verify that

$$A^\mu = (\phi, \vec{A}) \quad , \tag{2.18}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, , \tag{2.19}$$

such that they become

$$\Box A^\mu - \partial^\mu(\partial\cdot A) = J^\mu \tag{2.20}$$

where $\partial\cdot A \equiv \partial_\mu A^\mu$. Also, we know the potentials $\vec{A}$ and ϕ are not unique: the fields $\vec{B}$, $\vec{E}$ are unchanged by a 'gauge transformation'

$$\vec{A} \to \vec{A}' = \vec{A} - \vec{\nabla}\chi \quad , \quad \phi \to \phi' = \phi + \frac{\partial\chi}{\partial t} \tag{2.21}$$

which read for the 4-potential A^μ

$$A^\mu \to A^{\mu\prime} = A^\mu + \partial^\mu \chi \tag{2.22}$$

giving $F^{\mu\nu\prime} = F^{\mu\nu}$. Hence (2.20) 'has no solution' (really an infinity of solutions). The way out of this impass is to gauge transform such that A satisfies a 'gauge condition' – the obvious generalization of the Coulomb gauge $\vec{\nabla}\cdot\vec{A}' = 0$ is the Lorenz gauge condition

Under a Lorentz transformation

$$A^\mu(x) \to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \qquad (2.27)$$

so $A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$. For an infinitesimal transformation this can be written

$$A'^\mu(x) = [\,\delta^\mu{}_\nu - \tfrac{1}{2}\omega_{\alpha\beta}(\mathbb{M}^{\alpha\beta})^\mu{}_\nu\,]\,A^\nu(x)\,, \qquad (2.28a)$$

$$(M^{\alpha\beta})^\mu{}_\nu = i(x^\alpha\partial^\beta - x^\beta\partial^\alpha)\,\delta^\mu_\nu + i(g^{\mu\alpha}\delta^\beta{}_\nu - g^{\mu\beta}\delta^\alpha{}_\nu) \qquad (2.28b)$$

Now the Lorentz group contains rotations as a subgroup and indeed defining $\underset{\sim}{J}^i = \frac{1}{2}\epsilon_{ijk}\underset{\sim}{M}^{jk}$ acting on $\vec{A}$ as a column vector

$$\vec{\underset{\sim}{J}} = \vec{r}\times(-i\vec{\nabla}) + \vec{\underset{\sim}{I}}\,. \qquad (2.29)$$

One recognises the first part of (2.29) as the orbital angular momentum from quantum mechanics; the second term is the spin angular momentum. Thus the spin is $|S| = 1$ and $(\vec{\epsilon}(\vec{k},1) \pm \vec{\epsilon}(\vec{k},2))/\sqrt{2}$ correspond to projections $s = \pm 1$ along $\vec{k}/|\vec{k}|$ ('helicity'). Upon quantization these are the 'photons'.

Finally, for later use, suppose in (2.20) $J^\mu = -m^2 A^\mu$. In this case there is no gauge invariance because the right hand side changes but still taking the divergence $\partial\cdot A = 0$. The ansatz (2.25) gives $k^2 = m^2$ as for a massive particle. Further $k\cdot\epsilon = 0$ gives as well as (2.26)

$$\epsilon^{\mu}(k,3) = \frac{1}{m}\left(|\vec{k}|,\ \omega\frac{\vec{k}}{|\vec{k}|}\right) \qquad (2.30)$$

which is unaffected by rotations about $\vec{k}$ so having helicity 0.

Reference

The standard text is: J. D. Jackson, "Classical Electrodynamics", Wiley, New York, 3rd edition, 1975.

Problems

2.1. Determine the Lorentz invariants $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$ in terms of $\vec{E}$ and $\vec{B}$

2.2 Verify eqs. (2.28) and (2.29).

2.3 Prove that the polarization 4-vectors in (2.26) and (2.30) satisfy

$$\sum_{\lambda=1}^{3} \epsilon^{\mu}(k,\lambda)\,\epsilon^{\nu}(k,\lambda) = -g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{m^2} \qquad (2.31)$$

What happens as $m \to 0$?

2.4 How do the fields / potentials transform under the improper Lorentz transformations of parity, $\vec{x} \to -\vec{x}$, and time reversal $t \to -t$? What happens for 'charge conjugation, $q \to -q$?