Elements of Group Theory

Symmetries play a crucial role in the Standard Model, and the language in which to discuss them is that of Group Theory. An abstract group G consists of a set of elements g_i which are closed under an associative rule "$\circ$" for combining them, which contains a unique inverse element e and for every g_i a unique inverse element g_i^{-1}. Thus everyone knows a group: the integers under addition.

A little thought shows that the abstract two element group consisting of $\underline{e}$ and $\underline{a}$ must have $a^{-1} = a$. One knows two examples: spatial reflections or parity,

$$P\, f(\vec{x}) = f(-\vec{x}) \quad \Rightarrow \quad P^2 = I,$$

and the permutation of two objects, denoted S_2. Writing the objects as column vectors

$$\begin{pmatrix}1\\2\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}$$

one sees that S_2 may be <u>represented</u> by matricies

$$e \sim \begin{pmatrix}1&0\\0&1\end{pmatrix}, \quad a \sim \begin{pmatrix}0&1\\1&0\end{pmatrix}$$

with "$\circ$" becoming matrix multiplication. By contrast, parity as expressed above is a realization. Evidently the elements of parity and S_2 can be put in unique correspondence: they are said to be isomorphic (were the mapping between them many to one they would be homomorphic).

A little bit of toying with the matrix representation of S_3 reveals that (i) the composition of elements is noncommutative (ii) there are three representations of S_2 contained and (iii) every one of them is 'reducible' in the sense that the matrices are block diagonal. In general a group is 'abelian' iff $g_i \circ g_j = g_j \circ g_i$ for all its elements,[+] else it is nonabelian. A group G may contain a subset of elements which satisfy the group axioms and hence constitute a subgroup G'. The elements in G but not in G' constitute the coset 'G/G''. If g_i is in G/G' then the set of elements $g_i \circ G' g_i^{-1}$ form a group isomorphic to G' called the conjugate subgroup. Should it happen that $g_i \circ G' g_i^{-1} = G'$ then this is an invariant subgroup. A group with no proper invariant (abelian) subgroups is (semi-) simple.

Mainly our interest lies with groups whose elements depend on one or more continuous parameters. If the dependance is analytic the continuous group is a Lie group. Familiar examples from mechanics are translations and rotations; the (former) latter is '(non-) compact' because the parameters take on an (in-) finite range of values. Symmetry enters the game because we can

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[+] Every group has two improper subgroups: e alone and the entire group.

eg <u>define</u> the groups $O(3)$ as the invariances of $X^2+Y^2+Z^2$. Assembling X, Y and Z in a colum vector $\underline{X}$, if $\underline{X}' = \underset{\sim}{Q}\underline{X}$ leaves $\underline{X}'^T\underline{X}' = \underline{X}^T\underline{X}$ then $\underset{\sim}{Q}$ is an orthogonal matrix. Omitting parity the group is $SO(3)$, $\underline{X}' = \underset{\sim}{R}\underline{X}$ with $\underset{\sim}{R}$ orthogonal and of unit determinant. The importance of Lie groups is that we can study infinitesimal transformations (neighborhood of the identity) and build finite ones from these. For $SO(3)$ write for infinitesimal θ^a $(a=1,2,3)$

$$\underset{\sim}{R} = \underset{\sim}{I} + i\underset{\sim}{T_a}\theta^a \qquad (1.1)$$

where the "i" is convention, the T_a s are 'generators' in the defining or 'fundamental' representation and we are using the summation convention.[†] Reality, orthogonality imply $\underset{\sim}{T_a}^* = \underset{\sim}{T_a}^T = -\underset{\sim}{T_a}$ so

$$\underset{\sim}{T_1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (1.2)$$

The finite transformation is just

$$\underset{\sim}{R} = \lim_{n\to\infty} \left(\underset{\sim}{I} + \frac{i}{n}\vec{\theta}\cdot\vec{\underset{\sim}{T}}\right)^n = \exp(i\vec{\underset{\sim}{T}}\cdot\vec{\theta}) \qquad (1.3)$$

(observe how easily we switch notation!).

A similar analysis can be done when the 'basis' $\underline{X}$ is complex, the group being $U(n)$ in n dimensions. Here we omit a factor $U(1)$ represented by

$$\mathrm{diag}\,(e^{i\theta_1}, e^{i\theta_2}, \ldots e^{i\theta_n})$$

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† ie repeated indicies are summed over implicitly unless otherwise stated

and so deal with $SU(n)$. For $n=2$ we know the answer from quantum mechanics

$$\underset{\sim}{t}_a = \frac{\sigma_a}{2} , \tag{1.5}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

being the usual Pauli matricies. In the case $n = 3$ $t_a = \lambda_a/2$, $a = 1, 2, \ldots, 8$ with the Gell-Mann matricies

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{1.7}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

For Lie groups closure is represented by the 'Lie algebra' $\mathcal{G}$

$$[t_a, t_b] = i C_{abc} t_c \tag{1.8}$$

The 'structure constants' C_{abc} themselves form a representation of the algebra

$$(\underset{\sim}{T}_a)_{ab} = -i C_{abc} \tag{1.9}$$

by virtue of the Jacobi identity, termed the 'adjoint representation'. The structure constants of $SU(2)$ are just ϵ_{abc} (the totally antisymmetric tensor with

$\epsilon_{123} = +1$) and the adjoint representation has already been written down in eq (1.2)

Much of what was written earlier carries over from G to $\mathcal{G}$; an abelian algebra has $C_{abc} = 0$, subgroups become subalgebras etc. One point of caution: algebras with the same C_{abc} are locally isomorphic but may be different globally (eg SO(3) and SU(2) are homomorphic as one recalls from quantum mechanics). The maximal abelian subalgebra of $\mathcal{G}$ is the 'Cartan subalgebra' and (more or less) the number of generators it contains is the rank of the group. Taking SU(3) as an example: by inspection of eq (1.7) t_1, t_2 and t_3 form an SU(2) subgroup while t_1, t_8 constitute the Cartan subalgebra so the rank is <u>2</u>. The importance of the Cartan subalgebra comes in the construction of higher dimensional irreducible representations, generalizing that for angular momentum in quantum mechanics. Albeit such representations are crucial to eg the quark model of hadrons we will only have occasion to need the fundamental and adjoint representations so we eschew this topic.

<u>References:</u>

A good introduction to group theory is the last

chapter of: J. T. Cushing, "Applied Analytical Mathematics for Physical Scientists", Wiley, New York, 1975.

For the construction of irreducible representations and applications see: H. Georgi, "Lie Algebras in Particle Physics", Benjamin-Cummins, New York, 1982.

Problems:

1.1. Write down a matrix representation of S_3 and verify the three statements made in the text.

1.2 Show that $SO(n)$ and $SU(n)$ have, respectively $n(n-1)/2$ and n^2-1 generators

1.3 Writing $\underset{\sim}{X} = \vec{\sigma}\cdot\vec{x}$ show that $\underset{\sim}{X}' = \vec{\sigma}\cdot\vec{x}' = \underset{\sim}{U}^{+}\underset{\sim}{X}\,\underset{\sim}{U}$, U an element of $SU(2)$ yields $\underset{\sim}{X}' = \underset{\sim}{R}\,\underset{\sim}{X}$ with $\underset{\sim}{R} \in SO(3)$, and the mapping $SU(2) \to SO(3)$ is 2 to 1 (homomorphism). (Note that the transformation rule for $\underset{\sim}{X}$ signifies that $\vec{x}$ belongs to the adjoint representation of $SU(2)$.)