Honours Project Report: Cosmological k-essence in a Shear-Free Spherical Universe

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Abstract

We investigate the possibilities of the unification of the dark regime of the universe in a shear-free spherical model. We develop a Hamiltonian formulation for K-essence and attempt to map out the space-time numerically for the specific case of tachyon-type models. We also derive a second order partial differential equation for the scale factor which is solved and the solution used to determine the Hubble parameter and the lapse function.

Introduction

Primordial nucleosynsthesis constrains the energy density of baryons to make up only a small fraction of the universe. This implies that we only really have an theoretical understanding of a tiny portion of the universe. The missing energy density is thought to constitute an exotic form of matter (dark matter) and the yet unexplained dark energy. The existence of dark matter has long been accepted by the cosmological world, with much compelling evidence such as the rotational speed of galaxies, orbital velocities of galaxies in clusters and gravitational lensing to support it. The existence of dark energy however, was only proposed much more recently when observations of Type 1a Supernovae revealed, quite startlingly, that the Hubble expansion was accelerating. Although other theories, such as the existence of large voids in the universe have been proposed to explain these observations, dark energy is one of the forerunners to provide a solution. The current best-fit values for the fractions of closure density are $\Omega_B = 0.04$, $\Omega_{DM} = 0.22$ and $\Omega_{DE} = 0.74$.

The unification of the dark sector in the universe remains one of the foremost problems facing modern cosmology. The simplest such model merely combines baryons with conventional cold dark matter (CDM) and a cosmological constant Λ to provide the dark energy. This ΛCDM model, however neglects to provide a satisfactory explanation of why Λ is non-zero, but such that DM and DE are comparable today. This problem can be somewhat removed by replacing Λ with an evolving scalar field. Models of this type are given the name quintessence-CDM but like the ΛCDM model assumes that DM and DE are distinct entities. Another interpretation of these results is that DM and DE are different manifestations of a common structure. The first models of this type arose in the last few years and are based around the Chaplygin gas, a perfect fluid which obeys the equation of state

$$P = -\frac{A}{\rho}, \tag{1}$$

This paper will attempt to provide an insight into a particular such model called k-essence, a scalar field with noncanonical kinetic terms. This model was first introduced in an attempt to describe early universe inflation. The cosmological potential of equation (1) was first given by Kamenshchick et al [1], who observed that

$$\rho(R) = \sqrt{A + \frac{B}{a^6}}, \tag{2}$$

where R is the scale factor and B is a constant of integration. Thus we see that this potential interpolates between $\rho \sim \sqrt{B}a^{-3}$, $P \sim 0$ for small values of R, and a cosmological constant $\rho \sim \sqrt{A} \sim -P$ for large values of R. This gives the picture of an inhomogeneous universe with highly overdense regions such as galaxies and clusters providing dark matter and underdense voids which drive the expansion of the universe and pushes ρ to its limiting value $\sqrt{A}$ giving dark energy. It is interesting to also note that the Chaplygin gas has an equivalent scalar field formulation with a Lagrangian of the form [2],

$$\mathcal{L} = -\sqrt{A}\sqrt{1 - g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}}. \tag{3}$$

The first important feature of equation (3) is that it exhibits the same interpolating properties of equation (2). Secondly, tachyon models of this type appear in string theory which points to a connection between the dark regime and fundamental physics. Thirdly, it is apparent that to achieve dark matter and dark energy in the same epoch, thus agreeing with observation of the universe today, they must exist in distinct regions. For this paper, we turn to the shear-free spherical model and show that when combined with a Hamiltonian formulation of k-essence we end up with a simple set of equations to describe k-essence cosmology.

The Shear-Free Spherical Model

We assume that the universe can be modeled as a perfect fluid, in comoving coordinates, exhibiting spherical symmetry. Now on making the assumption of a vanishing shear and using the line element

$$ds^2 = N(r,t)^2 dt^2 - R(r,t)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2\theta \; d\phi^2)\right], \tag{4}$$

where $u^\mu = N^{-1}\delta^\mu_t$ is the fluid's 4-velocity, the Einstein Field equations in conjunction with the matter tensor taking the form of that of a perfect fluid we obtain the conservation equation

$$\dot{\rho} + 3\frac{\dot{R}}{R}(\rho + P) = 0, \tag{5}$$

and the Euler Equation

$$\frac{N_{,r}}{N} + \frac{P_{,r}}{\rho + P} = 0. \tag{6}$$

We continue with an analysis analogous to that of Kustaanheimo & Qvist [3] and Wyman [4]. Using the Einstein Field Equations in the form given in Tolman [5] we obtain a number of useful results. The G^t_r component vanishes leaving, on integration over r,

$$\frac{\dot{R}}{R} \equiv \frac{1}{N}\frac{R_{,t}}{R} = H(t), \tag{7}$$

where $H(t)$ is the Hubble parameter and is a function of integration of t only. Thus for the G^t_t equation, we get

$$3H^2 - \frac{1}{R^2}\left[2\frac{R_{,rr}}{R} - (\frac{R_{,r}}{R})^2 + \frac{4}{r}\frac{R_{,r}}{R}\right] = \rho. \tag{8}$$

We can the eliminate N from the G^r_r and the $G^\theta_\theta = G^\phi_\phi$ equations using

$$N_{,r} = (H^{-1}\frac{R_{,t}}{R})_{,r} = H^{-1}(\frac{R_{,r}}{R})_{,t}, \tag{9}$$

giving, for the $r - r$ equation

$$3H^2 + 2\dot{H} - \frac{1}{R^3 H}\left[\frac{(R_{,r})^2}{R} + \frac{2}{r}R_{,r}\right]^{\cdot} = -P \tag{10}$$

and for the $\theta - \theta$ equation,

$$3H^2 + 2\dot{H} - \frac{1}{R^3 H}\left[R_{,rr} - \frac{(R_{,r})^2}{R} + \frac{1}{r}R_{,r}\right]^{\cdot} = -P. \quad (11)$$

respectively, where we have used units $8\pi G = c = 1$. On comparing the previous two equations and taking the integral with respect to t we obtain the equation of pressure isotropy

$$\frac{1}{2}b(r) = R_{,rr} - 2\frac{(R_{,r})^2}{R} - \frac{1}{r}R_{,r}, \quad (12)$$

where $b(r)$ is a function of integration of r only. Using equation (12) to eleminate $R_{,rr}$ for the G^t_t equation we obtain an interesting form for a Friedmann-like equation which acts as a constraint for the Hubble parameter.

$$H^2 = \frac{1}{3}\left[\rho + \frac{b}{R^3}\right] + \frac{1}{R^2}\left[(\frac{R_{,r}}{R})^2 + \frac{2}{r}\frac{R_{,r}}{R}\right]. \quad (13)$$

Now, taking the r-derivative of the above equation and using equation (12) to once again eliminate $R_{,rr}$, we are left with

$$\rho_{,r} + \frac{1}{R^3}(b_{,r} + \frac{3b}{r}) = 0. \quad (14)$$

Thus we see that for a specified energy density ρ and $b(r)$ and using equation (14), it is possible to eliminate the $R_{,r}$ dependence of equation (13) which thus simply reduces to a purely algebraically function which should be easily solved should it be well-behaved. This will be shown later for the particular case of a tachyon-type model with $\mathcal{H} = \sqrt{p^2 + R^6 V^2(\varphi)}$, later in the paper. In order to determine the t-dependence of the scale factor to obtain $R = R(r,t)$ we need to develop a Hamiltonian formulation for k-essence.

A Hamiltonian Formulation of K-essence

We begin by considering a minimally coupled k-essence model [6] in which the action is described by

$$S = \int d^4x \, \sqrt{-g}\left[-\frac{R}{16\pi G} + \mathcal{L}(\varphi, Y)\right], \quad (15)$$

where $\mathcal{L}$ is the most general Lagrangian involving the field φ and its first derivatives and is shown in equation (3). Thus we have $\mathcal{L} = \mathcal{L}(\varphi, Y)$, where $Y = g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}$. The energy momentum tensor obtained from this action, equation(15 takes the form of a perfect fluid with $\mathcal{L}_Y$ defined as $\partial\mathcal{L}/\partial Y$,

$$\begin{aligned} T_{\mu\nu} &= 2\mathcal{L}_Y \varphi_{,\mu}\varphi_{,\nu} - \mathcal{L}g_{\mu\nu}, \\ &= (\rho + P)U_\mu U_\nu - P g_{\mu\nu}. \end{aligned}$$

We can thus identify the fluids 4-velocity, energy density and pressure

$$U_\mu = \frac{\varphi_{,\mu}}{Y}, \qquad P = \mathcal{L}, \qquad \rho = Y\frac{\partial\mathcal{L}}{\partial Y} - \mathcal{L}.$$

We also note that in a comoving coordinate system we have $\varphi = \varphi(t)$, a function of t only and $Y = \dot{\varphi}$. The above identification for ρ is reminiscent of a Legendre transform. However, the equation for the φ-field in comoving coordinates

$$\frac{1}{R^3}(R^3\frac{\partial\mathcal{L}}{\partial Y})^{\cdot} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

shows the correct definition to be

$$L(\varphi,\dot{\varphi},R) = R^3\mathcal{L},$$

which takes the role of the Lagrangian. Thus we see that the conjugate momentum, p to φ is in fact

$$p = R^3\frac{\partial\mathcal{L}}{\partial Y}.$$

Finally we see that

$$R^3\rho = \mathcal{H}(\varphi,p,R) = pY - R^3\mathcal{L}, \tag{16}$$

which is the Hamiltonian for k-essence. Hamilton's equations then follow from this and are given by

$$\dot{\varphi} = \frac{\partial\mathcal{H}}{\partial p} \qquad \text{and} \qquad \dot{p} = -\frac{\partial\mathcal{H}}{\partial\varphi}. \tag{17}$$

From here it is convenient to choose φ as the the time variable and from our definition for $Y = \dot{\varphi}$, we obtain

$$Y = \frac{\partial\mathcal{H}}{\partial p} \qquad \text{and} \qquad Yp_{,\varphi} = -\frac{\partial\mathcal{H}}{\partial\varphi}. \tag{18}$$

Now, using equation (7) in the form $H = Y\frac{R_{,\varphi}}{R}$ in conjunction with equations (18) it is possible to step forward incrementally in time and for each step calculate the value for H from the simplified version of equation (13) for specified $\mathcal{H}(\varphi,\dot{\varphi},R)$, $b(r)$ and $p(r,\varphi)$. In this way it should be possible for the entire space-time to be mapped out.

The Tachyon-type Model

We now return to the tachyon type model as mentioned previously. Using the relation given in equation (16) we see that the corresponding energy density for this Hamiltonian is

$$\rho = \sqrt{\frac{p^2}{R^6} + V^2(\varphi)}. \tag{19}$$

Taking the r-derivative of this gives

$$\rho_{,r} = \frac{1}{2\rho}\left[\frac{2pp_{,r}}{R^6} - \frac{6p^2R_{,r}}{R^7}\right],$$

and using this in equation (14) yields

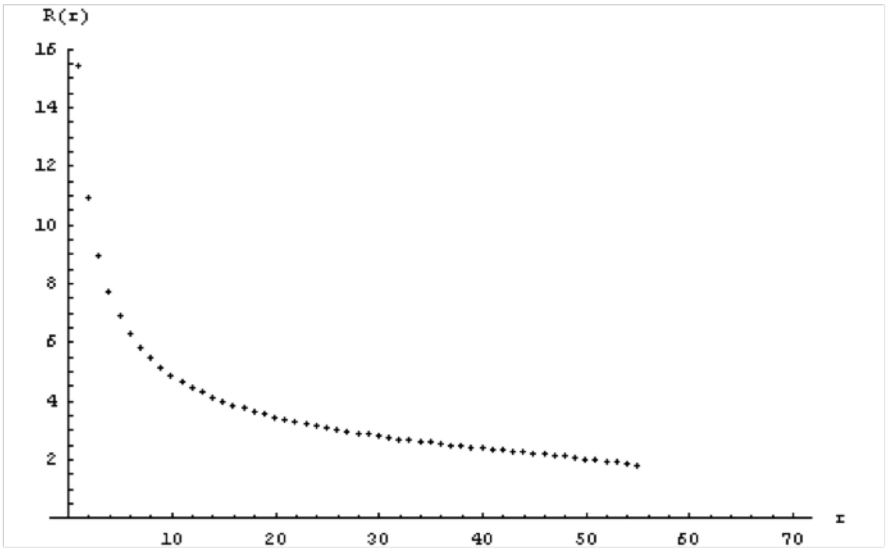

Figure 1: r-dependance for the scale factor R for a constant time slice: $b(r) = r^2$.

$$\frac{1}{2\rho}\left[\frac{2pp_{,r}}{R^6} - \frac{6p^2 R_{,r}}{R^7}\right] + \frac{1}{R^3}(b_{,r} + \frac{3b}{r}) = 0.$$

This can then be solved for $R_{,r}$ on a constant φ-slice giving

$$R_{,r} = \frac{Rp_{,r}}{3p} + \frac{\rho R^4}{3p^2}(b_{,r} + \frac{3b}{r}). \tag{20}$$

We can then substitute this back into equation (13) to give

$$H^2 = \frac{1}{3}\left[\rho + \frac{b}{R^3}\right] + \frac{1}{R^2}\left[(\frac{p_{,r}}{3p} + \frac{\rho R^3}{3p^2}(b_{,r} + \frac{3b}{r}))^2 + \frac{2}{r}(\frac{p_{,r}}{3p} + \frac{\rho R^3}{3p^2}(b_{,r} + \frac{3b}{r}))\right]. \tag{21}$$

This equation was then solved numerically in order to determine the r dependance of the scale factor R on a constant φ-slice using the particular case of the tachyon model where the momentum, p is independent of the radial component r and for $V(\varphi) = V_0$, a constant. This energy density is thus analogous to that of the Chaplygin gas and with the choice of $b(r) = r^2$ equation (21) is reduced to

$$H^2 = \frac{1}{3}\left[\rho + \frac{r^2}{R^3}\right] + \frac{1}{R^2}\left[(\frac{5}{3}\frac{\rho R^3}{p^2}r)^2 + \frac{10}{3}\frac{\rho R^3}{p^2}\right]. \tag{22}$$

The result of this numerical analysis as shown in Figure 1 for $R(r)$ versus r. This result fails to fall in line with that of which was expected. The graph has a singularity and the scale factor takes on an unphysical negative value at $r = 57$. On returning to equation (20) we notice, that with the same choices for p and $b(r)$ and in the limiting case for very large R such that $\rho \to \rho_\infty = constant$ that

$$R_{,r} = \frac{\rho_\infty R^4}{3p^2}(5r). \tag{23}$$

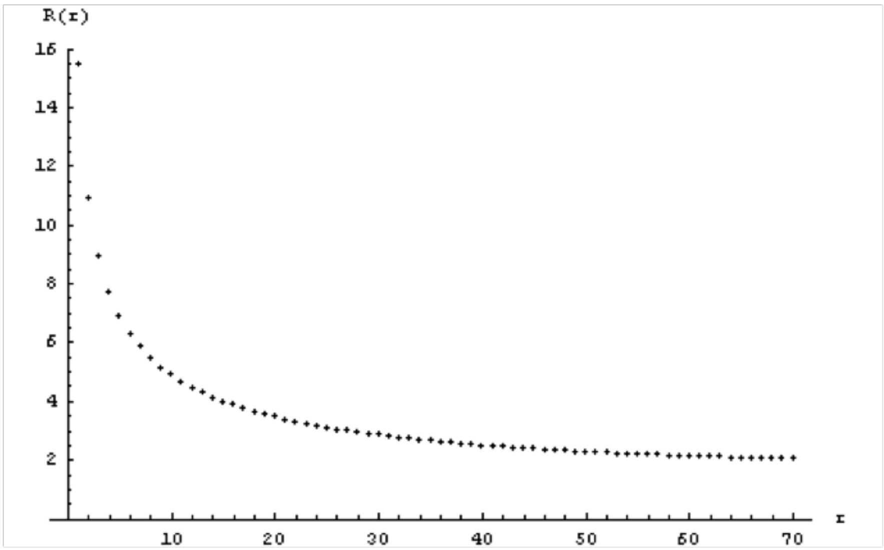

Figure 2: r-dependance for the scale factor R for a constant time slice $b(r) = -r^2$.

It is clear that $R_{,r}$ must be positive as it depends only on quantities which are themselves positive and also increasing as r increases which is clearly in contradiction with the plotted graph. We run into even more problems when attempting to solve equation (23) for $R(r)$. A simple separation of variables results in

$$R(r) = \left[-\frac{2p}{5\rho_\infty r^2}\right]^{\frac{1}{3}}. \tag{24}$$

which implies a scale factor R that is negative and which once again tends to zero for large r. The ambiguity created by equations (23) and (24) can be somewhat ameliorated by changing the definition of $b(r)$ to $b(r) = -r^2$. The numerical analysis performed with this new definition for $b(r)$ is given in Figure 2. This transforms equation (23) to

$$R_{,r} = -\frac{\rho_\infty R^4}{3p^2}(5r), \tag{25}$$

and equation (24) to

$$R(r) = \left[\frac{2p}{5\rho_\infty r^2}\right]^{\frac{1}{3}}, \tag{26}$$

both of which agree with the numerical analysis performed on equation (21) with the new definition for $b(r)$. These results agree for a large part with those obtained in the following section where we determine an expression for the scale factor algebraically and are able to plot it as a function of both the t and r coordinates. In theory one should now be able to time step forward using equations (7) and (18) and thus plot the scale factor for both the r and t-dependance. This was attempted for this project but no satisfactory results were obtained and should perhaps be left to an expert in the field of computational physics.

Analytic Determination of the Scale Factor

In order to take this analysis further we now turn to an alternative method in which we are able to determine the scale factor $R(r,t)$ from which we can determine the an expression for the equation of pressure isotropy and thus obtain the lapse function and Hubble parameter. Returning to the Einstein Field Equations, most specifically the G_r^r and $G_\theta^\theta = G_\phi^\phi$ equations we note that on equating equations (10) and (11) we see that

$$\left[\frac{(R_{,r})^2}{R}\right]^{\cdot} + \left[\frac{1}{r}R_{,r}\right]^{\cdot} = \left[\left[\frac{R_{,rr}}{R} - (\frac{R_{,r}}{R})^2\right]R\right]^{\cdot}. \tag{27}$$

Rearranging the terms we get

$$\left[R_{,rr} - 2\frac{(R_{,r})^2}{R} - \frac{1}{r}R_{,r}\right]^{\cdot} = 0.$$

Dividing through by r^2 yields

$$\left[\frac{R_{,r}}{r^3} + \frac{2}{r^2}\frac{(R_{,r})^2}{R} - \frac{R_{,rr}}{r^2}\right]^{\cdot} = 0,$$

from which it follows that

$$\begin{aligned} 0 &= \left[\frac{R_{,r}}{r^3} - \frac{R^2}{r^2}(\frac{R_{,r}}{R^2})_{,r}\right]^{\cdot}, \\ 0 &= \left[-R^2\frac{1}{r}\left[\frac{1}{r}\frac{R_{,r}}{R^2}\right]_{,r}\right]^{\cdot}, \\ 0 &= \left[R^2\frac{1}{r}\frac{\partial}{\partial r}\left[\frac{1}{r}(\frac{\partial R}{\partial r})\frac{1}{R^2}\right]\right]^{\cdot}. \end{aligned} \tag{28}$$

Now on making the substitution $x = \frac{r^2}{2}$, we get

$$\left[R^2\frac{\partial^2}{\partial^2 x}\frac{1}{R}\right]^{\cdot} = 0.$$

This can then be integrated to give

$$R^2\frac{\partial^2}{\partial^2 x}\frac{1}{R} = C(x), \tag{29}$$

where $C(x)$ is a function of integration of x only. We now consider the case where $C(x)$ is just a constant ($C(x) = C\alpha^2$). In the analysis of Kustaanheimo and Qvist, it is shown that for $C(x) = (ax^2 + bx + c)^{-\frac{5}{2}}$ it is possible to obtain solutions which are expressible in terms of elementary functions. Since we are considering in the x-derivative of R, we may consider equation (40) simply as an ordinary differential equation

$$R^2\frac{d^2}{d^2x}\frac{1}{R} = C(x). \tag{30}$$

This can then be integrated by making a substitution $Z = \frac{1}{R}$ leaving

$$\frac{d^2}{dx^2}Z = C\alpha^2 Z^2. \tag{31}$$

By making a further substitution, $v = \frac{dZ}{dx}$ and using the chain rule we obtain

$$v\frac{dv}{dz} = C\alpha^2 Z^2,$$

which can be trivially integrated to give

$$v = \frac{dz}{dx} = \sqrt{\frac{2}{3}C\alpha^2 Z^3 + \sigma_1(t)}, \tag{32}$$

where $\sigma_1(t)$ is a function of integration of t only. Thus we have

$$\frac{dx}{dz} = \left[\frac{2}{3}C\alpha^2 Z^3 + \sigma_1(t)\right]^{-\frac{1}{2}},$$

which can be integrated to give

$$x - x_0(t) = \int \left[\frac{2}{3}C\alpha^2 Z^3 + \sigma_1(t)\right]^{-\frac{1}{2}} dZ. \tag{33}$$

This equation has a solution in terms of the Weierstrass elliptic function as we shall show, without loss of generality, for the particular case where $\sigma_1(t) = 0$. Equation (32) thus reduces to

$$\frac{dz}{dx} = \alpha\sqrt{\frac{2}{3}C}Z^{\frac{3}{2}}. \tag{34}$$

This can now be rearranged and integrated to give

$$-2Z^{-\frac{1}{2}} = \alpha\sqrt{\frac{2}{3}C}x + \sigma_2(t).$$

Thus we have that

$$Z = \frac{1}{R} = \frac{1}{\left[\alpha\sqrt{\frac{1}{24}C}r^2 + \sigma(t)\right]^2}. \tag{35}$$

We now return to the G^t_t equation, equation (8) and rewrite it in terms of Z and its derivatives.

$$\begin{aligned}\rho &= 3H^2 - Z^2\left[2Z(2AZ^{-\frac{1}{2}} + 2A^2r^2) - Z^2(4A^2r^2Z^{-1}) + 4Z(2AZ^{-\frac{1}{2}})\right],\\ &= 3H^2 - Z^2\left[4AZ^{\frac{1}{2}} + 4A^2r^2Z - 4A^2r^2Z + 8AZ^{\frac{1}{2}}\right],\\ &= 3H^2 - 12AZ^{\frac{5}{2}},\end{aligned} \tag{36}$$

where we have set $A = \sqrt{\frac{C}{24}}\alpha$. In order to determine an explicit expression for the Hubble parameter $H(t)$ we return to equation (7). We note that

$$\dot{R} = \frac{1}{N}R_{,t}. \tag{37}$$

Using the results obtained for $R(r,t)$ in equation (35) in the above equation we see that

$$H(t) = \frac{\dot{R}}{R} = \frac{1}{N}\frac{2\sigma_{,t}\left[Ar^2 + \sigma(t)\right]}{\left[Ar^2 + \sigma(t)\right]^2}. \quad (38)$$

Now we see that in order for $H(t)$ to depend only on t we must have that $N = f(t)\left[Ar^2 + \sigma(t)\right]^{-1}$. We now consider the most simple case, when $f(t) = \sigma_{,t}$ and thus $H(t)$ is a constant in time. Taking the t-derivative of equation (36) we get

$$\begin{aligned}\dot{\rho} &= -30AZ^{\frac{3}{2}}\dot{Z},\\ &= 30AZ^{\frac{5}{2}}H,\\ &= -\frac{30}{12}\rho H + \frac{30}{4}H^3. \quad (39)\end{aligned}$$

Using equation (5), the conservation equation, we have

$$\begin{aligned}P &= -\rho - \frac{\dot{\rho}}{3H},\\ &= -\rho + \frac{10}{12}\rho - \frac{30}{12}H^2,\\ &= -\frac{1}{6}\rho - \frac{30}{12}H^2, \quad (40)\end{aligned}$$

which can be identified as the equation of state for the system and the adiabatic speed of sound is thus found to be

$$c_s^2 \equiv \frac{\partial P}{\partial \rho} = -\frac{1}{6}. \quad (41)$$

Now, on making the choice $\sigma(t) = t^2$ in order to obtain a scale factor and lapse function similar to that obtained by Wyman we see that $f(t)$ is a constant and

$$Z = \frac{1}{R} = \frac{1}{\left[A(t^2 - r^2)\right]^2}. \quad (42)$$

This is plotted in Figure 3 for $R = R(r,t)$. We note that we only need consider that part of the graph to the left of the line $r = t$ if we look only within our past null cone. Using this we can calculate the lapse function and the Hubble parameter

$$N = R^{-\frac{1}{2}} = \frac{1}{\left[A(t^2 - r^2)\right]}, \quad (43)$$

$$H = 4At. \quad (44)$$

We now return to equation (12) and substitute the relevant terms for the scale factor and its derivatives calculated from equation (35). Thus we calculate

$$\begin{aligned}b(r) &= 24A^2r^2 - 8A^2t^2 - 64A^2r^2 + 8A^2t^2 - 8A^2r^2,\\ &= -48A^2r^2. \quad (45)\end{aligned}$$

Thus we have the equation of pressure isotropy to be dependent only on r as required and it is negative for all real values of the constant A. We can now write the Einstein Field Equations in terms of $b(r)$ for the G^t_t and G^r_r components giving

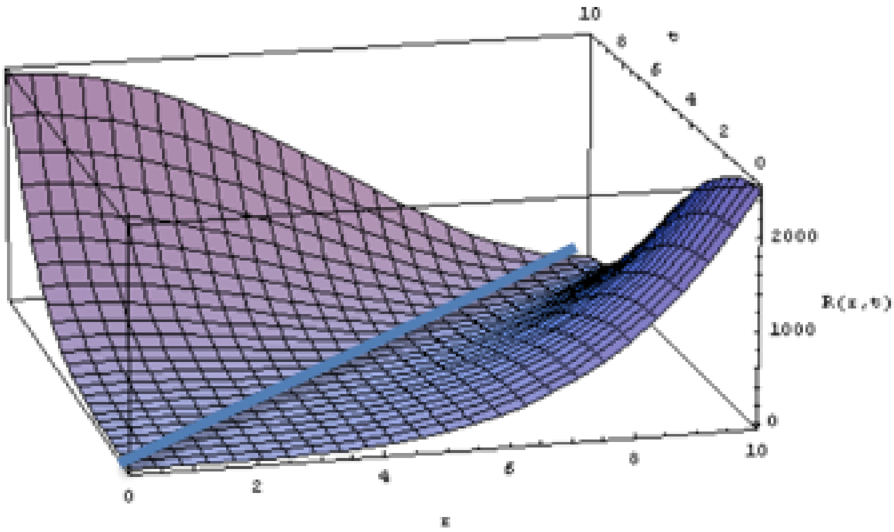

Figure 3: r and t-dependance of the scale factor, $R(r,t)$.

$$
\begin{aligned}
\rho &= 3H^2 - \frac{1}{R^2}\left[\frac{b}{R} + 3(\frac{R_{,r}}{R})^2 + \frac{6}{r}\frac{R_{,r}}{R}\right], \\
&= 3H^2 - 12AR^{-\frac{5}{2}},
\end{aligned}
$$

for the G^t_t equation and using equation (7),

$$
\begin{aligned}
P &= 3H^2 + 2\dot{H} - \frac{1}{R^3 H N}\frac{2}{r}(R_{,r})_{,t}, \\
&= 3H^2 + 2\dot{H} + 4AR^{-\frac{5}{2}}.
\end{aligned}
$$

A comparison of equations (46) and (46) we see that

$$
P = \rho + 16AR^{-\frac{5}{2}} + 2\dot{H}. \tag{46}
$$

We can now use these to 'look' along the past null cone, given by $tG^t_t + rG^r_r = 0$. For the Einstein Field Equations above we obtain

$$
\begin{aligned}
0 &= \rho t + Pr, \\
&= t\rho - \frac{1}{3}r\rho + 4H^2 + 2\dot{H}, \\
&= (t + \frac{r}{3})\rho + \frac{16A^2t^2}{RN^2} + \frac{8Ar}{N^2}\left[\frac{At^2}{R^{\frac{1}{2}}}(1 - t\frac{N_{,t}}{N})\right].
\end{aligned} \tag{47}
$$

where we used equation (7) to eliminate H and $\dot{H}$. We can now use equation (47) to determine the lapse function and hence the Hubble parameter.

Looking Forward

We now attempt to to determine constraint and evolution equations for the Hubble parameter. Returning to the Einstein Field Equations, equations (8) and (10) and using the equation of pressure isotropy in the $t-t$ equation we note that we can eliminate the H^2 terms by subtracting the two field equations leaving

$$\begin{aligned} -(\rho+P)-\frac{b(r)}{R^3} &= 2\dot{H}-\frac{1}{R^3H}\left[\frac{(R_{,r})^2}{R}+\frac{2}{r}R_{,r}\right]^{\cdot}+\frac{3}{R^3}\left[\frac{(R_{,r})^2}{R}+\frac{2}{r}R_{,r}\right], \\ &= 2\dot{H}-\frac{1}{H}\left[\frac{1}{R^3}\{\frac{(R_{,r})^2}{R}+\frac{2}{r}R_{,r}\}\right]^{\cdot} \\ &\quad +\frac{1}{H}(\frac{1}{R^3})^{\cdot}\left[\frac{(R_{,r})^2}{R}+\frac{2}{r}R_{,r}\right] \\ &\quad +\frac{3}{R^3}\left[\frac{(R_{,r})^2}{R}+\frac{2}{r}R_{,r}\right] \end{aligned} \tag{48}$$

This can then be simplified by making use of equation (7), leaving

$$-(\rho+P)-\frac{b(r)}{R^3}=2\dot{H}-\frac{1}{H}\left[(\frac{R_{,r}}{R^2})^2+\frac{2}{rR}(\frac{R_{,r}}{R^2})\right]^{\cdot}.$$

Now, making use of the substitution $Z=\frac{1}{R}$, as in the previous section, we get

$$3H^2-3\left[Z_{,r}^2-\frac{2}{r}ZZ_{,r}\right]=\rho+Z^3b(r), \tag{49}$$

for the G^t_t equation, and

$$2\dot{H}-\frac{1}{H}\left[Z_{,r}^2-\frac{2}{r}ZZ_{,r}\right]^{\cdot}=-(\rho+P)-Z^3b(r) \tag{50}$$

for equation (48). These two equations may be regarded as a constraint and an evolution equation for the Hubble parameter respectively. We can now obtain and expression for $Z_{,r}$ and thus eliminate it from the G^t_t equation and equation (50) and the equation of pressure isotropy becomes

$$\frac{b(r)}{2}=\frac{1}{r}Z_{,r}-Z^2Z_{rr}. \tag{51}$$

Dividing this through by $4r^2$ and choosing $b(r)=\frac{4\alpha}{5}r^2$, we get

$$\frac{1}{4r^2}Z_{,rr}-\frac{1}{4r^3}Z_{,r}+\frac{\alpha Z^2}{10}=0.$$

Now, substituting $x=r^2$ we get

$$\frac{\partial^2 Z}{\partial x^2}+\frac{\alpha}{10}Z^2=0,$$

and multiplying by $\frac{\partial Z}{\partial x}$ gives

$$\frac{\partial}{\partial x}\left[\frac{1}{2}(\frac{\partial Z}{\partial x})^2+\frac{\alpha}{30}Z^3\right]=0,$$

which on integrating with respect to x gives

$$\frac{1}{2}(\frac{\partial Z}{\partial x})^2 + \frac{\alpha}{30}Z^3 = C(t). \tag{52}$$

We can now use this to obtain and expression for the first derivative of Z

$$Z_{,r} = 2r\sqrt{C(t) - \frac{\alpha}{15}Z^3}. \tag{53}$$

Now using the same choice for $b(r)$ in equation (14) and substituting $Z = \frac{1}{R}$, we get

$$\rho_{,r} + Z^3 4\alpha r = 0, \tag{54}$$

which, on making the substitution $x = r^2$ becomes

$$\frac{\partial \rho}{\partial x} + 2\alpha Z^3 = 0. \tag{55}$$

We are now in a position to obtain expressions for the constraint and evolution equations of the Hubble parameter in terms of Z only and none of its derivatives. We should be able to map the scale factor and Hubble parameter for the entire space-time. Using equation (53) in equation (49) and taking $C(t) = C$ =constant, we get

$$3H^2 - 12\left[r^2(C - \frac{\alpha}{15}Z^3) - Z\sqrt{C - \frac{\alpha}{15}Z^3}\,\right] = \rho + Z^3 b(r),$$

which on substituting for $b(r)$ and rearranging terms becomes

$$3H^2 - 12Cr^2 - Z\sqrt{C - \frac{\alpha}{15}Z^3} = \rho.$$

Thus we have

$$H^2 = \frac{\rho}{3} + 4Cr^2 + \frac{Z}{3}\sqrt{C - \frac{\alpha}{15}Z^3}, \tag{56}$$

for the constraint on the Hubble parameter which can also be given in terms of the scale factor R,

$$H^2 = \frac{\rho}{3} + 4Cr^2 + \frac{1}{3R}\sqrt{C - \frac{\alpha}{15}\frac{1}{R^3}}. \tag{57}$$

Similarly, the evolution equation becomes

$$2\dot{H} - \frac{1}{H}\left[4r^2(C - \frac{\alpha}{15}Z^3) - \frac{2}{r}Z2r\sqrt{C - \frac{\alpha}{15}Z^3}\,\right]^{\cdot} = -(\rho + P) - Z^3\frac{4\alpha}{5}r^2,$$

which in terms of R is just

$$2\dot{H} - \frac{1}{H}\left[4r^2(C - \frac{\alpha}{15}\frac{1}{R^3}) - \frac{4}{R}\sqrt{C - \frac{\alpha}{15}\frac{1}{R^3}}\,\right]^{\cdot} = -(\rho + P) - \frac{1}{R^3}\frac{4\alpha}{5}r^2.$$

Applying the time-derivative where applicable this becomes

$$\begin{aligned} & 2\dot{H} - \frac{1}{H}\{\frac{4\alpha r^2}{5}\frac{\dot{R}}{R^4} + \frac{4\dot{R}}{R^2}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{\frac{1}{2}} - \frac{4\alpha}{10}\frac{\dot{R}}{R^5}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{-\frac{1}{2}}\}, \\ = \quad & 2\dot{H} - \frac{1}{H}\{\frac{4\alpha r^2}{5}\frac{H}{R^3} + \frac{4H}{R}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{\frac{1}{2}} - \frac{4\alpha}{10}\frac{H}{R^4}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{-\frac{1}{2}}\}, \end{aligned}$$

where we have used equation (7) to remove the $\dot{R}$ dependance. Thus for the evolution equation we obtain

$$\begin{aligned} 2\dot{H} \quad & - \quad \frac{4\alpha r^2}{5}\frac{1}{R^3} + \frac{4}{R}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{\frac{1}{2}} - \frac{4\alpha}{10}\frac{1}{R^4}\left[C - \frac{\alpha}{15}\frac{1}{R^3}\right]^{-\frac{1}{2}} \\ & = \quad -(\rho + P) - \frac{1}{R^3}\frac{4\alpha r^2}{5}. \end{aligned} \tag{58}$$

In principle we should now be able to specify a particular model, such as a tachyon type model and use the energy density in equation (57) for a constant time slice from which we can obtain the r-dependance of R for constant time.

Now returning the the Hamiltonian formulation for k-essence, in particular equation (16) from which we derived Hamilton's equations, equations (17) and (18), we note that we can use the solution for equation (57) in conjunction with equation (58) to determine the evolution of the Hubble parameter in time. In this way we should be able to map out the entire space-time.

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