A branelike approach to conformal Brans-Dicke gravity

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UCT

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Abstract

The braneworld description of the universe considers our 4-dimensional spacetime as a membrane embedded in a higher dimensional warped *bulk*. Matter is confined to the brane, however the propagation of the gravitational field in more than three dimensions changes the gravitational effects on the brane from those dictated by Einstein gravity. I shall introduce briefly the simple Randall-Sundrum brane model setup, and discuss a simple metric solution. Following this, I will discuss the motivations for Brans-Dicke theory which will provide a starting point for Chapter 2, in which this theory (with a conformal scalar field) will be further analysed. After this brief primer on Brans-Dicke theory, I shall allude to examples in the literature on black hole metric solutions - the Beckenstein solution via the conformal Einstein scalar equations, and the branelike approach solution by McFaddon and Turok. I shall then draw attention to a link between the Randall-Sundrum 2-brane setup and Brans-Dicke gravity by the work of Takeshi Chiba.

In the second section, I start with a four-dimensional Brans-Dicke action with a scalar field ansatz. After proving the conformal invariance of the theory, I will find the field equations for this theory. Completing this, a Schwarzschild-like solution for the metric and conformal scalar field will be found. This solution has geometry equivalent that of an extremal Reissner Nordstrom spacetime.

The third section attempts to test the theory (and compare it to conventional Einstein gravity) by considering the trajectories of null geodesics. Starting from the geodesic equation, the equations of motion for a photon in a static isotropic black hole spacetime will be found. Using this, the expression for the solution to the unbound orbit scenario will be derived. This leads directly to the deflection angle integral. The integral is in general elliptical and cannot be solved explicitly. Thus, we follow the treatment prescribed by Bozza in finding strong field limit solutions both for the Schwarzschild metric and the previously derived conformal Brans-Dicke metric. These solutions will be used to check the accuracy of the numerical evaluations of the deflection angle. Plots of the deflection angles and some examples of photon trajectories in high proximity to black holes will be presented. I will then discuss the lensing applications in both spacetimes, and attempt to put a lower bound on the telescope resolution required to observe the difference between the two spacetimes.

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Chapter 1

Initial motivations

1.1 Randall-Sundrum branes

The Randall-Sundrum brane model is arguably the barest brane model type, with matterless branes. The brane itself however posessses a tension which induces a gravitational field. We denote this energy density per unit volume or tension as σ(1.0). The action for this model contains 5-dimensional Einstein-Hilbert terms (including a cosmological constant term), in addition, there is a four dimensional term which effects a gravity due to the brane tension. The action [1] is

$$S_{RS} = -\frac{1}{16\pi G_{(5)}} \int d^4x dz \sqrt{-g^{(5)}} R^{(5)} - \Lambda \int d^4x dz \sqrt{-g^{(5)}} - \sigma \int d^4x \sqrt{-g^{(4)}} \tag{1.1}$$

Since we expect no curvature on the brane (since there is no matter on the brane), we require a condition on the cosmological constent to ensure four dimensional flatness. This enforces a condition on the cosmological constant, requiring a negative cosmological constant [3], related to the tension via

$$\Lambda = -\frac{4\pi}{3} G_{(5)} \sigma^2 \tag{1.2}$$

This condition coresponds to a four dimensinally flat solution with form

$$ds^2 = a^2(z)\, \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \tag{1.3}$$

where a is the *warp factor* which effectively changes the four dimensional curvature as a function of distance in the z direction. In order to ensure pure minkowskian curvature on the brane, we 'place' the brane at a certain z_{br} such that the warp factor is unity at $z = z_{br}$. This warp factor has the form

$$a(z) = e^{-k|z|} \tag{1.4}$$

with

$$k = \frac{4\pi}{3} G_{(5)} \sigma \tag{1.5}$$

The brane is therefore located at $z = 0$.To ensure (1.3) is an allowed solution in this theory, and to prove that the warp factor is in fact given by (1.4), we substitute (1.3) into the Einstein Field Equations (which must hold since the action to this theory contains an Einstein-Hilbert consideration). The Einstein tensor is

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 3\left(a'^2 + aa''\right) \tag{1.6a}$$

$$G_{z\mu} = 0 \tag{1.6b}$$

$$G_{zz} = 6\left(\frac{a'}{a}\right)^2 \tag{1.6c}$$

Now the Einstein equations here are equal to

$$G_{\mu\nu} = 8\pi G_{(5)}\Lambda g_{\mu\nu} + 8\pi G_{(5)}\sigma g_{\mu\nu}\delta\left(z\right) \tag{1.7a}$$

$$G_{z\mu} = 0 \tag{1.7b}$$

$$G_{zz} = 8\pi G_{(5)}\Lambda g_{zz} \tag{1.7c}$$

By substituting (1.4) into the above equations, it is a straightforward calculation to show that the the field equations are satisfied provided (1.2) holds true and that the warp strength is given by (1.5). The brane here is situated at $z= 0$, the hypersurface with minkowskian metric. There is an alternative Randall-Sundrum scenario, which still has solution form (1.3). This approach makes the extra dimension compact by introducing a two-brane setup, one with a positive tension situated at $z = 0$, the other has negative tension, and is situated at $z = z_c$. This theory is usually referred to as RS1. The RS2 theory is similar, with one brane placed infinitely far away from the first.

1.2 Brans-Dicke Theory

The Brans-Dicke theory of gravity was presented in 1961 as an alternative to Einstein gravity. Einstein gravity describes a tensor field. Brans-Dicke theory contains, in addition to this tensor field, a scalar field which couples to matter. This theory was originally presented as a possible resolution to Mach's principle. Mach's principle perportes that the motion or possible rest of a body has to be defined in relation to a kind of *absolute space* and *cannot* be defined in terms of *relative* motion to that of other bodies. The problem here is of course, what exactly does one mean by *absolute space*. What are the possible features of such?

Consider the Newton bucket argument. We fill a bucket with water and place it in some kind of uniform (locally, at least) gravitational field. If one spins the bucket, the water inside will (viscosity and time allowing) assume a parabolic shape due to the introduction of a centrifugal force. However were one to somehow rotate the surrounding laboratory, Mach would have argued, the physics should remain unchanged, and once again, the water would assume the parabolic shape. Mach contends that the only meaningful motion is that relative to the rest of the matter in the universe, and that, [5] *"the inertial reaction experienced in a laboratory accelerated relative to the distant matter of the universe may be interpreted equivalently as a gravitational force acting on a fixed laboratory due to the presence of distant accelerated matter."*This *absolute space* therefore, refers to distant matter in the universe. Einstein gravity does not take this rather strict Machi(velli?)an decree on inertia into consideration. Brans and Dicke argue that a possible theory of gravity which takes Mach's principle into account would require a non-constant gravitational 'constant'. It would vary with space, and (very loosely) depend on the ratio of a bodies mass to its distance from that particular point in space. Thus, the scalar field concept was introduced to conventional Einstein gravity. The inertial mass of a body should depend somehow on the particles interaction with this scalar field.

So how would this approach change the equations? The Einstein-Hilbert action with a term for the matter lagrangian (i.e. a term which provides the inclusion of other fields into the theory, like electromagnetism) has the form [6]

$$S = k \int d^4x\sqrt{-g}\,(R + L_M) \tag{1.8}$$

Here,

$$k = \frac{c^4}{16\pi G} \tag{1.9}$$

R is the Ricci scalar, and L_m is the matter lagrangian. The modification that Brans-Dicke prescribes is as follows. We let $\phi \approx 1/G$- be the scalar field which depends on the four-coordinates, and simply associate it with the Ricci scalar term, as opposed to the constant term. The second term is simply the lagrangian density of the scalar field, with an additional ψ- in the denominator to ensure the nondimensionality of the constant ω. Thus we have

$$S = k \int d^4x\sqrt{-g}\,(\psi R - \omega\frac{\partial_a\phi\partial^a\psi}{\psi} + L_M) \tag{1.10}$$

with

$$k = \frac{c^4}{16\pi} \tag{1.11}$$

Upon variation with respect to the metric tensor and then with respect to the scalar field, one can obtain the equations of motion. It is easily apparant then that as $\omega \to \infty$, Brans-Dicke gravity becomes Einstein gravity [6]. Recent measurements attempting to fix a value for the Brans-Dicke paramater give a lower bound at 40 000. In a reasonable theory, one would expect the paramater to have order of magnitude of one. These findings weaken the strength of *classical* Brans-Dicke models, and strengthen the position of Einstein gravity. [1]

1.3 Black Hole solutions in the literature

A theory can only be tested by considering the motion of test particles (mass or massless) in the spacetime. In this regard, metric solutions to the field equations of a theory are required. Generally, these solutions are either black-hole like - generally isotropic, with a mass centered at the origin; or they are cosmological - homogenous and isotropic. We aim to somehow distinguish Einstein gravity from alternative theories. In this text, we shall seek a manifestation of these deviations by considering the way light is bent when it passes in high proximity to a very large mass. The solutions that we require are therefore of the blackhole type.

The McFaddon and Turok approach considers tensionless branes with metric Ansatz [8]

$$ds_5^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + \Phi(x)dY^2 \tag{1.12}$$

The coefficient to the 5th component term is the Radion. Effectively, it is a measure of the *size* of the extra dimension as a function of the four-coordinates. Note that in this theory, the bulk warp factor is not present. When this metric is substituted into the five-dimensional Einstein-Hilbert action ((1.8) only with 5-D) and integrated over the 5th dimension, an effective four dimensional action is obtained:

$$S = m_P^2 \int d^4x\sqrt{-g}\phi R \tag{1.13}$$

Here m_P is the Planck mass. Upon finding the field equations, and assuming isotropy, the following black hole metric results:

$$ds_4^2 = -dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.14}$$

[1] It is notable to mention that there are arguments against the restoration of Einstein gravity in the limit $\omega \to \infty$ [7]

with associated radion

$$\Phi = \sqrt{1 - \frac{2m}{r}} \tag{1.15}$$

McFaddon and Turok then take this approach with the Randall-Sundrum two brane scenario. A warp factor is included in the metric ansatz (1.12). The four-metric is modified to include this warp factor as a function of the radion. In addition, the two branes have different curvature depending on sign of the tension. The changes are:

$$g^{+}_{\mu\nu} = \frac{1}{4}(1 + \Phi)^2 g_{\mu\nu} \tag{1.16a}$$

$$g^{-}_{\mu\nu} = \frac{1}{4}(1 - \Phi)^2 g_{\mu\nu} \tag{1.16b}$$

By following a similar procedure used previously, and by making various coordinate transformations, one finds spacetime a geometry that is extremal Reissner-Nordstrom like.

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \tag{1.17}$$

This metric is a solution to the standard Einstein field equations when the black hole has not only a mass, but a charge. It is this solution which is of particular interest in this project, for we shall derive a metric geometrically equivalent, only via a completely different starting point. While McFaddon and Turok use the Randall-Sundrum scenario to derive this effective 4-dimensional metric, there are other theories to which this metric is a solution. In a different approach by Beckenstein [9], an argument for a metric geometrically identical to (1.17) is presented, only in a completley different theory. Beckenstein's solution comes from the weyl invariant Einstein scalar equations.

1.4 The Chiba connection

Of course, the simple fact that the same metric is solution to different theories under different simplifications is of little consequence with regard to the underlying fundamental simularities that these different theories may share. No powerful connection at this level can be made. However, work done by T. Chiba [12] finds a link between the Randall-Sundrum two brane scenario, and that of Brans-Dicke gravity at the action level. Starting with the action for the RS two-brane setup (a modification of (1.1)), with similar constraints on the consmological constant and the brane tensions, a 5-dimensional metric ansatz with warp factor *and* effective radion is inserted into this action.

$$S = 2^4 x \int_c^r {}_0 d\phi \sqrt{-g_5} \left[\frac{M^3}{2} R_5 - 2\Lambda\right] - \sigma_+ \int d^4x \sqrt{-g_+} - \sigma_- \int d^4x \sqrt{-g_-} \tag{1.18}$$

The metric is:

$$ds^2 = e^{-2kT(x)|z|} \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu + T(x)^2 dx^2 \tag{1.19}$$

where $\bar{g}_{\mu\nu}$ is a general non-Minkoskian metric, and T(x) is the radion. After insertion into the RS action, and integrating over the 5-th dimension, an effective four dimensional action on the positive tention brane is obtained:

$$S_{Chiba} = \int d^4x \sqrt{-g} \left[-\frac{M^3(1 - e^{-2kr_c T(x)})}{2k} R + 3kr_c^2 M^3 e^{-2kr_c T(x)} (\nabla T(x))^2\right] \tag{1.20}$$

here r_c is the postition of the positive tension brane in the 5th dimension. If we make the identifications $\frac{1}{K} = \frac{M^3}{k}$, $K = 8\pi G_N$, and identifying the combination $\sqrt{(6)}e^{-kr_cT(x)}$ with a scalar field $\psi(x)$, we get

$$\mathcal{S} = \frac{1}{2K}\int d^4x\sqrt{-g}\left[-\left(1-\frac{\Psi^2}{6}\right)R + g^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu}\right] \tag{1.21}$$

The effective four dimensional action on the positive brane in the Randall-Sundrum model is therefore like, and tells us that on the brane the gravitational effects have a Brans-Dicke form. We shall continue to analyse this action.

Chapter 2

Finding a black hole metric solution

2.1 A conformal Brans-Dicke action and associated field equations

Consider as ansatz a Brans-Diche scalar-tensor theory with a conformally coupled scalar field; the action for this theory:

$$\mathcal{S} = \frac{1}{2K}\int d^4x\sqrt{-g}\left[-\left(1-\frac{\Psi^2}{6}\right)R + g^{\mu\nu}\Psi_{,\mu}\Psi_{,\nu}\right] \tag{2.1}$$

with

$$k = 8\pi\, G_N \tag{2.2}$$

Before attempting to find the equations of motion, we shall prove that the above action is indeed invariant under conformal transformations. The conformal transformation is

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \tag{2.3}$$

Where Ω is a function of the four-coordinates. The Ricci tensor transforms like:

$$R \rightarrow \tilde{R} = \Omega^2\left[R + 6g^{\alpha\beta}\left(\ln\Omega\right)_{;\alpha;\beta} - 6\frac{g^{\alpha\beta}}{\Omega^2}\Omega_{;\alpha}\Omega_{;\beta}\right] \tag{2.4}$$

In addition, we require the scalar field to transform like a *conformal* scalar field:

$$\psi \rightarrow \tilde{\psi} = \Omega\psi \tag{2.5}$$

The action under this transformation is (we ignore the first term - effectively the Einstein-Hilbert contribution to this action which is certainly conformally invariant).

$$\tilde{\mathcal{S}} = \int d^4x\sqrt{-g}\left(\frac{\tilde{\psi}^2}{6}\tilde{R} + \tilde{g}^{\mu\nu}\tilde{\psi}_{,\mu}\tilde{\psi}_{,\nu}\right) \tag{2.6}$$

Sunstituting (2.3), (2.4) and (2.5) into (2.6) we get

$$\tilde{\mathcal{S}} = \int d^4x\sqrt{-g}\Omega^{-4}\left\{\frac{\psi^2}{6}\Omega^4\left[R + 6\left(g^{\alpha\beta}\frac{\Omega_{,\alpha}}{\Omega}\right)_{;\beta} - 6\frac{\alpha\beta}{\Omega^2}\Omega, \alpha\Omega_{,\beta}\right] + \Omega^2 g^{\mu\nu}\left(\Omega\psi\right)_{,\mu}\left(\Omega\psi\right)_{,\nu}\right\} \tag{2.7}$$

$$= \int d^4x\sqrt{-g}\left\{\frac{\psi^2}{6}R + \psi^2\left(g^{\alpha\beta}\frac{\Omega_{,\alpha}}{\Omega}\right)_{;\beta} - \psi^2\frac{g^{\alpha\beta}}{\Omega^2}\Omega_{,\alpha}\Omega_{,\beta} + g^{\mu\nu}\left[\psi_{,\mu}\psi_{,\nu} + 2\psi\psi_{,\mu}\frac{\Omega_{,\nu}}{\Omega} + \psi^2\frac{\Omega_{,\mu}\Omega_{,\nu}}{\Omega^2}\right]\right\} \tag{2.8}$$

Simplifying further:

$$\tilde{S} = \int d^4x\sqrt{-g}\left[\frac{\psi^2}{6}R + g^{\mu\nu}\psi_{,\mu}\psi_{,\nu} + \left(\psi^2 g^{\alpha\beta}\frac{\Omega_{,\alpha}}{\Omega}\right)_{;\beta}\right] = S \tag{2.9}$$

The third term will give no contribution to the equations of motion because it is a surface term - so will vanish upon integration. The action (2.1) is therefore conformally invariant. We will now proceed to find the field equations. Two equations of motion will be obtained, the first from the variation of (2.1) with respect to the conformal scalar, the second with respect to the metric tensor. Variating with respect to the scalar field:

$$\delta_\Psi S = \frac{1}{K}\int d^4x\sqrt{-g}\left[\frac{\Psi}{6}R - (g^{\mu\nu}\Psi_{,\nu})_{;\mu}\right]\delta\Psi \tag{2.10}$$

Requiring

$$\delta S = 0, \tag{2.11}$$

the relationship between the metric and the scalar field is obtained:

$$(g^{\mu\nu}\Psi_{,\nu})_{;\,\mu} = \frac{\Psi}{6}R \tag{2.12}$$

And variating with respect to the metric tensor gives

$$\delta_g S = \frac{1}{2K}\int d^4x\sqrt{-g}\left[\left(1-\frac{\Psi^2}{6}\right)\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) + \left(1-\frac{\Psi^2}{6}\right)_{;\mu;\nu} - g^{\alpha\beta}\left(1-\frac{\Psi^2}{6}\right)_{;\alpha;\beta} g_{\mu\nu}\right.$$
$$\left. + \Psi_{,\mu}\Psi_{,\nu} - \frac{g_{\mu\nu}}{2}g^{\alpha\beta}\Psi_{,\alpha}\Psi_{,\beta}\right]\delta g^{\mu\nu} \tag{2.13}$$

Using (2.11), we get the matter analogue to (2.12), and let it represent a source term. This source term is a result of taking the functional derivative of S with respect to the metric tensor. So the source term by definition represents the energy-momentum tensor.

$$kT^{\mu\nu} = \left(1-\frac{\Psi^2}{6}\right)\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) - \left(1-\frac{\Psi^2}{6}\right)_{;\mu;\nu} + g^{\alpha\beta}g_{\mu\nu}\left(1-\frac{\Psi^2}{6}\right)_{;\alpha;\beta} +$$
$$-\Psi_{,\mu}\Psi_{,\nu} + \frac{g_{\mu\nu}}{2}g^{\alpha\beta}\Psi_{,\alpha}\Psi_{,\beta} \tag{2.14}$$

Taking derivatives and noting that the covariant derivative of a scalar is simply reduces to the gradient,

$$kT^{\mu\nu} = \left(1-\frac{\Psi^2}{6}\right)\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) - g^{\alpha\beta}g_{\mu\nu}\frac{\Psi}{3}\Psi_{,\alpha;\beta} + \frac{g_{\mu\nu}}{6}{}^{\alpha\beta}\Psi_{,\alpha}\Psi_{,\beta} + \frac{\Psi}{3}\Psi_{,\mu;\nu} - \frac{2}{3}\Psi_{,\mu}\Psi_{,\nu} \tag{2.15}$$

Taking the trace, we find

$$-R\left(1-\frac{\Psi^2}{6}\right) - \Psi g^{\alpha\beta}\Psi_{;\alpha;\beta} = kT^{\mu}_{\mu} \tag{2.16}$$

Now using our first equation of motion (2.12), and expanding the covariant derivative, recalling that the covariant derivative of the metric tensor is zero in every frame; we substitute into 2.16 and obtain

$$R = -kT^{\mu}_{\mu} \tag{2.17}$$

A most useful metric solution for this spacetime is one which can be compared with a known solution to the Einstein field equations. The Schwarzschild metric is a spherically symmetric, static and isotropic solution to the Einstein field equations in *empty* space. By restricting our solution to be of this type, we aim to obtain a Schwarzschild analogue to our conformal Brans-Dicke theory. The energy-momentum tensor is zero for such an empty spacetime, giving $R = 0$ and reducing the equations of motion to

$$\left(g^{\mu\nu}\Psi_{,\nu}\right)_{;\mu} = 0 \tag{2.18}$$

and

$$\left(1-\frac{\Psi^2}{6}\right)\left(R_{\mu\nu}-\frac{g_{\mu\nu}}{2}R\right) - g^{\alpha\beta}g_{\mu\nu}\frac{\Psi}{3}\Psi_{,\alpha;\beta} + \frac{g_{\mu\nu}{}^{\alpha\beta}}{6}\Psi_{,\alpha}\Psi_{,\beta} + \frac{\Psi}{3}\Psi_{,\mu;\nu} - \frac{2}{3}\Psi_{,\mu}\Psi_{,\nu} = 0 \tag{2.19}$$

An isotropic metric, as a function of the radial coordinate only, with the angular dependence equivalent to a 2-sphere metric has the general form

$$ds^2 = e^{A(r)}dt^2 - e^{B(r)}dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2.20}$$

We can quickly find the equation of motion for the scalar field (2.18). $g^{\mu\nu}\Psi_{,\nu}$ is a contravariant tensor, so (2.18) is the covariant divergence of a contravariant tensor. The covariant divergence of a contravariant tensor V can be shown [6] to be equal to:

$$V^{\mu}_{;\mu} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\mu}}\sqrt{g}V^{\mu} \tag{2.21}$$

where $g = \delta_{\mu\nu}g^{\mu\nu}$. This gives for (2.18)

$$\frac{1}{e^{\frac{A+B}{2}}r^2\sin\theta}\left[-e^{\frac{A-B}{2}}r^2\sin\theta\psi'\right]_{,r} = 0 \tag{2.22}$$

Giving

$$r^2 e^{\frac{A-B}{2}}\psi' = k \tag{2.23}$$

To find the solution for the scalar field as a function of the radial coordinate, we require the solutions to the metric coefficients *A(r)* and *B(r)*. We shall find these by solving the second field equation (2.19). Using the relation:

$$\Psi_{;\mu;\nu} = \Psi_{,\mu;\nu} = \Psi_{,\mu,\nu} - \frac{\partial\Psi}{\partial r}\Gamma^{r}_{\mu\nu} \tag{2.24}$$

we are able to express the covariant derivatives in terms of the Christoffel symbols. In addition, the second field equation depends on the Ricci tensor, which is defined by an index contraction over the Riemann tensor :

$$R_{\mu\nu} = R^{\gamma}_{\mu\gamma\nu} = \Gamma^{\gamma}_{\mu\nu,\gamma} - \Gamma^{\gamma}_{\mu\gamma,\nu} + \Gamma^{\gamma}_{\epsilon\gamma}\Gamma^{\epsilon}_{\mu\nu} - \Gamma^{\gamma}_{\epsilon\mu}\Gamma^{\epsilon}_{\nu\gamma} \tag{2.25}$$

Both (2.24) and (2.25) depend on the affine connection, which can be expressed in terms of the metric tensor

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}\left(g_{\beta\delta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}\right) \tag{2.26}$$

The nonvanishing Christoffel symbols for the metric (2.20) are

$$\Gamma^{t}_{tr} = \Gamma^{t}_{rt} = \frac{A'}{2} \tag{2.27a}$$

$$\Gamma^{r}_{tt} = \frac{A'}{2}e^{A-B} \tag{2.27b}$$

$$\Gamma^{r}_{rr} = \frac{B'}{2} \tag{2.27c}$$

$$\Gamma^r_{\theta\theta} = -re^{-B} \tag{2.27d}$$

$$\Gamma^r_{\phi\phi} = -r\sin^2\theta e^{-B} \tag{2.27e}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \tag{2.27f}$$

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{2.27g}$$

$$\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \tag{2.27h}$$

where the dash denotes the partial derivative with respect to r. The components of the Ricci tensor are:

$$R_{tt} = \frac{1}{2}e^{A-B}\left[A'' + \frac{2A'}{r} + \frac{A'}{2}(A' - B')\right] \tag{2.28a}$$

$$R_{rr} = \frac{1}{2}\left[-A'' + \frac{2B'}{r} - \frac{A'}{2}(A' - B')\right] \tag{2.28b}$$

$$R_{\theta\theta} = e^{-B}\left[\frac{r}{2}(B' - A') - 1\right] + 1 \tag{2.28c}$$

$$R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{2.28d}$$

$$R_{\mu\nu} = 0, \mu \neq \nu \tag{2.28e}$$

Since both the Ricci tensor and the metric tensor are diagonal, and since $\psi = \psi(r)$, the second equation of motion (2.19) has four non-trivially satisfied components. Substituting the Christoffels and the Ricci tensor, we find four second order differential equations with unknowns ψ, A and B. The t-t component:

$$\left(1 - \frac{\psi^2}{6}\right)\frac{1}{2}e^{A-B}\left[A'' + \frac{2A'}{r} + \frac{A'}{2}(A' - B')\right] - \frac{1}{6}e^{A-B}\psi'^2 - \frac{\psi\psi'}{6}A'e^{A-B} = 0 \tag{2.29}$$

The r-r component:

$$\left(1 - \frac{\psi^2}{6}\right)\frac{1}{2}\left[-A'' + \frac{2B'}{r} - \frac{A'}{2}(A' - B')\right] - \frac{1}{2}\psi'^2 + \frac{\psi}{3}\left(\psi'' - \psi'\frac{b'}{2}\right) = 0 \tag{2.30}$$

$\theta - \theta$:

$$\left(1 - \frac{\psi^2}{6}\right)\left\{e^{-b}\left[\frac{r}{2}(B' - A') - 1\right] + 1\right\} + \frac{r^2}{6}e^{-B}(\psi')^2 + \frac{\psi\psi'}{3}re^{-B} = 0 \tag{2.31}$$

$\phi - \phi$:

$$\left\{\left(1 - \frac{\psi^2}{6}\right)\left\{e^{-b}\left[\frac{r}{2}(B' - A') - 1\right] + 1\right\} + \frac{r^2}{6}e^{-B}(\psi')^2 + \frac{\psi\psi'}{3}re^{-B}\right\}\sin^2\theta = 0 \tag{2.32}$$

Note that the last equation is simply $\sin^2\theta$ times the $\theta - \theta$ equation. We have an overdetermined system of differential equations, and the last one gives us nothing new, so we therefore ignore it. Since we are dealing with an empty spacetime, the Ricci scalar is 0.

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 0 \tag{2.33}$$

Writing the Ricci scalar in terms of the metric coefficients:

$$R = e^{-B}\left[A'' + \frac{2}{r}(A' - B') + \frac{A'}{2}(A' - B')_{\frac{2}{r^2}}\right] - \frac{2}{r^2} = 0 \tag{2.34}$$

Now taking the partial derivative of the first order differential equation for the scalar field (2.23) with respect to r, we obtain

$$\psi'' + \left(\frac{A' - B'}{2} + \frac{2}{r}\right)\psi' = 0 \tag{2.35}$$

We can substitute equations (2.34) and 2.35) into the components of the field equation in order to eliminate the second derivatives.

$$e^{-B}\left[\left(1 - \frac{\psi^2}{6}\right)\left(\frac{B'}{r} + \frac{e^B - 1}{r^2}\right) - \frac{\psi'^2}{6} - \frac{\psi\psi' A'}{6}\right] = 0 \tag{2.36}$$

$$e^{-B}\left[\left(1 - \frac{\psi^2}{6}\right)\left(\frac{A'}{r} + \frac{1 - e^B}{r^2}\right) - \frac{\psi'^2}{2} - \frac{\psi\psi'}{3}\left(\frac{A'}{2} + \frac{2}{r}\right)\right] = 0 \tag{2.37}$$

$$e^{-B}\left[\left(1 - \frac{\psi^2}{6}\right)\left(\frac{B' - A'}{2r} + \frac{e^B - 1}{r^2}\right) + \frac{\psi'^2}{6} + \frac{\psi\psi'}{3r}\right] = 0 \tag{2.38}$$

Because $R = 0$, and since the ricci scalar is the trace (because of the symmetry), there is a linear dependency in (2.38), so we discard it. We have three unknowns - two metric coefficients and the scalar field, and we have reduced our overdetermined system to three coupled differential equations - (2.23), (2.36) and (2.37).

2.2 Finding a Schwarzschild-like solution

Recall the Schwarzschild solution has the form

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{2.39}$$

with $d\Omega^2$ the metric on the two-sphere, and the speed of light set to unity.

$$d\Omega^2 = d\phi^2 + \sin^2\theta d\phi^2 \tag{2.40}$$

Note that the Schwarzschild solution has $A = -B$. In our search for an analogous metric in our space-time with a conformally coupled scalar, we will enforce this on the field equations. So (2.23) becomes

$$\psi' = k\frac{e^{-A}}{r^2} \tag{2.41}$$

This restriction also reduces (2.36) and (2.37) to

$$-\frac{2}{3}\psi'\left[\psi' + \left(\frac{A'}{2} + \frac{1}{r}\right)\psi\right] = 0 \tag{2.42}$$

This DE gives two sets of solutions, the first is the trivial solution for ψ and,

$$A' = -\frac{2}{r} \Rightarrow e^A = e^{-B} = 1 - \frac{2GM}{r} \tag{2.43}$$

The second solution to (2.42) is

$$\psi = q\frac{e^{-A/2}}{r} \tag{2.44}$$

Note that we now have two constants associated with ψ, $k$and q. To find the relationship between the two, we differentiate (2.44) and equate it with (2.41).

$$k\frac{e^{-A}}{r^2} = -\left(\frac{A'}{2} + \frac{1}{r}\right)q\frac{e^{-A/2}}{r} \tag{2.45}$$

Which has solution

$$e^{A/2} = -\frac{k}{q} - \frac{C}{r} \tag{2.46}$$

We require that the leading terms in the expansion of (2.43) are identical to the newtonian case. This forces

$$k = -q \tag{2.47}$$

and

$$C = GM \tag{2.48}$$

To find the constant q, we substitute, (2.41), (2.44), (2.47) and (2.48) in (2.43). This sets

$$q = \sqrt{6}GM \tag{2.49}$$

Finally we have,

$$ds^2 = \left(1 - \frac{GM}{r}\right)^2 dt^2 - \left(1 - \frac{GM}{r}\right)^{-2} dr^2 - r^2 d\Omega^2 \tag{2.50}$$

with associated scalar field

$$\psi = \sqrt{6}\frac{GM}{r - GM} \tag{2.51}$$

This metric is geometrically equivalent to an extremal Reissner-Nordstrom black hole, and is of the same type as those discussed in section 1, only derived from yet another starting point. At first glance, we note that the radius at which the singularity occurs is at $r = GM$ as opposed to the Schwarzschild radius of $r = 2GM$. To analyse the geometry of this manifold in more detail, we shall consider the geodesics made by null test particles (photons) in the vicinity of a black hole. We shall investigate therefore the lensing of light in this Conformal Brans-Dicke spacetime, and compare it with light propagating in its classic Einstein analogue - the Schwarzschild spacetime.

Chapter 3

Testing the theory - Strong lensing by black holes

3.1 Geodesics in static isotropic gravitational fields

Because I intend to show the effects of matter on both CBD and Schwarzschild null geodesics, I shall derive a general formula for the deflection angle for photons in an isotropic spacetime. The general metric has the form

$$ds^2 = e^A(r)dt^2 - e^B(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \tag{3.1}$$

Since the metric solutions that I wish to investigate both have angular metric dependence equal to that on the two-sphere, we do not lose generality by imposing r^2 as the metric angular terms coefficients. The geodesic equation is

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\gamma}\frac{dx^\nu}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \tag{3.2}$$

where the affine paramater on the manifold is λ. The Chrisoffel symbols for this metric have already been calculated (2.27a)-(2.27h). Substituting these into (3.2), an equation of motion for each of the four-coordinates is acquired:

$$\frac{d^2r}{d\lambda^2} + \frac{B'(r)}{2}\left(\frac{dr}{d\lambda}\right)^2 - re^{-B(r)}\left(\frac{d\theta}{d\lambda}\right)^2 - re^{-B(r)}\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{1}{2}A'(r)e^{A(r)-B(r)}\left(\frac{dt}{d\lambda}\right)^2 = 0 \tag{3.3a}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{3.3b}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\phi}{d\lambda}\frac{dr}{d\lambda} + \cot\theta\frac{d\phi}{d\lambda}\frac{d\theta}{d\lambda} = 0 \tag{3.3c}$$

$$\frac{d^2t}{d\lambda^2} + A'(r)\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \tag{3.3d}$$

Due to the metric isotropy, without losing generality we can confine the photon to the equatorial plane, setting $\theta = \frac{\pi}{2}$. This warrants (3.3b) to be trivially satisfied. Dividing (3.3c) by d$\phi/d\lambda$, and (3.3d) by dt/dλ, we find two constants of motion:

$$\frac{d}{d\lambda}\left(\ln\frac{d\phi}{d\lambda} + \ln r^2\right) = 0 \tag{3.4a}$$

$$\frac{d}{d\lambda}\left(\ln\frac{dt}{d\lambda} + A(r)\right) = 0 \tag{3.4b}$$

(3.4a) yields conservation of angular momentum:

$$J = r^2 \frac{d\phi}{d\lambda} \tag{3.5}$$

(3.4b) gives

$$\frac{dt}{d\lambda} = e^{-A(r)} \tag{3.6}$$

Substituting (3.5), and (3.6) into (3.3a) gives the radial equation of motion as

$$\frac{d^2r}{d\lambda^2} + B'(r)\left(\frac{dr}{d\lambda}\right)^2 - J^2\frac{e^{-B(r)}}{r^3} + \frac{1}{2}A(r)e^{-A(r)+B(r)} = 0 \tag{3.7}$$

Multiplying this equation with *2A(r)dr/dλ*:

$$\frac{d}{d\lambda}\left[e^{B(r)} + \left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - e^{-A(r)}\right] = 0 \tag{3.8}$$

We identify this constant of motion as the energy:

$$-E = e^{B(r)} + \left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - e^{-A(r)} \tag{3.9}$$

From (3.1), (3.5), (3.6) and (3.9) we find that

$$E = \frac{ds^2}{d\lambda^2} \tag{3.10}$$

Since we are interested in the motion of massless particles only, the constant of proportionality between the affine paramater λ and proper distance s vanishes. Hence, the energy for a photon is zero. Now, by solving (3.6) for $d\lambda$, and inserting this into (3.5) we get

$$r^2\frac{d\phi}{dt} = Je^{-A(r)} \tag{3.11}$$

This is independent of the affine parameter. Similarly using (3.6) in (3.9) we find

$$e^{B(r)-2A(r)}\left(\frac{dr}{dt}\right) + \frac{J^2}{r^2} - e^{-A(r)} = 0 \tag{3.12}$$

If one uses (3.11) and (3.9) to eliminate the temporal component, we have a differential equation for ϕ as a function of the radial component only.

$$\frac{e^{B(r)}}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} - \frac{e^{A(r)}}{J^2} = 0 \tag{3.13}$$

This is a seperable first order differential equation, so the solution can be written as an integral:

$$\phi(r) = \int_r^\infty \frac{e^{B(r)/2}}{r^2}\left(\frac{e^{A(r)}}{J^2} - \frac{1}{r^2}\right)^{-\frac{1}{2}} \tag{3.14}$$

This means that given a radial distance from the black hole, its angular displacement at that radius can be found subject to initial conditions. It is J that *holds* the content given by the initial conditions. Since we are interested in the lensing applications only, we need consider only the case of *unbound* orbits. We therefore assume that the particle approaches the black hole form infinity. In this limit, the metric becomes Minkowskian, that is, $A(\infty) = B(\infty) = 1$. The impact parameter b is (see figure 3.1)

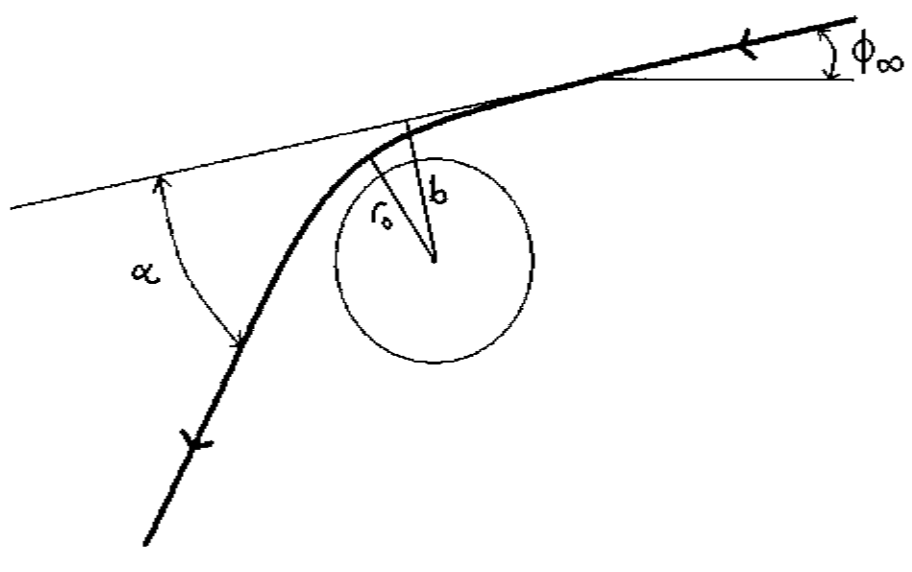

Figure 3.1: Basic deflection of light setup in the proximity of a large mass.

$$b \approx r\sin(\phi - \phi_\infty) \approx r(\phi - \phi_\infty) \tag{3.15}$$

where ϕ_∞ is the incident photon direction. Also, because the field is infinitly weak here, the velocity is constant, so

$$-v \approx \frac{d}{dt}(r\cos(\phi - \phi_\infty)) \approx \frac{dr}{dt}) \tag{3.16}$$

Inserting these into (3.11) and (3.12) we find $v = 1$ and $J = bV^2 = b$. So for photons, the impact paramater is the angular momentum. We now have found an equation for this unbound orbit (3.14) as a function of impact parameter. For our purposes, it is more useful to give the orbit equation as a function of the distance of closest approach r_0, rather than the impact parameter. To this end, we substitute $r = r_0$ into (3.12), and use the fact that at this point, the rate of change of the radius with angle is 0. $\frac{dr}{d\phi} = 0$ (The photon has approached its maximum proximity to the black hole, and will now begin its recession from the black hole.) This gives

$$J = r_0 e^{\frac{-A(r)}{2}} \tag{3.17}$$

Note that (3.14) describes *half* of the photons complete geodesic. There is a complete symmetry in ϕ around $r = r_0$. In other words, if one changes from a radial distance $r_0 + D$ to $r_0 - D$, the deflection angle is the same for a particular D. This implies then, that the total deflection angle, i.e. the deflection of the photon from a straight line is

$$\alpha \equiv \Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi \tag{3.18}$$

For sake of accordance with the notation usually used in the literature, and indeed for the purpose of simplicity, before proceeding to the analysis of lensing in the CBD spacetime, I shall take this oppertunity to make a slight change to the metric coefficients, namely $a(r) = e^{A(r)}$ and $b(r) = e^{B(r)}$. Inserting (3.17) into the orbit equation and making the notation change, we have finally for the orbit:

$$\phi(r) = \phi_\infty + \int_r^\infty \frac{b^{1/2}(r)}{r}\left[\left(\frac{r}{r_0}\right)^2\left(\frac{a(r_0)}{a(r)}\right) - 1\right]^{-1/2} dr \tag{3.19}$$

3.2 Deflection angles for CBD and SCH

Before proceeding with the deflection angle calculations explicitly in terms of the CBD or Schwarzschild metric coefficients, we shall find the *photon spheres*. While the event horizons of the CBD and Schwarzschild black holes are $r_{CBD} = 1$ and $r_{Sch} = 2$ (letting $GM = c = 1$) , the photon sphere radii are greater. The photon sphere is an unstable photon orbit. Any incident photon with an impact paramater such that the distance of closest approach is *less* than the photon sphere will be *sucked* into the black hole, never to emerge. Because the photon sphere is effectively infinitly thin, this orbit is unstable - no photon will stay just above or below it for long. Those grazing the photon sphere will - because of their high proximity to the black hole, experiance maximum deflection, possibly even circling the black hole multiple times before emerging. The photon sphere is given by the largest solution to [15]

$$\frac{a'(r)}{a(r)} = \frac{c'(r)}{c(r)} \tag{3.20}$$

where $a(r)$ is the temporal metric coefficient and $c(r)$ the angular coefficient, (for both our black hole cases, $c(r) = r^2$). Recall the Schwarzschild and CBD and metrics are given by (2.39) and (2.50) respectivly. The simple photon sphere calculation gives $r_{ps} = 3$ and $r_{pc} = 2$. The CBD photon sphere radius is therefore equal to the Schwarzschild event horizon radius. It is not completely imprudent to at this stage remark the following: Since the Schwarzschild photon sphere is smaller than the CBD one, we can expect the total deflection angle for photons incident on either black hole to be *greater* in the CBD case than in the Schwarzschild case. This is due to the higher proximity that the CBD black holes allow the incident photons.

From a lensing perspective, the photon orbits (3.19) are not as useful to us as the deflection angles as a function of the distance of closest approach. We are interested in (3.18). In terms of the metric coefficients, the total defelction angle is:

$$\alpha = 2\int_{r_0}^\infty \frac{b^{1/2}(r)}{r}\left[\left(\frac{r}{r_0}\right)^2\left(\frac{a(r_0)}{a(r)}\right) - 1\right]^{-1/2} dr - \pi \tag{3.21}$$

In the treatment of the analysis of the above equation, I shall initially find approximations to the weak and strong field limits for both cases using the procedure outlined by Bozza [15]. Following this, the results for the direct evaluation of the above integral will be shown, and compared with the approximate ones.

Making the substitutions $u = \frac{1}{r}$, and $u = u_0 x$, the deflection angles for the Schwarzschild and CBD cases are given by

$$\alpha_{Sch} = 2\int_0^1 \frac{dx}{\sqrt{1 - x^2 + r_s u_0\left(x^3 - 1\right)}} - \pi \tag{3.22a}$$

$$\alpha_{CBD} = 2\int_0^1 \frac{dx}{\sqrt{(1 - r_c u_0^2) - (x - r_c u_0 x^2)^2}} - \pi \tag{3.22b}$$

where r_s and r_c are the event horizon radii. The Schwarzschild case has under the square root on the denominator a cubic polynomial, while the CBD case expression is quintic. Both are irregular elliptical integrals, and cannot be evaluated analytically. Note that both expressions are regular over the integration interval, however there it a complex pole at the lower limit. This pole represents the asymptotic behavoir of the photon defelction as it approaches the photon sphere. Expanding in x, and dropping higher order terms, we get an approximation for the deflection angles for the Schwarzschild and CBD cases respectively in the weak field:

$$\alpha_{Sch} \approx 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left[1 + \frac{r_s u_0}{2}\frac{(1-x^3)}{1-x^2}\right] - \pi \tag{3.23a}$$

$$\alpha_{CBD} \approx 2\int_0^1 \left[1 + r_c u_0 \frac{1-x^3}{1-x^2}\right] - \pi \tag{3.23b}$$

Now since $r_s = 2r_c$, these two integrals are identical with solution

$$\alpha_{weak} = 4u_0 = \frac{4}{r_0} \tag{3.24}$$

This result shows that in the weak field limit, the deflection angles of photons in either spacetimes are identical. Therefore, only photons with a high enough proximity (those experiencing a strong enough field) to a black hole will show results that will differentiate between the two spacetimes. In analyzing the strong field limits approximation to the deflection angle, we continue following the Bozza procedure. We transform the independent variable $x \to 1 - x$, giving

$$\alpha_{Sch} = 2\int_0^1 \frac{dx}{\sqrt{(2 - 3r_s u_0)x + (3r_s u_0 - 1)x^2 - r_s u_0 x^3}} - \pi \tag{3.25a}$$

$$\alpha_{Sch} = 2\int_0^1 \frac{dx}{\sqrt{2(1-2r_c u_0)(1-r_c u_0)x + [2r_c u_0(1-r_c u_0) - (1-2r_c u_0)^2]x^2 - 2r_c u_0(1-2r_c u_0)x^3 - (r_c u_0)^2 x^4}} - \tag{3.25b}$$

These equations hold for $u_o < u_p$, the divergence occurs when they are equal. Bozza seperates the integral into two terms, one which diverges I_D and a regular term I_R. We have then $\alpha = I_D + I_R - \pi$ with

$$I_{D_{Sch}} = 2\int_0^1 \frac{dx}{\sqrt{(2 - 3r_s r_0)x + (3r_s u_0 - 1)x^2}} \tag{3.26a}$$

$$I_{D_{CBD}} = 2\int_0^1 \frac{dx}{\sqrt{2(1-2r_c u_0)(1-r_c u_0)x + [2r_c u_0(1-r_c u_o) - (1-2r_c u_0)^2]\,x^2}} \tag{3.26b}$$

These terms diverge when the distance of closest approach is the photon sphere. Howerver, the following terms do not:

$$I_{R_{Sch}} = 2\int_0^1 \frac{dx}{x}\left(\frac{1}{\sqrt{1-\frac{2x}{3}}} - 1\right) \tag{3.27a}$$

$$I_{R_{CBD}} = 2\sqrt{2}\int_0^1 \frac{dx}{x}\left(\frac{1}{\sqrt{1-\frac{x^2}{2}}} - 1\right) \tag{3.27b}$$

Evaluating these two pairs of integrals gives

$$I_{D_{Sch}} = 2\ln\left(\frac{2}{1-\frac{u_0}{u_p}}\right) \tag{3.28a}$$

$$I_{D_{CBD}} = 2\sqrt{2}\ln\left(\frac{2}{1-\frac{u_0}{u_p}}\right) \tag{3.28b}$$

and

$$I_{R_{SchH}} = 2\ln\left[3\left(\sqrt{3}-1\right)^2\right] \tag{3.29a}$$

$$I_{R_{CBD}} = 2\sqrt{2}\ln\left(\frac{4}{2+\sqrt{2}}\right) \tag{3.29b}$$

Thus, provided the the distance of closest approach is close enough to the photon sphere, the total defelction angles are given by

$$\alpha_{Sch} = -2\ln\left(1-\frac{u_0}{u_p}\right) + 2\ln\left(6(\sqrt{3}-1)^2\right) - \pi \tag{3.30a}$$

$$\alpha_{CBD} = -2\sqrt{2}\ln\left(1-\frac{u_0}{u_p}\right) + 2\sqrt{2}\ln\left(\frac{4\sqrt{2}}{1+\sqrt{2}}\right) - \pi \tag{3.30b}$$

3.3 Simulation results

The evaluation of (3.21) was performed in Mathematica version 4.1. The Mathematica integration algorithm uses gaussian quadrature with adaptive stepping. The integration gave the following results for the deflection angle:

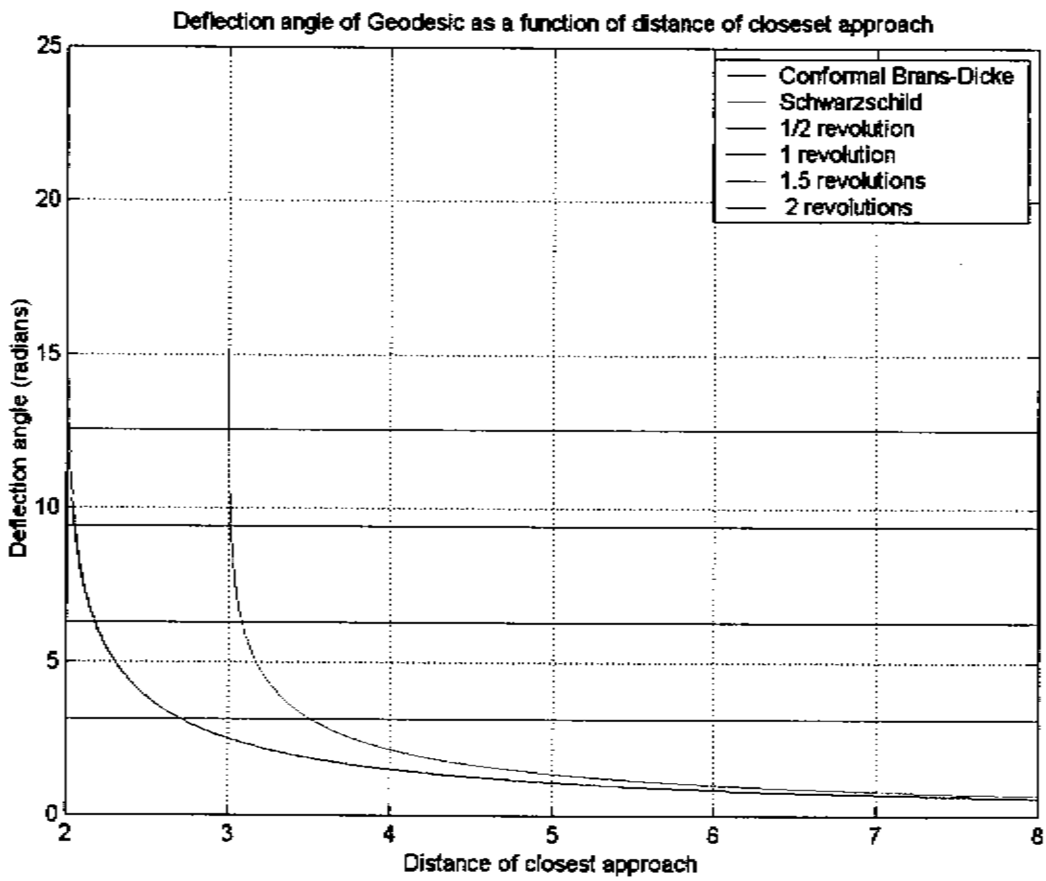

Figure 3.2: Deflection angle against distance of closest approach for both cases ($GM = c = 1$). Observe the correct asymptotic behavior as the geodesics approach the photon sphere. The photon spheres are at $r = 3$ and $r = 2$ for the Schwarzschild and CBD cases respectively. The horizontal lines indicate the angles for which the photon has done 1/2 a revolution, 1 revolution etc. around the black hole.

In order to check the strength of the integration routine, we compare it with the expressions for the strong field limit (3.30a) and (3.30b). First the Schwarzschild case comparison:

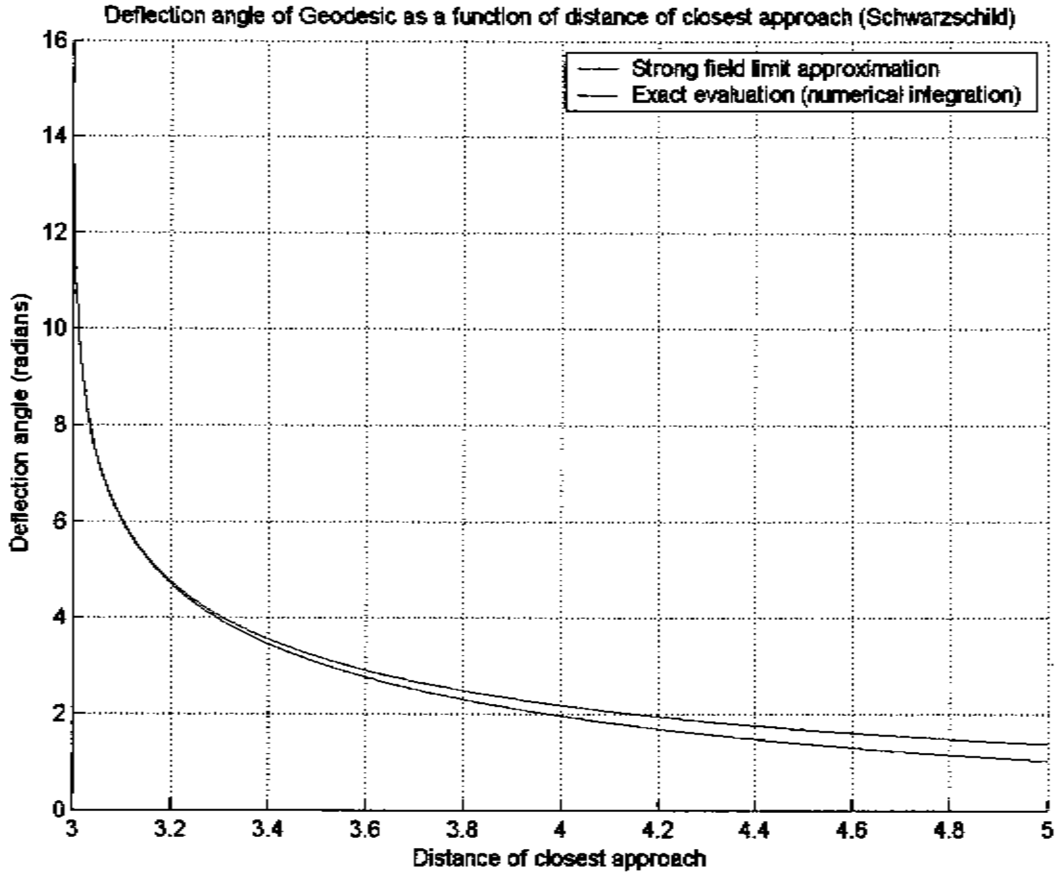

Figure 3.3: This plot shows the relationship between the deflection angle and the distance of closest approach for both the direct integral evaluation and the strong field limit relationship.

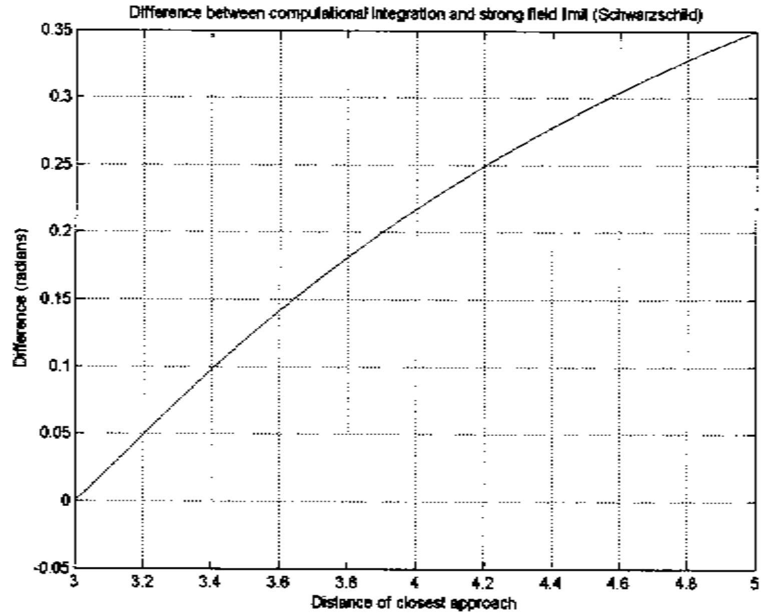

Figure 3.4: The difference between the strong field limit expression and the exact numerical expression. We expect the difference in the limit as the distance of closest approach tends to the photon sphere radius to be zero if the integration routine is successful.

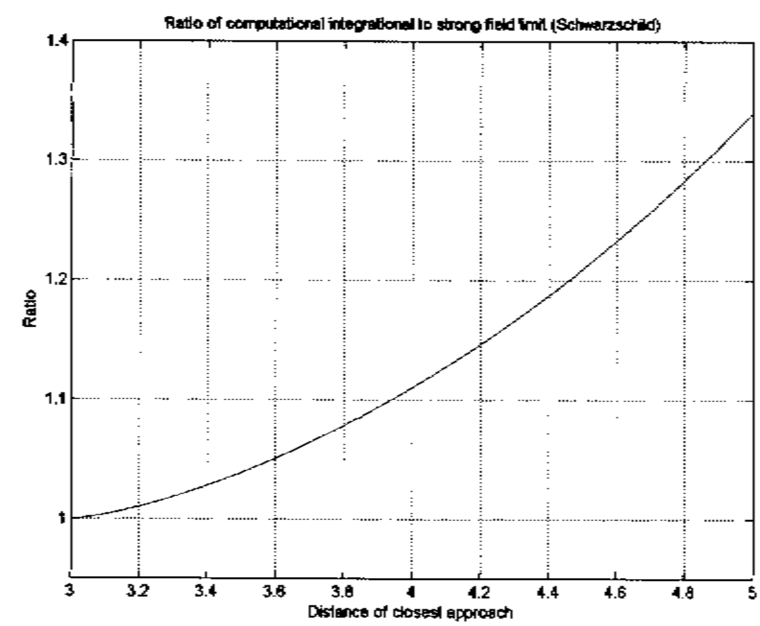

Figure 3.5: Once again, a check on the accuracy of the integration method.

In order to fix an upper bound for the relative accuracy, we *zoom into* the photon sphere region by $\approx 10^3$ times.

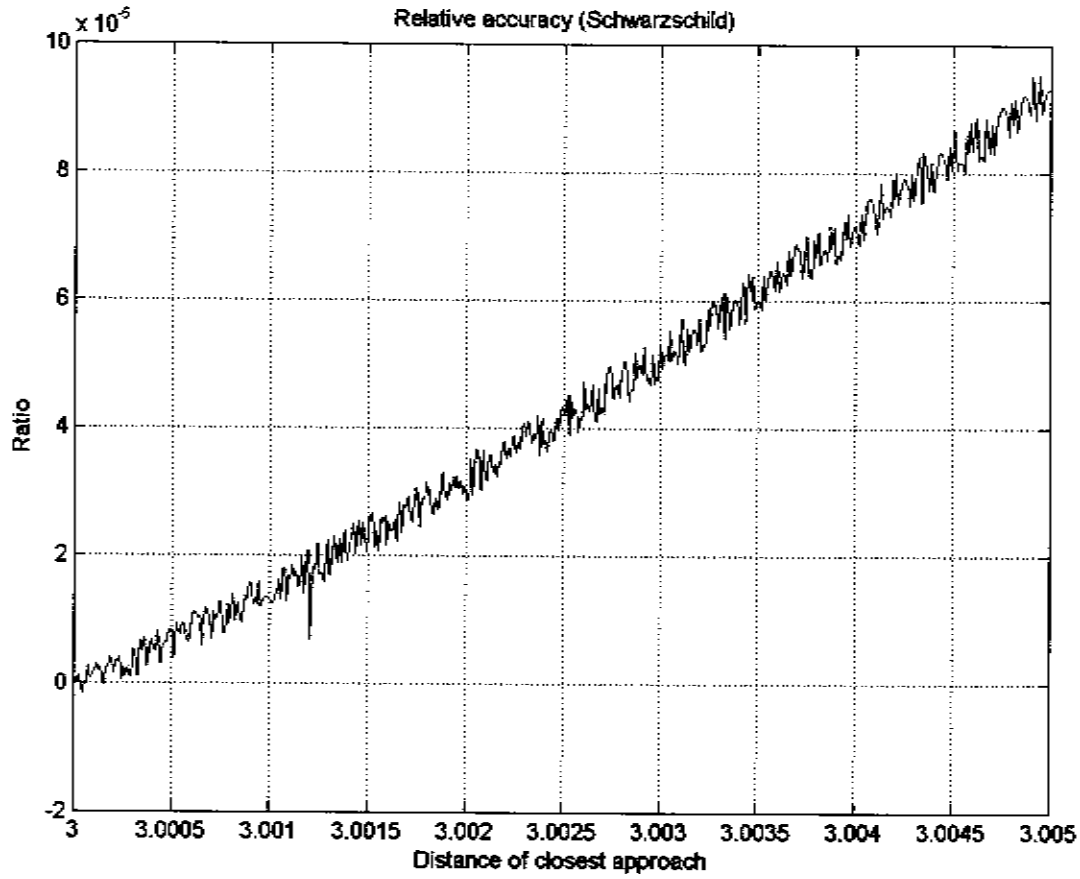

Figure 3.6: The *relative* accuracy of the integration routine with the strong field limit in high proximity to the photon sphere. At this resolution, small fluctuations from the correct values are visible in the integration routine. This places an upper bound on the relative accuracy at 1/10000.

The integration routine comparison for the CBD spacetime:

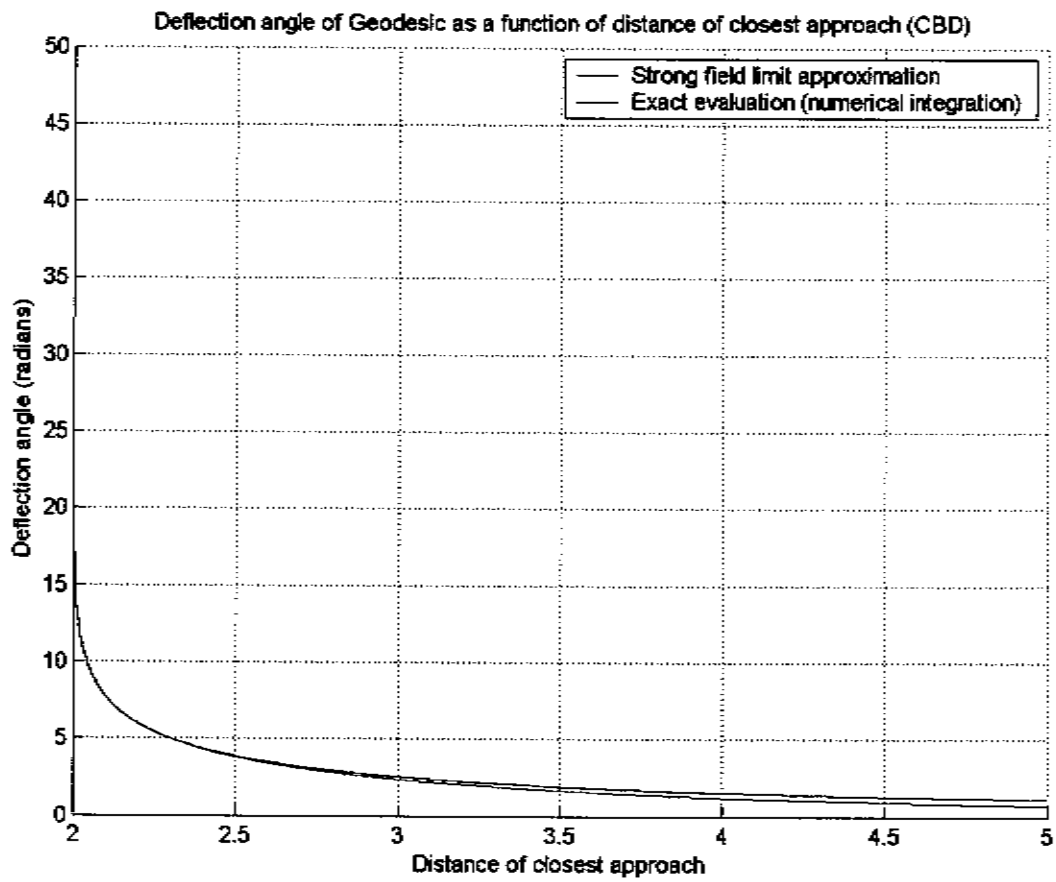

Figure 3.7: This plot shows the deflection angles for a given distance of closest approach. (CBD)

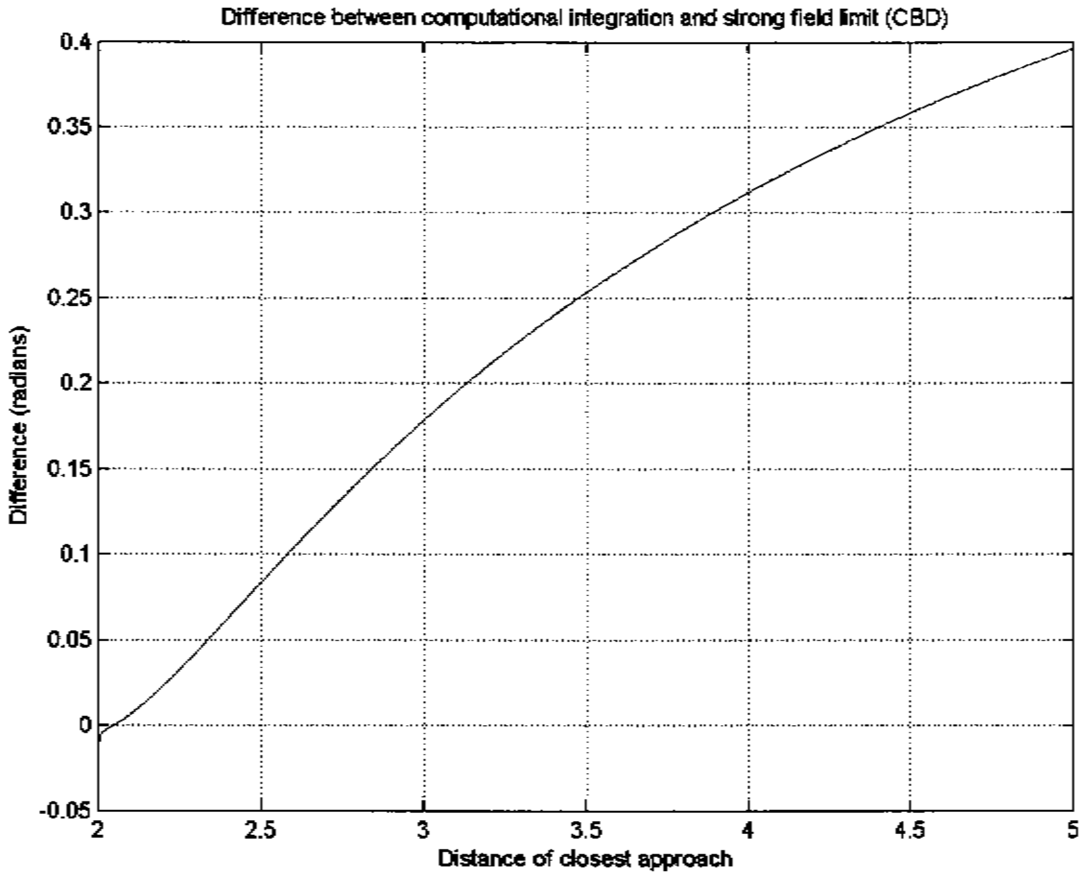

Figure 3.8: The difference between the strong field limit expression and the exact numerical expression. We expect the difference in the limit as the distance of closest approach tends to the photon sphere radius to be zero if the integration routine is successful.

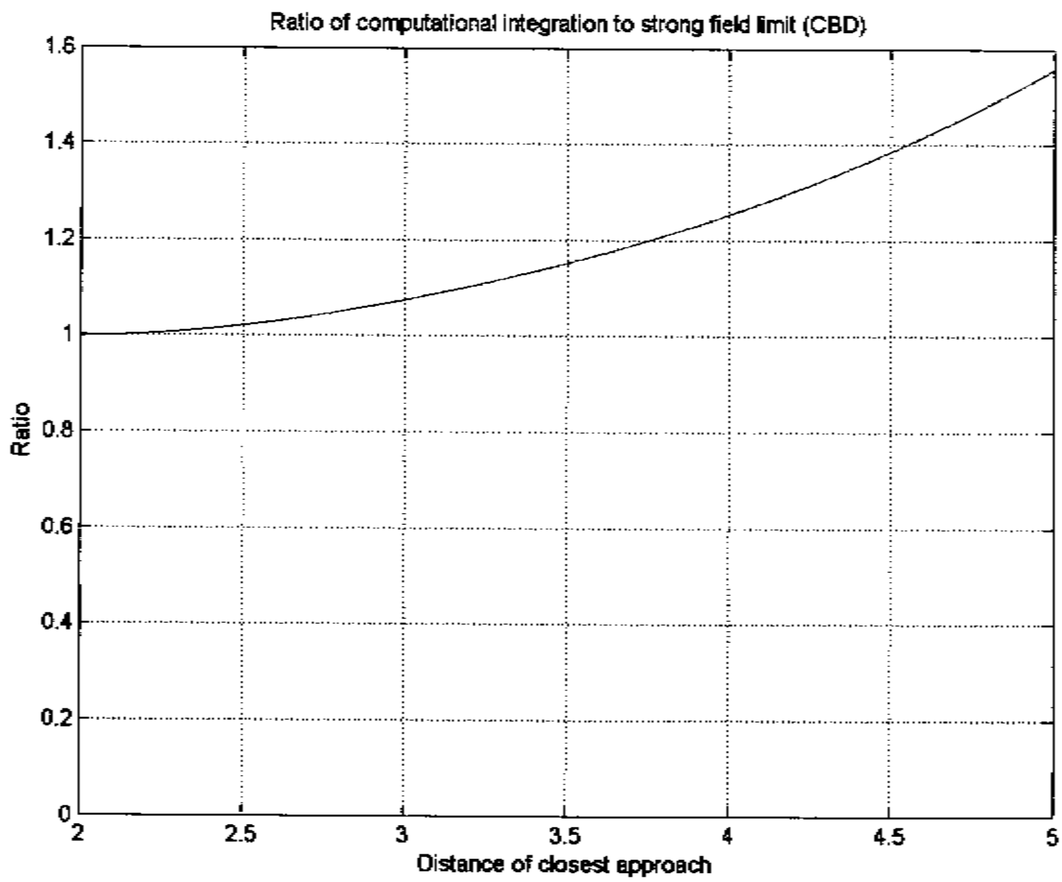

Figure 3.9: The ratio between the two methods (Should both be unity in the photon sphere limit if the integration limit does not diverge).

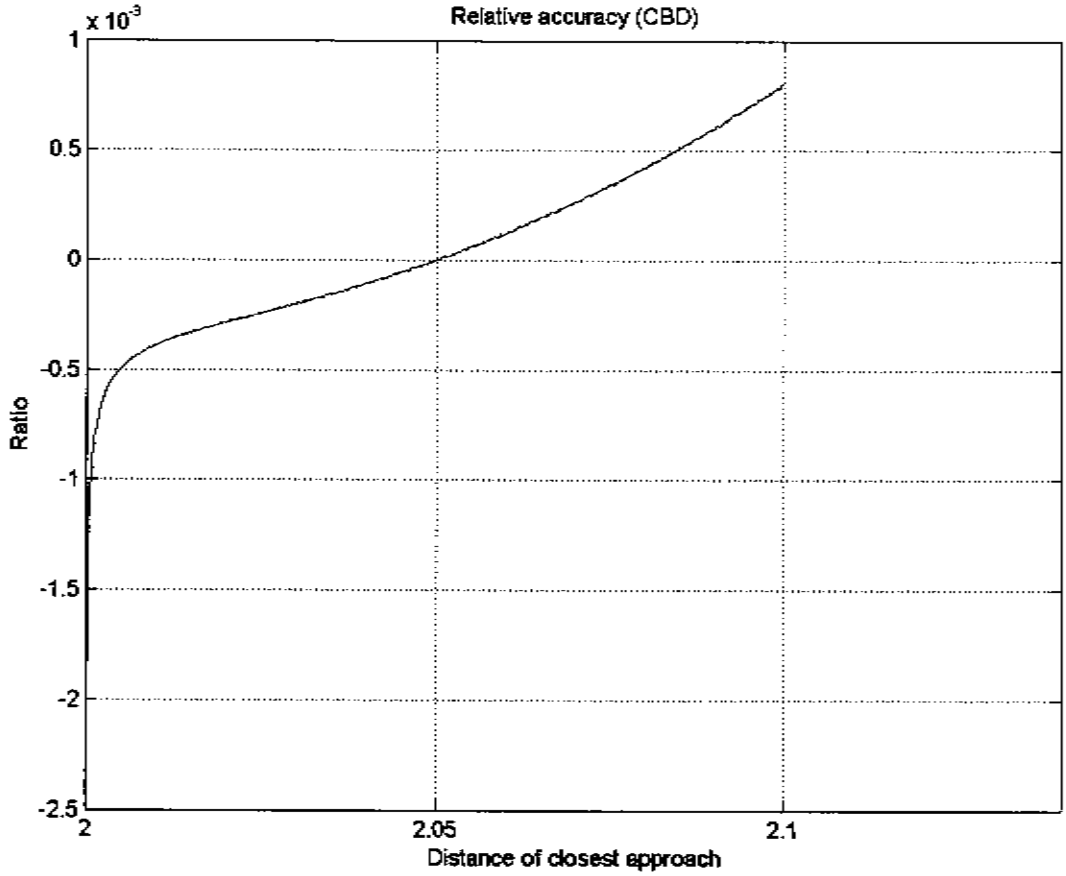

Figure 3.10: The *relative* accuracy of the integration routine with the strong field limit in high proximity to the photon sphere. We have zoomed in to the region just above the photon sphere. At this proximity the integration routine's accuracy begins to diverge. The integration is not as strong here as for the Schwarzschild case. The relative accuracy here is about 1/1000. This can be expected considering the slightly more complex nature of the CBD integral.

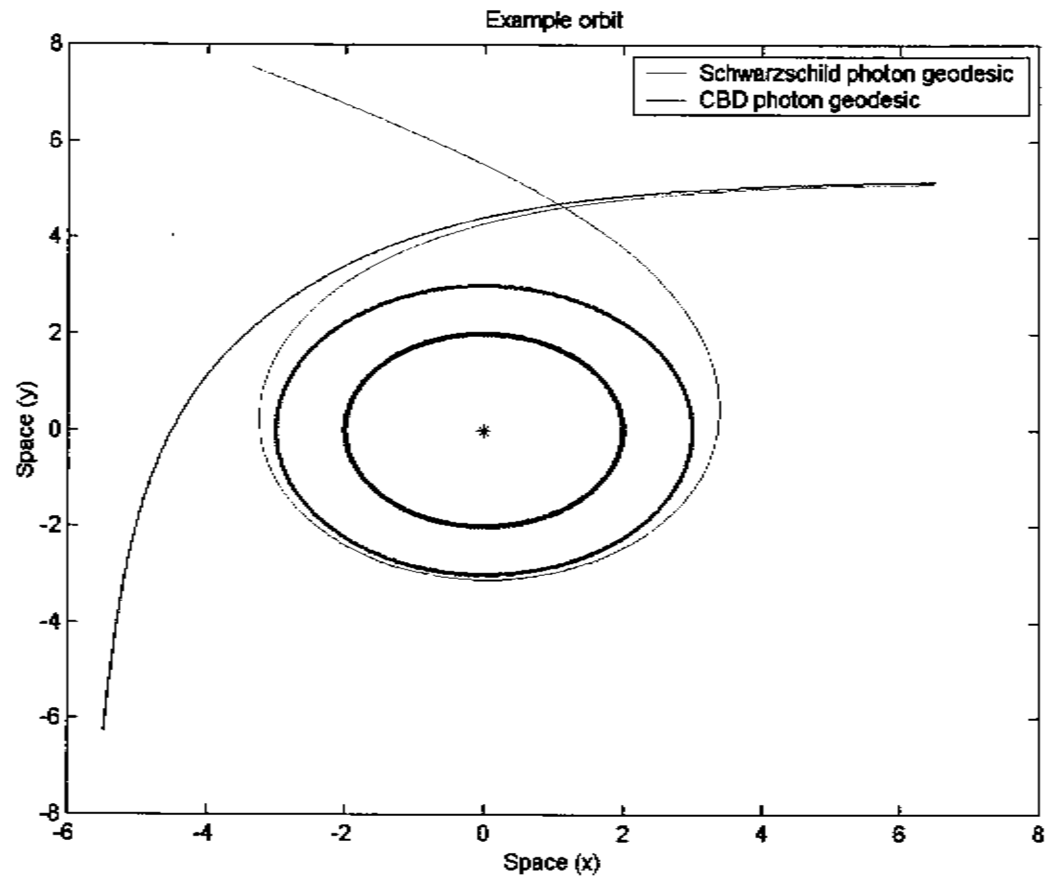

Figure 3.11: This shows an example of the photon trajectory through space, simply the evaluation of (3.19) for a particular impact paramater and associated distance of closest approach. Here, the distance of closest approach of the photon in the Schwarzschild spacetime is 3.1m. The distance of closest approach of the CBD photon is 3.9m (simply chosen because this gives approximately the same impact parameter as the Schwarzschild photon). The Schwarzschild photon is close enough to circle its black hole, while the CBD photon is simply bent by its black hole - due to a low proximity to its own black hole. The two bold circles in this plot are the black hole photon spheres. The larger belongs to the Schwarzschild black hole with radius r = 3m and the smaller belongs to the CBD black hole with radius = 2m.

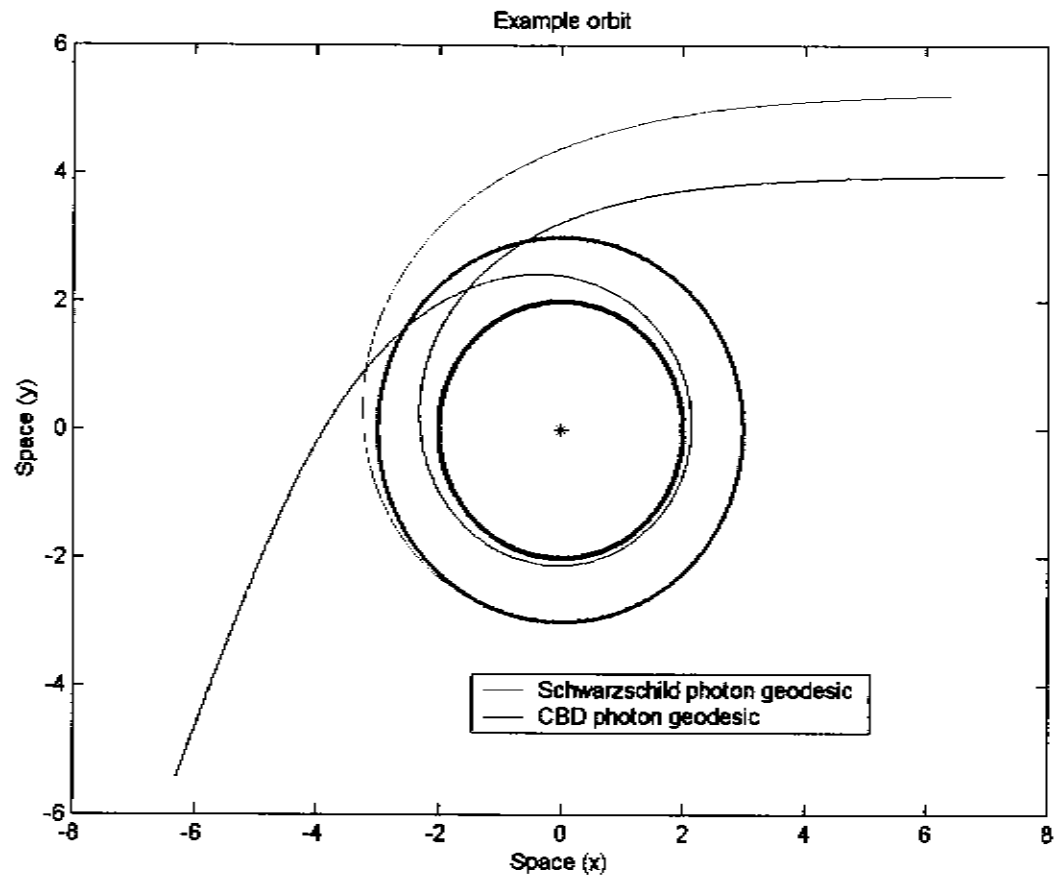

Figure 3.12: An example similar to the above, only here the incident photons have lower distances of closest approach. The CBD photon's distance of closest approach to its black hole is 2.1m. The Schwarzschild photon has impact paramater too low to orbit its black hole - thus, it crosses the photon sphere and is swallowed. Since both spacetimes have only one solution to (3.20), and so only one orbit, a photon which crosses this orbit will eventually cross the event horizon and will therefore never emerge. The reason we do not see the photon trajectory on the above plot after it crosses the photon sphere, is because of the assumption made in the derivation of the orbit equation. The orbit equation integrand becomes complex when the photon crosses the photon sphere. It was assumed that the relationship between the angular position and the radial position had a stationary point. A photon crossing the photon sphere has a monotonically decreasing radius.

3.4 Application to lensing

Leaving geometrized units behind and turning to SI units, we shall look at lensing in our own galaxy. The stars orbiting the galactic center provide us with ideal candidates for strong lensing. We aim to distinguish the two spacetimes from one another, and attempt to find a lower bound on the required telescope resolution.

The most general lensing setup is as follows:

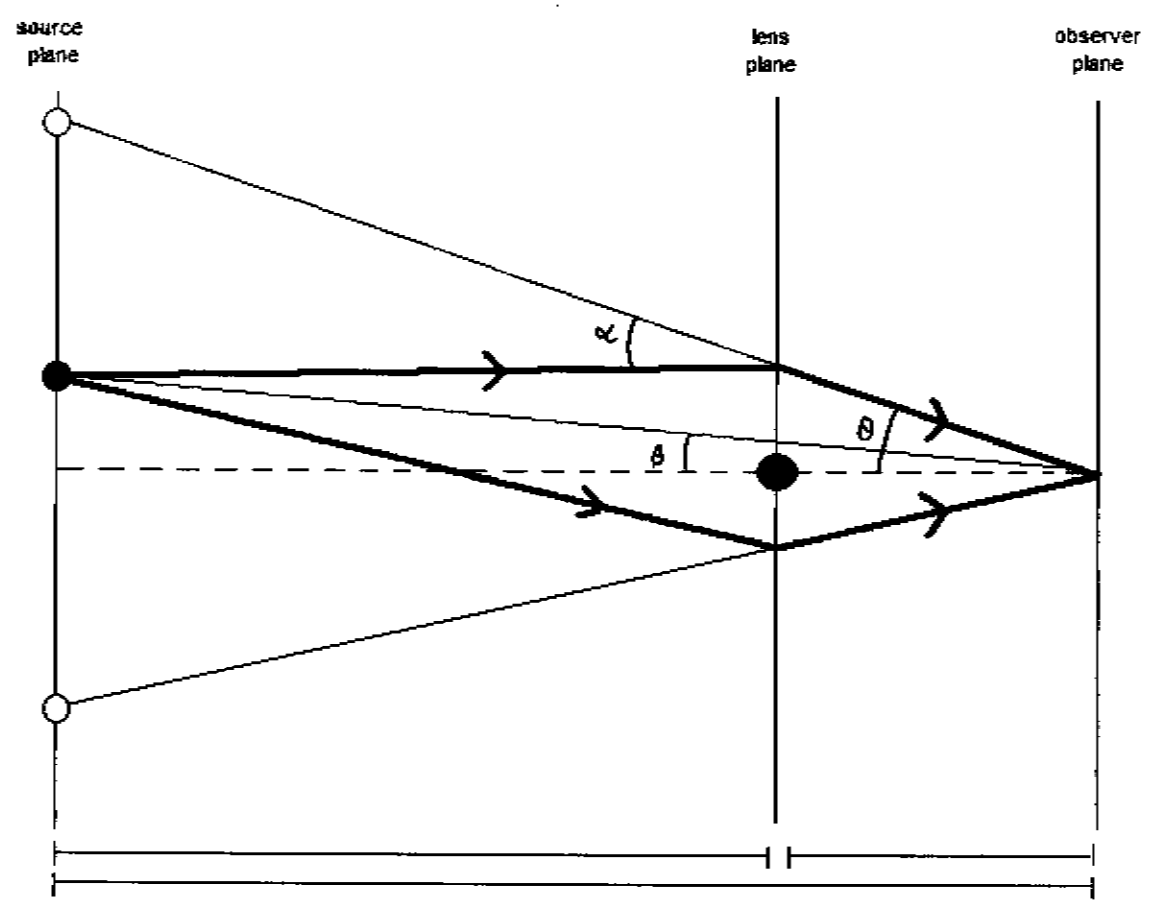

Figure 3.13: A generic lensing setup. While here the source plane is further away from the observer plane than the lens plane is; lensing is not restricted to this. In addition, this diagram shows only the first relatavistic image. (Essentially a *distorted* Einstein Ring because the source is not collinear with the observer and lens.)

α in the above diagram is the deflection angle, b is the impact parameter, β is the angular position of the source in the absence of the lens, and θ is the angular position of the source as seen by the observer. The diagram shows only the first image that reaches the observer. There are an infinite number of images, the higher the image number, the lower the impact parameter and deflection angle (after subtracting 2π an appropriate number of times). After calculation of the deflection angle for a given distance of closest approach (already done above), and given the above setup with known r_0, β, D_ls and D_os; we need to find the corresponding deflection angle α.

Using simple geometry one can show that the relationship between θ and α is given by [19]

$$\tan\theta = \frac{D_l s}{D_o s}\left[\tan\theta + \tan(\alpha-\theta)\right] + \tan\beta \tag{3.31}$$

For small β, this simplifies to

$$\theta = \beta + \frac{D_l s}{D_o s}\alpha \tag{3.32}$$

Using the above *lens equation*, once we have found α, we can find θ - our observable. We shall use as our example the star S2 in orbit around the black hole Sagitarius A*. The distance to the Sagitarius A* is $\approx 8.51 kPc$. S2 has [20] an orbital period of 15.24 ± 0.36 years, the inclination of the orbit normal with respect to line of sight is 131.9 ± 1.3 degrees, the semimajor axis has angular size 0.1226 ± 0.0025 arcseconds, and the eccentricity of the orbit is 0.876 ± 0.0072. The simulations give the following results:

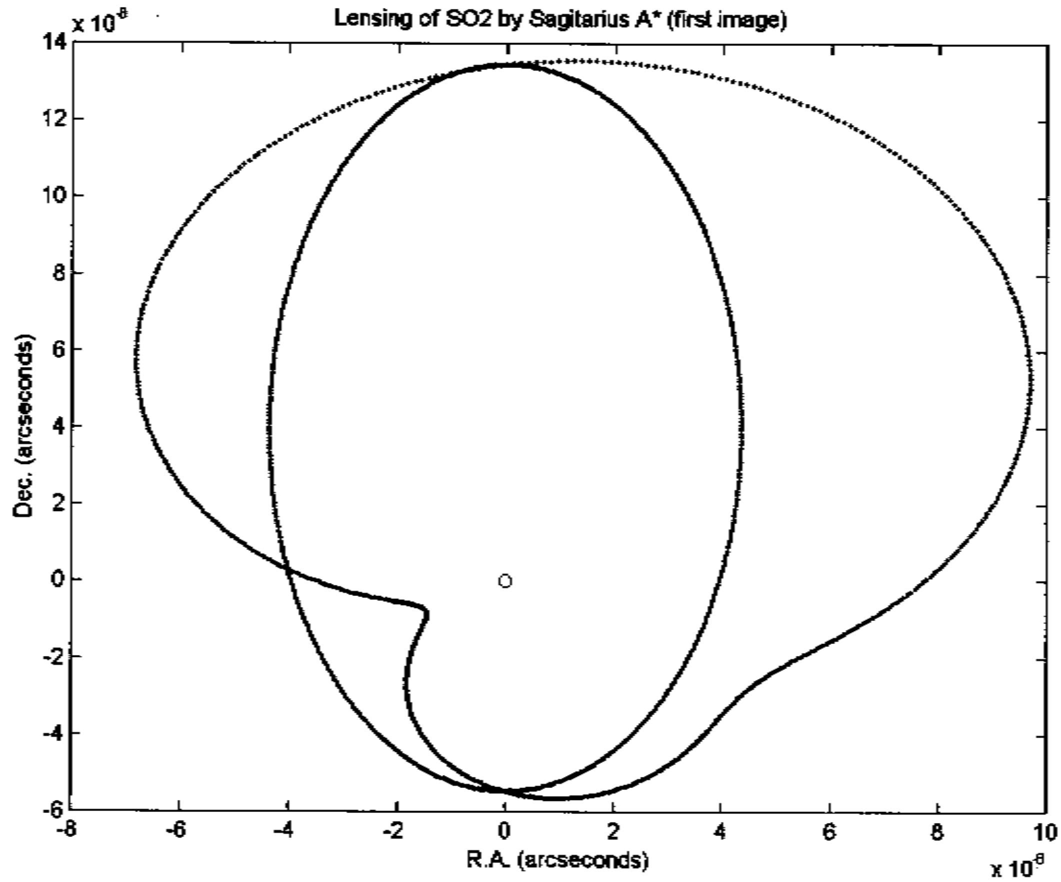

Figure 3.14: The blue curve shows the angular positions of the images from the star S2 throughout its orbit in the absence of the black hole. The red curve shows the angular positions of the first relatavistic images from the star throughout its orbit. Unfortunately, even at this proximity to the black hole, the Schwazschild and CBD deflection angle cases give the same results.(Both lie on the red curve.) The asymmetry in the relatvistic images is a consequence of the tilt in the star's orbit. (The orbit is not flat-on.)

3.5 Final conclusions

The Chiba connection between Randall-Sundrum branes and a conformal Brans-Dicke theory allows us to see the manifestation of branelike features on our brane. Beginning with this link, the CBD action was variated, giving the corresponding field equations. Using these we were able to find a black hole metric solution analagous to the Schwarzschild metric - static and isotropic. This metric had geometry equivalent to that of an extremal Reissner-Nordstrom spacetime. The deflection of photons by black holes in the Schwarzschild spacetime differs from the CBD spacetime in the strong field - the weak field gives identical results. It is for this reason, that only the observation of *extreme* phenomena, like the deflection of light in high proximity to a black hole would provide a testable oppertunity to observe the black-hole like features of our universe. Upon consideration of the null geodesics in generic spacetimes with only radial dependence on metric coefficients, an expression for the deflection of light by black holes was obtained. This involved an integral which could not be solved analytically, so two approaches were used. The first was a direct numerical evaluation of the integral. The second method, valid only in the strong field limit was found following the approach outlined by Bozza. Approximate solutions of these types were found both for the Schwarzschild spacetime and the CBD spacetime. The strong field limit solutions were used to check the strength of the integration routine. The implementation of the routine to the Schwarzschild case exhibited a higher *relative* accuracy (10^{-4}) than the CBD case integration ($about 10^{-3}$). Both however, are well within the required accuracy range. The application of the deflection of light to lensing was considered. The example used was the star S2 in orbit around Sgr A*, the black hole at the centre of our galaxy. It was found that even in this case there was no observed difference between the two spacetimes for the first relatavistic images. Thus, the required telescope resolution is certainly beyond the technology available today. Should astronomers be able to observe with high resolution the positions of relatavistic images, we would be provided with a strong tool for determining the nature of black holes - and thus hint at the strengths of various theories of gravity.

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