Honours Project: Black-holes in the Horava-Witten Braneworld

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Abstract

The nature of black holes in the Horava-Witten Braneworld is examined. The behaviour of particles and nature of singularities are examined and compared with the more familiar dilaton black hole and Extremal Reissner-Nordström models. Various parameters are investigated to see if singularities in the curvature are real or co-ordinate in nature. We show that the model implies a naked singularity not hidden behind a horizon.

1 Introduction

While String theory attempts to provide a description of the universe we live in that is mathematically very elegant, it is questionable whether it is in fact a scientific theory, as it has not thus far produced any falsifiable predictions. One area that does hold promise is in looking at the accretion around black holes. Black holes are the best laboratory available for testing out theories of gravity, as we are able to see the effects of the strong gravity limit, such as accretion and gravitational lensing. By investigating these observables, it will be possible to determine the nature of gravity, whether it is best described by Einstein's GR, string theory, or something else. In order to determine this, one needs to establish the metric, which describes how particles move through space-time, which is different for each model, and is also frame dependent. One would expect the different metrics describing black-holes would predict different accretion patterns. On Earth, we are only able to see the weak gravity limit of these theories, which all reduce to Newtons gravity. By looking at the properties of black-holes in several different models, one can compare their predicted accretion processes with experimental data. Although our current technology is not capable of resolving the accretion flow near black holes to the precision required, it is hoped that in a couple of years, the technology will be in place to determine which of these models is the most accurate.

At a very simplified level, string theory takes the basic constituents of nature to be extended objects called strings, rather than point particles. In this picture, particles are taken to be quantized vibrational modes on the string. The strings may be open or closed, depending on the particular theory. However, this theory is not unique, and in fact there are 5 distinct string theories, all of which exhibit supersymmetry. For example, type IIA string theory is obtained by multiplying fermions of opposite chirality, as opposed to multiplying fermions of the same chirality, which yields type IIB string theory.

M-Theory is a more recent development, which includes many different 11-dimensional extended objects. It is possible to obtain all 5 string theories from this picture, and in fact one can transform between the different string theories by dimensionally lifting them from 10 dimensions to 11-D M-Theory. For more detail, see[5]

Although the Schwarzchild solution to the Einstein Field Equations is the most commonly used metric describing space-time around a black hole, it is not the only one. I will be examining the properties of several other solution, their curvature and singularities. In particular, I will be considering a metric calculated by Tupper, Kim and Viollier by compactifying the dimensions of 11-d M-Theory in the Horava-Witten Braneworld model, and examining the behaviour of geodesics near the singularities.

It is important to understand the nature of the objects that we will considering. A black hole is understood to be a region of space-time from which nothing, not even light can escape. In the Schwarzchild case, this region, containing a singularity, is contained within a horizon. Outside of this horizon, light can still escape. This implies that the black-hole is only the region inside the horizon. However, in the case of the Horava-Witten Braneworld, the space time is only define in a region from which light can escape, although singularities are present. Hence this is not a true "black hole".

2 Extremal Reissner-Nordström model for black holes

In order to gain insight into the results that will be obtained for the Horava-Witten braneworld, is is useful to compare it to other alternative black hole solutions. On such alternatve is the Reissner-Nordström model, which describes a black-hole that is static (i.e. no angular momentum), containing some electric charge. By contrast, The Schwarzchild solution describes a black-hole with no charge or angular momentum. This solution can be obtained from both GR and conformal Brans-Dicke theory, which is an alternative gravitaional field theory. In the case of the extremal Reissner-Nordström model, where the charge is maximal the line element is given by

$$ds^2 = \left(1 - \frac{R_s}{2r}\right)^2 dt^2 - \frac{1}{\left(1 - \frac{R_s}{2r}\right)^2} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{1}$$

and

$$g_{ab} = \begin{pmatrix} \left(1 - \frac{R_s}{2r}\right)^2 & 0 & 0 & 0 \\ 0 & -\frac{1}{\left(1-\frac{R_s}{2r}\right)^2} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}$$

Computing the covariant Ricci Tensor, one obtains

$$R_{ab} = \begin{pmatrix} \frac{(2r-Rs)^2Rs^2}{16r^6} & 0 & 0 & 0 \\ 0 & -\frac{Rs^2}{r^2(2r-Rs)^2} & 0 & 0 \\ 0 & 0 & \frac{Rs^2}{4r^2} & 0 \\ 0 & 0 & 0 & \frac{Rs^2sin^2\theta}{4r^2} \end{pmatrix}$$

$$R = 0 \tag{2}$$

$$R_{\mu\nu}R^{\mu\nu} = \frac{Rs^4}{4r^8} \tag{3}$$

While the singularity is there in the R_{rr} component of the Rici tensor, this model doesn't contain any singularities in the Ricci scalar or contracted Ricci tensor at $2r = Rs$, implying that it is a co-ordinate singularity, rather than an actual singularity, although the singularity at $r = 0$ is a real one. Here, and for the rest of the paper, R_s refers to the Schwarzchild radius of the black-hole, given by $R_s = 2GM/c^2$.

3 Dilatonic black holes

When dealing with the Einstein equations, it is often useful to be able to work in several different frames. This is not unlike performing particle scattering calculations first in the CM frame, then transforming back to the lab frame. The two frames of interest are the Einstein frame, and the Jordan (or string) frame. Calculations are typically much easier to perform in the Einstein frame, but looking at particles in this frame does not tell you much about the physics involved, as particles in the Einstein frame do not follow geodesics. It is best to start in the string frame, where particles DO follow geodesics, then perform a conformal transformation to the Einstein frame, calculate the quantities one requires, such as the curvature, then perform another conformal transformation to take the results back to the string frame and examine them there.

A conformal transformation maps a metric in one frame to another via

$$\hat{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \tag{4}$$

where Ω is a smooth continuous function.

Another formulation that has been used is one involving the dilaton, a particle of the scalar field associated with gravity in string theory, coupled with matter. This model is obtained by considering 10-d String theory, dimensionally reduced down to 4-d. By performing a Kaluza-Klein dimensional reduction, one can obtain a 4-d effective action for both the String and Einstein frames.

$$S_{EF} = \frac{1}{2}\int d^4x\sqrt{(-g)}\left[\frac{1}{\ell_p^2}\left(R_{(g)} - \frac{1}{2}g^{\mu\nu}\nabla_{\mu\phi}\nabla_{\nu\phi}\right) - \frac{1}{\alpha^2}e^{-\phi}F^2\right] \tag{5}$$

and

$$S_{SF} = \frac{1}{2}\int d^4x\sqrt{(-G)}e^{-\phi}\left[\frac{1}{\lambda_s^2}\left(R_{(G)} + G^{\mu\nu}\nabla_{\mu\phi}\nabla_{\nu\phi}\right) - \frac{1}{\alpha^2}F^2\right] \tag{6}$$

where $g^{\mu\nu}$ is the metric in the Einstein frame, $G^{\mu\nu}$ is the metric in the Einstein frame, R relates to the curvature, λ_s is the string length and α is the electromagnetic coupling constant. $G^{\mu\nu}$ is related to $g^{\mu\nu}$ via the conformal transformation

$$G_{ij} = e^{\phi} \tag{7}$$

If one then looks at the extremal limit solution (the maximum charge that the black hole can have an still be a physics solution) in 10-D reduced down to 4 dimensions, on obtains in the Einstein frame a metric of the form

$$g_{ab} = \begin{pmatrix} 1 - \frac{R_-}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{R_-}{r}} & 0 & 0 \\ 0 & 0 & -r^2\left(1 - \frac{R_-}{r}\right) & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta\left(1 - \frac{R_-}{r}\right) \end{pmatrix}$$

The Ricci tensor and Ricci scalar for this solution are

$$R_{ab} = \begin{pmatrix} \frac{R_-^2}{2r^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{R_-^2}{2r^2} & 0 \\ 0 & 0 & 0 & \frac{R_-^2 sin^2\theta}{2r^2} \end{pmatrix}$$

$$R = \frac{R_-^2}{2r^3(r - R_-)} \tag{8}$$

While the Ricci Tensor is regular at r = R_-, R does contain a singularity there. In addition, the dilaton field itself, given by

$$e^{-\phi} = \left(1 - \frac{R_-}{r}\right)^{-1} \tag{9}$$

is singular at $r = R_-$. This frame is then meanigless for considering geodesic behaviour, as particles in the Einstein frame do not follow Geodesic motion. One needs to make the conformal transformation to the String frame.

In the string frame, the metric has the form

$$g_{ab} = \begin{pmatrix} \left(1 - \frac{R_-}{r}\right)^2 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 1r^2\left(1 - \frac{R_-}{r}\right)^2 & 0 \\ 0 & 0 & 0 & r^2 sin^2\theta \left(1 - \frac{R_-}{r}\right)^2 \end{pmatrix}$$

The Ricci tensor and Ricci scalar for this solution are

$$R_{ab} = \begin{pmatrix} \frac{2R_-^2}{r^4} & 0 & 0 & 0 \\ 0 & \frac{2R_-}{r^2(r-R_-)} & 0 & 0 \\ 0 & 0 & -\frac{R_-}{r} & 0 \\ 0 & 0 & 0 & -\frac{R_- sin^2\theta}{r} \end{pmatrix}$$

$$R = \frac{4R_-^2}{r^2(r - R_-)^2} \tag{10}$$

4 The Horava-Witten model

While there are several ways of performing dimensional reduction on higher dimensional (n > 4) theories, for obtaining this metric, 10-d string theory is lifted up to 11-d M-theory. In this picture, 2 10-d branes are separated by a distance, which constitutes the 11th dimension. The branes are then dimensionally reduced down to 4-d, with the extra separation dimension still there. This is then further reduced down to effective 4-d actions (S_{bulk} and S_{brane}). The metric under consideration is obtained from the effective action of the brane.

$$S_{eff} = \frac{1}{2K}\int d^4x\sqrt{g}\left[\frac{3(\nabla\psi)^2}{2\cosh\psi} - R\cosh\psi\right] + S_m \tag{11}$$

where the matter contribution is given by

$$S_m = \int d^4x\sqrt{-g}(\cosh\psi)^{\frac{3}{2}}\left[|D\Phi|^2 - \left|\frac{\partial W}{\partial\Phi}\right|^2 - \frac{F^2}{4}\right] \tag{12}$$

In the Eistein frame, this is given by [1]

$$S_{eff} = \int d^4x\sqrt{-g}\left[-\frac{R}{2K} + \frac{3}{16K}\frac{1}{(1+\varphi)\varphi}(\nabla\varphi)^2\right] + S_m \tag{13}$$

The most general form of the line-element is in the Einstein frame is [3]

$$ds^2_{EF} = U^{2\alpha}dt^2 - U^{-2\alpha}\left[dr^2 + r^2U^2d\Omega^2\right] \tag{14}$$

$$U = \sqrt{1 - \frac{R_s}{\alpha r}} \tag{15}$$

$$\phi = \sqrt{2\left(1-\alpha^2\right)}\ln U \tag{16}$$

This definition of U and α sets that the line-element is only defined for $R_s/\alpha \leq r \leq \infty$ and $0 \leq \alpha \leq 1$.

Transforming to the String frame via the conformal transformation via

$$\hat{g}_{\mu\nu} = (1+\varphi)^{-\frac{3}{2}}\, g_{\mu\nu} \tag{17}$$

with

$$\varphi = \sinh^2\psi \tag{18}$$

and

$$\varphi = \sinh^2\sqrt{\frac{2}{3}}\phi \Longrightarrow \psi = \sqrt{\frac{2}{3}}\phi \tag{19}$$

Here ϕ represents the scalar field.

$$ds^2_{SF} = \frac{2\left[\left(1-\frac{R_s}{\alpha r}\right)^{\alpha+\epsilon}dt^2 - \left(1-\frac{R_s}{\alpha r}\right)^{\epsilon-\alpha}dr^2 - r^2\left(1-\frac{R_s}{\alpha r}\right)^{1+\epsilon-\alpha}d\Omega^2\right]}{1+\left(1-\frac{R_s}{\alpha r}\right)^{2\epsilon}} \tag{20}$$

where

$$\epsilon = \sqrt{\frac{1-\alpha^2}{3}} \tag{21}$$

Computing the Ricci Scalar for this metric with $u = 1 - R_s/\alpha$ gives the leading term as

$$R = \frac{\alpha^4 - \alpha^2}{2(1+u^{2\epsilon})R_s^2 u^{2+\epsilon-\alpha}} \tag{22}$$

which indicates that there will be not be a singularity at $r = R_s/\alpha$ provided $2+\epsilon-\alpha < 0 \Longrightarrow \alpha - \epsilon > 2$ However, from the restrictions on α and ϵ, such a combinastion is not possible, as the largest value of $\alpha-\epsilon$ is $1-0=1$.

Since we are unable to avoid a situation with a singularity, we examine the case where $\alpha = 1/2$.

In this case, the result in the String frame for the line element concerned is

$$ds^2 = \frac{1-\frac{2R_s}{r}}{1-\frac{R_s}{r}}dt^2 - \frac{1}{1-\frac{R_s}{r}}dr^2 - \frac{r^2\left(1-\frac{2R_s}{r}\right)}{1-\frac{R_s}{r}}d\theta^2 - \frac{r^2 sin^2\theta\left(1-\frac{2R_s}{r}\right)}{1-\frac{R_s}{r}}d\phi^2 \tag{23}$$

The metric itself is diagonal:

$$g_{ab} = \begin{pmatrix} \frac{1-\frac{2R_S}{r}}{1-\frac{R_S}{r}} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{R_S}{r}} & 0 & 0 \\ 0 & 0 & -\frac{r^2\left(1-\frac{2R_S}{r}\right)}{1-\frac{R_S}{r}} & 0 \\ 0 & 0 & 0 & -\frac{r^2 sin^2\theta\left(1-\frac{2R_S}{r}\right)}{1-\frac{R_S}{r}} \end{pmatrix}$$

The metric contains a singularity at r = 2R$_s$. However, the metric is only defined for r $\geq 2R_s$. Recall from (14) that $2R_s/ \leq r \leq \infty$ for $\alpha = 1/2$. The question at this stage is to determine if the singularity at r = $2R_s$ is real , or merely a result of our choice of co-ordinates. The best way of doing this is to look at invariants of this metric. If the singularities are merely a result of our choice of co-ordinates, they should not be present when we examine the invariants, such as the Ricci scalar (R^μ_μ) and the contracted Ricci ($R^{\mu\nu}R_{\mu\nu}$). These can be calculated from the Christoffel sybols, which in turn are calculated from the metric.

The covarient Ricci tensor R_{ab} is found to be

$$\begin{pmatrix} -\frac{R_s^2(3R_s-r)}{2r^2(R_s-r)^2(2R_s-r)} & 0 & 0 & 0 \\ 0 & \frac{R_s^2(10r^2-21R_sr+8R_s^2)}{2r^2(R_s-r)^2(2R_s-r)^2} & 0 & 0 \\ 0 & 0 & \frac{R_s^2(r^2-3R_sr+4R_s^2)}{2r(R_s-r)^2(2R_s-r)} & 0 \\ 0 & 0 & 0 & \frac{R_s^2 sin^2\theta(r^2-3R_sr+4R_s^2)}{2r(R_s-r)^2(2R_s-r)} \end{pmatrix}$$

The Ricci scalar is

$$R = -\frac{9R_s^2}{2r^2(2R_s - r)(R_s - r)} \tag{24}$$

$$R^{\mu\nu}R_{\mu\nu} = \frac{R_s^4\left[2(r^2 - 3rR_s + 4R_s^2)^2 + r^2(3R_s - r)^2 + (10r^2 + 8R_S^2 - 21R_sr)^2\right]}{4r^6(R_s - r)^2(2R_s - r)^4} \tag{25}$$

The contracted Ricci also contains the same singularity at r = $2R_s$.

It is interesting to contrast this against the results from the Randall-Sundrum model, which is also a braneworld, but put together "by hand“, not derived from string theory. At low energies, this reduces to conformal Brans-Dicke theory. An acceptable solution of this in the Einstein frame is the Extremal Reissner-Nordström with $\alpha = \frac{1}{2}$. With $\alpha = \frac{1}{2}$ it is always possible to conformally transform to a frame where the solution looks like the extremal RN solutionwhich is non-singular. However, as was noted at the beginning, it is only in the String (Jordan) frame that the particles follow geodesics, so transforming to this new frame still doesn't give much information about the particle behaviour unless it corresponds to the Jordan frame.

If one compares this the Ricci Tensor of the Dilaton model, one sees that the singularities are are present in both cases, with the square of the bracket containing the singularity appearing in the denominator of the R_{rr} component.

5 Geodesic behaviour

Of particular interest is the behaviour of Geodesics around the singularity. To calculate the null radial geodesics (always in the String frame) we set $d\theta = d\phi = ds = 0$ From this we obtain

$$\left(1 - \frac{R_s}{\alpha r}\right)^\alpha = \frac{dr}{dt} \tag{26}$$

$$\int_{t_0}^{t} dt = \int \frac{dr}{\left(1 - \frac{R_s}{\alpha r}\right)^\alpha} \tag{27}$$

$$= \int_{r_0}^{r} \frac{r^\alpha dr}{\left(r - \frac{R_s}{\alpha}\right)^\alpha} \tag{28}$$

As $r \to R_s/\alpha$, the integral approaches a finite value if $\alpha < 1$, but diverges if $\alpha = 1$ In the case concerned, with $\alpha = 1/2$, this inplies that it takes a finite amount of co-ordinate time for the particle to travel from the horizon to a point further away.

By comparison, the behaviour of the null radial geodesic in the Schwarzchild case goes like

$$\left(1 - \frac{2GM}{r}\right) = \frac{dr}{dt} \tag{29}$$

$$\int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{dr}{\left(1 - \frac{2GM}{r}\right)} \tag{30}$$

which diverges as $r_0 \to 2GM$, implying that it takes an infinite amount of time for light to escape from the horizon.

For the Extremal Reissner-Nordström case

$$\left(1 - \frac{GM}{r}\right)^2 = \frac{dr}{dt} \tag{31}$$

$$\int_{t_0}^{t} dt = \int_{r_0}^{r} \frac{dr}{\left(1 - \frac{GM}{r}\right)^2} \tag{32}$$

which also diverges as $r_0 \rightarrow GM$. All of these cases are just specific cases of (25), with $\alpha = 1$ and 2 for the Schwarzchild and extremal RN solutions respectively.

6 Discussion

In examining both the Dilaton and the Horava-Witten cases, String theory has been used as the starting point. However the methods of dimensional reduction are very different. Both of these cases contain singularities in their metrics and their curvature scalars (R). An important difference in the two results is that that while the KTV metric contains 'tidal' charges, the Dilaton model requires an extremal electric charge, a highly unlikely situation.

We see that both of these string theory approximations for a black hole suffer from the same problem, namely a naked singularity (one not hidden behind a horizon)[5]. Such a situation is not permitted by General Relativity. However this is not GR. It is unclear, though, exactly what these objects are. If such objects exist, then it would be possible to detect them via gravitational lensing.

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