Phantom Gravastar: An explanation to compact dark matter objects?

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Abstract

The understanding of dark matter objects is one of the modern mysteries. Usually one interpret them as Black Holes, but there is no empirically evidence only a lack of alternatives. Proposing the idea of a Phantom Gravastar we will give an alternative explanation to these dark matter objects without supposing them to be Black Holes, in particular to SgrA* and the MACHOs.

Contents

1 Introduction

The existence of dark matter and compact massive dark matter objects is beyond doubt. Dark matter even makes up most of the matter in the universe. Usually these kind of massive compact dark matter objects are interpreted as Black Holes, but this interpretation is purely based on a lack of different explanations. To assume a Black Hole describing these objects might seem convenient, since these objects don't emit radiation or at least too little for a baryonic object, but this might also be due to an extended non-baryonic object. Some interesting candidates for dark matter objects are MACHOs (Massive Halo Objects) and the heavy compact object in the center of the Milkyway.

In the last years the central object in the Milkyway (Sgr A*) has been subject to many speculations, which resulted in a quite well determined mass. Furthermore, observations of stars orbiting around SgrA* led to quite stringent restrictions to it's size. It is commonly believed that SgrA* is a massive Black Hole, but there have also been other explanations, like a massive ball of uncharged fermions [5], which can't be excluded.

The MACHOs are compact dark objects orbiting in the galactic Halo with masses below one solarmass. These objects cause a microlensing event, of a star in a neighboring galaxy, when the Macho crosses the line of sight between observer and star. An increase of brightness of the star can be observed. Macho are also commonly expected to be Black Holes, but again this is only due to a lack of alternative explanations.

A promising attempt to construct non-black-hole objects has been done by Mazur and Mottola [14], who proposed the idea of the de-Sitter gravastar, described by an equation of state $p = -\rho$. This object is contributed out of a quantum vacuum which undergoes a phase transition at the event horizon. The interior is de-Sitter matched to the Schwartzschild spacetime near the event horizon, leading to the result that the de-Sitter gravastar has no singularity at the origin and no event horizon. A striking feature is that this gravastar has a fixed size.

In the line of ideas similar to the de-Sitter gravastar, N. Bilic, G. Tupper and R.D. Viollier proposed the idea of a Phantom Gravastar [7], which can explain a very large scale of dark matter objects up to the most massive ones without being a Black Hole. It is basically described by the equation of state $p = -\frac{A}{\rho}$ where $\rho^2 < A$, leaving the kind of matter unspecified.

Since we are only dealing with one particular cosmological constant A to describe all objects, it is convenient to use a more general equation of state $p = -\frac{A}{\rho^n}$ for $\rho < A^{\frac{1}{n+1}}$. This gives us more flexibility in the mass-radius-relation of the object. The condition $\rho < A^{\frac{1}{n+1}}$ leads to the equation $p - \rho < 1$, which is the regime of the "phantom energy" [15], hence we will call the resulting object again a Phantom Gravastar. As we will see later on, it's possible to derive a mass-radius-relation, which gives us the opportunity to test, if these compact dark matter objects can be considered as Phantom Gravastars.

2 Phantom Gravastar

2.1 Einstein Field Equation for spherical symmetric spacetime

In this part we are going to consider how the metric for the Gravastar shall look like and what the implications for the Field-Equations are.

The Schwartzschild-metric, which reads

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2)$$

describes a pointmass in space-time, with a singularity at the origin, where the singularity at $r = 2GM$ is just a result of choice of coordinates [12]. While there is little doubt that the metric describes the outside physics of the Schwartzschild-radius R_S correctly, it is still unclear if it describes the physics of the inside correctly. There is no definite observational evidence for the existence of such singularities, therefore we are going to construct a metric without such singularities. Hence we substitute the constant M by $M(r)$, due to the varying mass inside the gravastar, and the time-component of the Schwartzschild-metric by an arbitrary positive function $\zeta(r)$, which will be determined in the next chapter.

$$ds^2 = \zeta^2(r)dt^2 - \frac{dr^2}{1 - 2G\frac{M(r)}{r}} - r^2(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2) \qquad (1)$$

Our choice for the energy momentum tensor is $T_\mu^\nu = diag(\rho, -p, -p, -p)$, since we want to consider a simple matter distribution. Using this together with (1) Einstein's field equation become [1]

$$M' = 4\pi r^2 \rho \qquad (2)$$

$$\zeta' = G\zeta \frac{M + 4\pi r^3 p}{r(r - 2GM)} \qquad (3)$$

while $T^\nu_{\mu;\nu} = 0$ gives

$$p' = -(\rho + p)\frac{\zeta'}{\zeta} \qquad (4)$$

We are going to take

$$p = -\frac{A}{\rho^n} \qquad (5)$$

where $n > 0$ real, as the equation of state for the Gravastar. Furthermore we will restrict our calculations to the phantom regime $\rho < A^{\frac{1}{n+1}}$ as we are going to motivate in the next chapter.

2.2 The Lapsefunction

We are going to derive the Lapsefunction and give reason to the condition $\rho < A^{\frac{1}{n+1}}$.

We want to have a density which is decreasing from $\rho_0 \equiv \rho(0)$, the center of the gravastar, up to $\rho(R) = 0$ where R marks the radius of the star. This is a convenient condition for the physical experience of well known matter structures. Furthermore we would like the Lapsefunction $\zeta(r)$ to hold the following three conditions, since we want to match (1) to the Schwartzschild-metric and avoid any singularities. At first $\zeta(0) = 1$, where we chose the value of 1 simply in analogy to the Schwartzschild-metric, which time-component $(1 - \frac{2GM}{r}) < 1$ everywhere outside the Schwartzschild-radius R_S. Furthermore we would like to have ζ decreasing up to the Schwartzschild-radius where $\zeta(R_S) = 0$,which is necessary since we are going to match (1) with the Schwartzschild-metric somewhere outside R_S.

In the following we are going to consider three different conditions for ρ with respect to A, and conclude in the end that we can drop two of them.

If $\rho = A^{\frac{1}{n+1}}$ equation (5) leads to

$$P = -\rho$$

which is the equation of state of the de-Sitter gravastar [4], where the Lapsefunction is given by

$$\zeta(r) = \left(1 - \frac{r^2}{{R_s}^2}\right)^{\frac{1}{2}}$$

Therefore we won't consider this condition anymore.

If $\rho < A^{\frac{1}{n+1}}$ we can calculate the Lapsefunction in the following way.

Starting from equation (4) and substituting (5) one obtains:

$$p' = -(\rho + p)\frac{\zeta'}{\zeta}$$

$$\Longleftrightarrow \quad \frac{-An\rho^{-n-1}}{\rho - \frac{A}{\rho^n}}\rho' = \frac{\zeta'}{\zeta}$$

$$\Longleftrightarrow \quad \frac{-n}{\frac{1}{A} - \frac{1}{\rho^{n+1}}}\frac{1}{\rho^{n+2}}\rho' = \frac{\zeta'}{\zeta}$$

$$\Longleftrightarrow \quad \int_{\rho_0}^{\rho(r)} \frac{-n}{\frac{1}{A} - \frac{1}{\rho^{n+1}}}\frac{1}{\rho^{n+2}}d\rho = \int_{\zeta(0)}^{\zeta(r)} \frac{1}{\zeta}d\zeta$$

where $\rho_0 \equiv \rho(0)$ is the central density. By rescaling t one can set $\zeta(0) = 1$, therefore

$$
\begin{aligned}
\ln(\zeta(r)) &= \int_{\rho_0}^{\rho(r)} \frac{-n}{\frac{1}{A} - \frac{1}{\rho^{n+1}}} \frac{1}{\rho^{n+2}} d\rho \\
&= \int_{\rho_0}^{\rho(r)} \frac{\frac{-n}{n+1}}{\frac{1}{\rho^{n+1}} - \frac{1}{A}} \frac{d}{d\rho}\left(\frac{1}{\rho^{n+1}} - \frac{1}{A}\right) d\rho \\
&= \frac{n}{n+1}\left[-\ln\left(\frac{1}{\rho^{n+1}} - \frac{1}{A}\right)\right]_{\rho_0}^{\rho(r)} \\
&= \frac{-n}{n+1} \ln\left(\frac{\frac{1}{\rho^{n+1}} - \frac{1}{A}}{\frac{1}{\rho_0{}^{n+1}} - \frac{1}{A}}\right) \\
&= \frac{-n}{n+1} \ln\left(\frac{\rho_0{}^{n+1}}{\rho^{n+1}} \frac{A - \rho^{n+1}}{A - \rho_0{}^{n+1}}\right) \\
&= \frac{n}{n+1} \ln\left(\left(\frac{\rho}{\rho_0}\right)^{n+1} \frac{A - \rho_0{}^{n+1}}{A - \rho^{n+1}}\right)
\end{aligned}
$$

Therefore the lapse function is given by:

$$
\zeta(r) = \left(\frac{\rho}{\rho_0}\right)^{n} \left(\frac{A - \rho_0{}^{n+1}}{A - \rho^{n+1}}\right)^{\frac{n}{n+1}} \tag{6}
$$

The Lapsefunction for $\rho > A^{\frac{1}{n+1}}$ is given by the same expression, since one can basically run through the above calculation, simply multiplying $\frac{1}{A} - \frac{1}{\rho^{n+1}}$ by -1, whenever it is necessary to ensure the positivity of the argument of the logarithm.
To see that we can drop the condition $\rho > A^{\frac{1}{n+1}}$, just substitute (5) into (4) and calculate the derivative of P, then:

$$
\frac{\frac{-nA}{\rho^{n+1}}}{1 - \frac{A}{\rho^{n+1}}} \frac{\rho'}{\rho} = \frac{\zeta'}{\zeta}
$$

One can see that $\zeta' < 0$ only holds, if $\rho < A^{\frac{1}{n+1}}$, since we also assumed ρ to be decreasing. While $\rho > A^{\frac{1}{n+1}}$ leads to either an increasing ζ or an increasing density, which we both want to avoid as mentioned in the beginning of this chapter. Therefore we can restrict the equation of state of the gravastar to the condition $\rho < A^{\frac{1}{n+1}}$ and drop the other two.

2.3 Derivation of the density

The next step is to obtain a differential equation for the density with respect to r.

The combination of equation (3) and (4) leads to the Oppenheimer-Tollmann-Volkoff-Equation [13]:

$$p' = -(\rho + p)G\frac{M + 4\pi r^3 p}{r(r - 2GM)} \tag{7}$$

substituting (5) into (7) leads to

$$\begin{aligned} An\rho^{-(n+1)}\rho' &= \left(\frac{A}{\rho^n} - \rho\right) G\frac{M - 4\pi r^3 \frac{A}{\rho^n}}{r(r - 2GM)} \\ \Longleftrightarrow \quad \rho' &= \rho^{n+1}\left(\frac{1}{\rho^n} - \frac{\rho}{A}\right) \frac{G}{n}\frac{M - 4\pi r^3 \frac{A}{\rho^n}}{r(r - 2GM)} \\ \Longleftrightarrow \quad \rho' &= \left(1 - \frac{\rho^{n+1}}{A}\right) \frac{G}{n}\frac{M - 4\pi r^3 \frac{A}{\rho^n}}{r(r - 2GM)}\rho \end{aligned} \tag{8}$$

which we take as the differential equation for the density.

Now we solve (8) in the small central density approximation $\rho_0 \ll A^{\frac{1}{n+1}}$. This leads with (5) to a much higher pressure than the density, which is contrary to the validity of Newtonian physics, where the pressure is much smaller than the density. Hence this approximation reflects the intention to describe an object for which the effects of general relativity are significant; we might also call it "antinewtonian" approximation. Furthermore we want to describe an object which is not a black hole, i.e.: which radius is larger than the Schwartzschild-radius $2GM$, hence we make the additional assumption: $2GM(r) << r$. We are going to show the self-consistency of this assumption later on.

Substitution of $\rho_0 \ll A^{\frac{1}{n+1}}$ into (8) leads to

$$\begin{aligned} \rho' &= \frac{G}{n}\rho\frac{M - 4\pi r^3 \frac{A}{\rho^n}}{r(r - 2GM)} \\ &= -\frac{G}{n}\rho\frac{4\pi r^3 \frac{A}{\rho^n}}{r(r - 2GM)} \end{aligned}$$

and using $2GM << r$ one obtains:

$$\rho' = -\frac{G}{n}A\frac{4\pi r}{\rho^{n-1}} \tag{9}$$

$$
\begin{aligned}
\iff & \quad n\rho^{n-1}\rho' = -4G\pi r A \\
\iff & \quad \int_{\rho_0}^{\rho(r)} n\rho^{n-1} d\rho = -\int_0^r 4G\pi r A dr \\
\iff & \quad \rho^n(r) - {\rho_0}^n = -2\pi G A r^2 \\
\iff & \quad \rho(r) = \rho_0 \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{n}} \qquad (10)
\end{aligned}
$$

with

$$
R^2 \equiv \frac{\rho_0^n}{2\pi G A} \qquad (11)
$$

is defined as the radius of the star, since the density vanishes there.

2.4 The Mass-radius-relation

In the following we derive the mass-radius-relation of the phantom gravastar, which we will use to test the predicting power of the Phantom Gravastar concerning SgrA* and the MACHOs. Substituting (10) into (2) one obtains:

$$M' = 4\pi r^2 \rho_0 \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{n}}$$

Integrating this expression from 0 to R, the radius of the star, gives the total mass $M \equiv M(R)$ of the star.

$$M = \int_o^R 4\pi r^2 \rho_0 \left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{n}} dr$$

Changing the variable $r = zR$ leads to:

$$M = 4\pi\rho_0 R^3 \int_0^1 z^2 (1 - z^2)^{\frac{1}{n}} dz$$

The solution of this integral [2]is given by

$$M = 2\pi\rho_0 R^3 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n} + 1\right)}{\Gamma\left(\frac{1}{n} + \frac{5}{2}\right)} \tag{12}$$

Using that according to (11) $\rho_0 = (2\pi G A R^2)^{\frac{1}{n}}$ one obtains that

$$M = (2\pi)^{\frac{n+1}{n}} (GA)^{\frac{1}{n}} R^{3+\frac{2}{n}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n} + 1\right)}{\Gamma\left(\frac{1}{n} + \frac{5}{2}\right)} \tag{13}$$

Therefore the mass-radius-relation reads:

$$\frac{M}{R^{3+\frac{2}{n}}} = (2\pi)^{\frac{n+1}{n}} (GA)^{\frac{1}{n}} \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n} + 1\right)}{\Gamma\left(\frac{1}{n} + \frac{5}{2}\right)} \tag{14}$$

or in an equivalent notation:

$$\frac{M^{\frac{n}{3n+2}}}{R} = (2\pi)^{\frac{n+1}{3n+2}} (GA)^{\frac{1}{3n+2}} \left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n} + 1\right)}{\Gamma\left(\frac{1}{n} + \frac{5}{2}\right)}\right)^{\frac{n}{3n+2}} \tag{15}$$

Obviously this gives us a radius distribution of the Phantom Gravastar depending on the mass:

$$R(M) = M^{\frac{n}{3n+2}}(2\pi)^{-\frac{n+1}{3n+2}}(GA)^{-\frac{1}{3n+2}}\left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(\frac{1}{n}+\frac{5}{2}\right)}\right)^{-\frac{n}{3n+2}} \quad (16)$$

One can easily see from the calculation above that the mass inside the radius of the gravastar (i.e.: $0 \leq r \leq R$) is given by:

$$M(r) = 4\pi\rho_0 R^3 \int_0^{\frac{r}{R}} z^2(1-z^2)^{\frac{1}{n}} dz \quad (17)$$

since one only integrates from 0 to r.

Now we are going to consider the mass radius relation for very big and small n. From equation (14) it is obvious that for large n

$$R \propto M^{\frac{1}{3}}$$

While for $n \to 0$ we obtain a radius, which is independent of the enclosed mass. This is a similar behaviour than that of the de Sitter gravastar.

To see that the assumption $2GM << r$ is self consistent, we simply substitute (12) into $2GM$.

$$\frac{2GM}{R} = 4\pi G\rho_0 R^2 \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}+\frac{5}{2})}$$

Substituting (11) leads to:

$$\frac{2GM}{R} = 2\frac{{\rho_0}^{n+1}}{A}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}+\frac{5}{2})}$$

Finally use $\rho_0 << A^{\frac{1}{n+1}}$ to see that

$$\frac{2GM}{R} << 1$$

2.5 Comparison with Polytropes

We are going to compare the equation of state for polytropes with (5) via the mass radius relation, but first a short overview about polytropes.
The equilibrium of a star is described by the following equations [3].

$$\frac{dp}{dr} = -\frac{GM(r)}{r^2}\rho$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

or

$$\frac{1}{r}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

while for polytropes one assumes the following equation of state:

$$p = K\rho^{1+\frac{1}{m}}$$

Substituting this into the above equation and defining $\rho \equiv \lambda\Theta^m$, $r = a\zeta$ with $a = \left(\frac{(m+1)K}{4\pi G}\lambda^{\frac{1}{m}-1}\right)^{\frac{1}{2}}$ we obtain the so called Lane-Emden-Equation.

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\Theta}{d\zeta}\right) = -\Theta^m$$

Polytropes are described by solutions of this equation. From the second equation of stellar equillibrium and the Lane-Emden-Equation one can derive an mass-radius-relation, which reads

$$GM^{\frac{m-1}{m}}R^{\frac{3-m}{m}} = \frac{(m+1)K}{(4\pi)^{\frac{1}{m}}}\left[-\zeta^{\frac{m+1}{m-1}}\frac{d\Theta}{d\zeta}\right]^{\frac{m-1}{m}}_{\zeta=\zeta_1}$$

Where ζ_1 denotes the first zero of Θ. So we can read this mass radius relation as:

$$M \propto R^{1-\frac{2}{m-1}}$$

where m is a real number.

Since we also obtained a mass-radius-relation for the gravastar, it might be interesting to compare the behaviour of the relation for polytropes, which qualitatively describes the behaviour of even relativistic stellar objects quite often.

In the small density approximation (14) the mass-radius-relation of the gravastar obeys the following relation:

$$M \propto R^{3+\frac{2}{n}}$$

We want to obtain a relation between n and m, for the situation that in both cases the mass is related to the radius with the same power. Thus a comparison of the powers leads to:

$$\begin{aligned} & 3 + \frac{2}{n} = 1 - \frac{2}{m-1} \\ \Longleftrightarrow \quad & \frac{1}{n} = \frac{1}{1-m} - 1 \\ \Longleftrightarrow \quad & \frac{1}{n} = \frac{m}{1-m} \\ \Longleftrightarrow \quad & n = \frac{1}{m} - 1 \\ \Longleftrightarrow \quad & m = \frac{1}{n+1} \end{aligned} \tag{18}$$

which gives us information about how both equations of state have to be related, to lead to a similar mass-radius-relation. Additionally we obtain the condition that $0 < m < 1$, since $n > 0$.

2.6 Junction condition for the shell

This chapter is dedicated to the matching of the metric (1) inside the Schwartzschild-metric. The aim is to derive the junction equation, which describes the surface density and the surface tension of the shell. The shell is a hypersurface in 3 + 1 dimensional spacetime, caused by fixing the variable r at $r = R_0$. We choose R_0 in such a way that the density is small but still nonzero, since for a vanishing density the pressure will become infinite. The junction equation will be used to derive an equation of state for the shell in the next chapter.

We denote with R_0 the radius of the shell, which should fulfill $R_S < R_0 < R$ since we want to avoid the singularities of the Schwartzschild-metric. Furthermore shall the radius R_0 be constant, since we are only going to consider a static gravastar. Let M_0 be defined as the mass of the shell plus $M(R_0)$, which is the mass of the gravastar star inside the radius of the shell R_0. Finally denote by σ the surface density and by θ the surface tension.

To match the metrics we use Israel's junction condition [6]:

$$\left[\left[K^b_a - \delta^b_a K^c_c\right]\right] = 8\pi G S^b{}_a \tag{19}$$

where $S^a{}_b \equiv diag(\sigma, \theta, \theta)$ denotes the surface stress energy tensor and $[[f]]$ denotes the discontinuity in f on the shell, i.e.

$$[[f(r)]] = \lim_{x \to 0}(f(R_0 + x) - f(R_0 - x))$$

The extrinsic curvature is defined as [6]:

$$K_{ij} \equiv e_j \nabla_{e_i} n$$

where $i, j = 1, 2, 3$ and e_1, e_2, e_3 form an orthonormal basis of the hypersurface. For the shell $i, j = t, \vartheta, \varphi$ and the basisvectors become $e_i{}^\alpha = \delta^\alpha{}_i$. The unitnormal to the hypersurface is denoted with n, for the shell we choose the unitnormal to be

$$n^\alpha = \sqrt{1 - \frac{2GM(r)}{r}}\delta^\alpha{}_r \tag{20}$$

Before we actually calculate the extrinsic curvature, we are going to derive an equivalent expression which avoids the derivative of the unitnormal n. Writing down K_{ij} in terms of indices and using Leibnitz rule one finds:

$$K_{ij} = (e_j{}^\alpha n_\alpha)_{;\beta} e_i{}^\beta - n_\alpha e_{j;\beta}{}^\alpha e_i{}^\beta$$

Since n and e_j are orthogonal we find

$$K_{ij} = -n_\alpha e_{j;\beta}{}^\alpha e_i{}^\beta$$

Substituting $e_i{}^\alpha = \delta^\alpha{}_i$ into this equation and writing the kovariant derivative in terms of Christoffel-symbols gives us the extrinsic curvature for the inside of the shell.

$$K_{ij} = -n_\alpha \left(\delta^\alpha{}_{i,j} + \Gamma^\alpha{}_{\beta\gamma}\delta^\beta{}_i\delta^\gamma_j\right)$$

Actually doing the given contractions, substituting (18) with the index down and using that $\delta^\alpha{}_{i,j} = 0$ shows that

$$K_{ij} = \frac{1}{\sqrt{1 - \frac{2GM(r)}{r}}}\Gamma^r{}_{ij} \tag{21}$$

Therefore the next step is to calculate the Christoffel-Symbols for the metric (1). The general expression for them in terms of the metric tensor is

$$\Gamma^b{}_{ij} = \frac{1}{2}g^{ab}(g_{aj,i} + g_{ai,j} - g_{ij,a})$$

Since $i, j = t, \vartheta, \varphi$ and g is diagonal this simplifies to

$$\Gamma^r{}_{ij} = -\frac{1}{2}g^{ar}g_{ij,a}$$

and we only have three nonzero Christoffel-Symbols:

$$\Gamma^r{}_{\vartheta\vartheta} = -r\left(1 - \frac{2GM(r)}{r}\right)$$

$$\Gamma^r{}_{\varphi\varphi} = -sin^2(\vartheta)r\left(1 - \frac{2GM(r)}{r}\right)$$

$$\Gamma^r{}_{tt} = \zeta\zeta'\left(1 - \frac{2GM(r)}{r}\right)$$

Substituting this into equation (19) and raising an index shows us that all three nonzero components of the extrinsic curvature inside the shell are given by

$$K^t_t = \frac{\zeta'}{\zeta}\left(1 - \frac{2GM(r)}{r}\right)^{\frac{1}{2}} \tag{22}$$

and

$$K_\varphi^\varphi = K_\vartheta^\vartheta = \frac{1}{r}\left(1 - \frac{2GM(r)}{r}\right)^{\frac{1}{2}} \tag{23}$$

With a similar calculation one finds for the extrinsic curvature of the Schwarzschild-metric outside the shell:

$$K_t^t = \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-\frac{1}{2}} \tag{24}$$

$$K_\varphi^\varphi = K_\vartheta^\vartheta = \frac{1}{r}\left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} \tag{25}$$

Considering the junction equation (19) again one finds for the surface density

$$[[-2K_\vartheta^\vartheta]] = 8\pi G\sigma$$

Substituting (23) and (25) gives

$$\left(1 - \frac{2GM(R_0)}{R_0}\right)^{\frac{1}{2}} - \left(1 - \frac{2GM_0}{R_0}\right)^{\frac{1}{2}} = 4\pi G R_0 \sigma$$

Using that the gravastar is much bigger than a black hole, i.e. $\frac{2GM_0}{R_0} << 1$, we can expand this equation by adding $\left(\frac{GM_0}{R_0}\right)^2$ and $\left(\frac{GM(R_0)}{R_0}\right)^2$ inside each square root respectively. Hence we obtain for the first junction equation:

$$\frac{M(R_0)}{R_0} - \frac{M_0}{R_0} = -4\pi R_0 \sigma \tag{26}$$

Considering (19) for the surface tension leads us to:

$$[[-K_t^t - K_\vartheta^\vartheta]] = 8\pi G\theta$$

Substituting (22), (24) and directly using equation (26) leads to

$$\frac{\zeta'}{\zeta}\left(1 - \frac{2GM(R_0)}{R_0}\right)^{\frac{1}{2}} - \frac{GM_0}{R_0^2}\left(1 - \frac{2GM_0}{R_0}\right)^{-\frac{1}{2}} - \frac{GM(R_0)}{{R_0}^2} + \frac{GM_0}{{R_0}^2} = 8\pi G\theta$$

Using (3) we obtain

$$\frac{G}{R_0^2}\frac{M(R_0) + 4\pi R_0^3 P}{\left(1 - \frac{2GM(R_0)}{R_0}\right)^{\frac{1}{2}}} - \frac{GM_0}{R_0^2}\left(1 - \frac{2GM_0}{R_0}\right)^{-\frac{1}{2}} - \frac{GM(R_0)}{{R_0}^2} + \frac{GM_0}{{R_0}^2} = 8\pi G\theta$$

We expand the square roots with the same method than for equation (26), which gives us finally

$$\frac{1}{2}R_0 p = \theta \tag{27}$$

2.7 Equation of state for the shell

In this chapter we are going to derive the equation of state of the shell. Starting with (26) gives us for the surface density

$$\frac{M(R_0)}{R_0} - \frac{M_0}{R_0} = -4\pi R_0 \sigma$$
$$\Longleftrightarrow \quad \sigma = \frac{1}{4\pi}\left(\frac{M_0}{{R_0}^2} - \frac{M(R_0)}{{R_0}^2}\right)$$

Inserting (17):

$$\sigma = \frac{M_0}{4\pi {R_0}^2} - \frac{R^3}{{R_0}^2}\rho_0 \int_0^{\frac{R_0}{R}} z^2(1-z^2)^{\frac{1}{n}} dz \qquad (28)$$

Now we are going to calculate $\frac{M_0}{4\pi {R_0}^2}$. Take $R_0 = xR$ with $x \in (0,1)$ and $M_0 = \mu M$ with $\mu > 0$. Hence

$$\frac{GM_0}{R_0} = \frac{GM}{R}\frac{\mu}{x}$$

Substituting (12) and using the notation $\Gamma = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{n}+1)}{\Gamma(\frac{1}{n}+\frac{5}{2})}$ leads to

$$\begin{aligned}\frac{GM_0}{R_0} &= 2\pi G R^2 \rho_0 \Gamma \frac{\mu}{x} \\ &= \frac{{\rho_0}^{n+1}}{A}\Gamma\frac{\mu}{x}\end{aligned}$$

$$\begin{aligned}\Longrightarrow \frac{M_0}{4\pi {R_0}^2} &= \frac{1}{4\pi R_0 G}\frac{M_0 G}{R_0} \\ &= \frac{{\rho_0}^{n+1}}{4\pi R_0 G A}\Gamma\frac{\mu}{x}\end{aligned}$$

Using again the notation $R_0 = xR$ one obtains

$$\frac{M_0}{4\pi {R_0}^2} = \frac{1}{4\pi RGA}{\rho_0}^{n+1}\Gamma\frac{\mu}{x^2}$$

while substituting (11) leads to

$$\frac{M_0}{4\pi {R_0}^2} = \frac{1}{2}\Gamma R \rho_0 \frac{\mu}{x^2}$$

Inserting this again into equation (28) and using $\frac{R_0}{R} = x$ leads to the surface density:

$$\sigma = R\rho_0 \left(\frac{1}{2}\Gamma \frac{\mu}{x^2} - \frac{1}{x^2} \int_0^x z^2 (1-z^2)^{\frac{1}{n}} dz \right) \tag{29}$$

From (27) we obtain for the surface tension:

$$\theta = \frac{1}{2} R_0 p$$

Inserting into this expression the basic condition (5) leads to:

$$\theta = -\frac{1}{2} R_0 \frac{A}{\rho^n}$$

Simply substituting (10) and using again the definition of x leads to

$$\theta = -\frac{1}{2} x R \frac{A}{\rho_0{}^n (1-x^2)} \tag{30}$$

It is desirable to have an equation of state which is independent of the central density and the radius of the shell. Thus we try to find powers of σ and θ which cancel these parameters. We can use for this again the relation (11) $R^2 \propto \rho_o{}^n$.
Obviously $\frac{2}{n} + 1$ is a sufficient power, since

$$\sigma \theta^{\frac{2}{n}+1} \propto R\rho_0 \frac{R^{\frac{2}{n}+1}}{\rho_0{}^{n\frac{2}{n}+1}} \propto \rho_0{}^{\frac{1}{2}[n+2+n(\frac{2}{n}+1)-2n(\frac{2}{n}+1)]} = const.$$

Taking this to the power of $\frac{n}{2+n}$ leads to the shell equation of state.

$$\sigma^{\frac{n}{2+n}} \theta = -\frac{1}{2} \frac{Ax}{1-x^2} \left(\frac{1}{2\pi G A} \right)^{\frac{n+1}{n+2}} \left(\frac{1}{2}\Gamma \frac{\mu}{x^2} - \frac{1}{x^2} \int_0^x z^2 (1-z^2)^{\frac{1}{n}} dz \right)^{\frac{n}{2+n}} \tag{31}$$

3 Application of the Phantom Gravastar

3.1 Describing the most massive object

The uniqueness of the cosmological constant restricts the $n = 1$ model concerning the mass-radius-relation, for an arbitrary $n > 0$ these restrictions become less stringent and we gain a more flexible mass-radius-relation. In this chapter we are going to determine the cosmological constant A.
The increase of mass is accompanied by an increase of density. Hence there will be a mass, such that $\rho = A^{\frac{1}{n+1}}$, while (8) shows us that the density will then be constant. $\rho = A^{\frac{1}{n+1}}$ leads us to the case of the de-Sitter-gravastar, where due to a phase transition at the shell only a unique constant A exists [14] to describe all objects. Furthermore the radius is independent of the enclosed mass and density,being equal to the event-horizon. Hence it is convienient to suppose the density reaching the value $A^{\frac{1}{n+1}}$ for the most massive objects in the universe. Substituting into the Einstein Field Equation (2) the above expression for the density and integrating this, we find

$$M(r) = \frac{4}{3}\pi r^3 A^{\frac{1}{n+1}} \tag{32}$$

To obtain an expression for the radius, only depending on A and G, we consider the Lapse-function for the de-Sitter-Gravastar, which must vanish at the event-horizon. Hence we get the equation:

$$\sqrt{1 - \frac{2GM(r)}{r}} = 0$$

Therefore the radius, where the Phantom Gravastar will collapse to become a Black Hole, is given by:

$$R = \sqrt{\frac{3}{8\pi G A^{\frac{1}{n+1}}}}$$

Substituting this into equation (32) and isolating A leads us to:

$$A = \left(\frac{3}{32\pi G^3 M^2}\right)^{n+1} \tag{33}$$

According to [8] the most massive objects will be found to be around $3 * 10^9 M_\odot$, hence we take $4 * 10^9 M_\odot$, which will result in a stronger condition for the gravastar radius, since it will become larger.

Substituting this mass and $G = 4.926 * 10^{-6} \frac{c^3 sec}{M_\odot}$ into (33) leads us to the $A(n)$.

$$A = \left(\frac{3}{16}\right)^{n+1} \frac{1}{G} \frac{1}{(2\pi)^{n+1}} \frac{1}{4^{2n+2}} \frac{1}{(4.926)^{3n+2}} 10^{-6} \tag{34}$$

Which reads in units of keV:

$$A = (8.394 keV)^{4(n+1)}$$

Substituting (34) into the radius distribution of the antinewtonian approximation and dividing by 499 gives us the radius in astronomical units [a.u.], if M is in units of solarmasses.

$$R(M) = \frac{4.926}{499} \left(\frac{256}{3}\right)^{\frac{n+1}{3n+2}} 10^{\frac{6}{3n+2}} \left(\frac{\Gamma\left(\frac{1}{n}+\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{n}+1\right)}\right)^{\frac{n}{3n+2}} M^{\frac{n}{3n+2}} \tag{35}$$

Thus the radius-function for the special case $n = 1$ reads:

$$R(M) = 1.206834174 M^{\frac{1}{5}} \tag{36}$$

3.2 Describing Sgr A*

The mass and the radius of the super massive object in the center of the milkyway are both well known. Thus it is convenient to use this data to set restrictions to the values of n. The mass of SgrA* is $3.7(\pm 0.2)10^6 M_\odot$ [8] while the radius must be less than 45a.u. [9].
For the values of n we find from the inequality $R(M) < 45$ a.u. the condition:

$$n > 0.4$$

If n satisfies this condition, the gravastar will lay well within the upper bound of SgrA*. For big n the radius approximates the value of

$$R = 7.695 a.u.$$

being far beyond the Schwartzschild-radius.

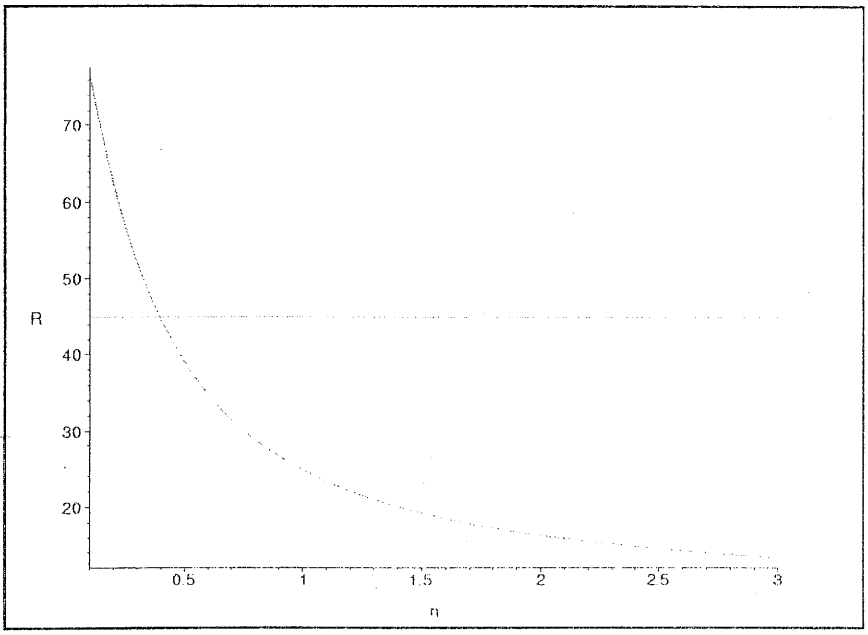

Figure 1: Radial behaviour of SgrA* with varying n, the straight line marks the upper bound of the size of SgrA*; R is in a.u.

3.3 Description of the MACHOs

The occurrence of Gravitational Microlensing events make the existence of compact dark matter objects in the Halo of the Milkyway very likely. A recent paper [11] give strong restrictions to low mass MACHOs, while [10] the existence of MACHOs with masses $m > 0.1M_\odot$ is very likely. Therefore we will consider the radial behaviour of the gravastar with masses between $0.1 and 0.9M_\odot$.
To give a restriction to the size of these MACHOs we will use [12] that they move with a velocity of $220km/s$ and that the duration of the microlensing-effect is given by $t = 130\sqrt{\frac{m}{M_\odot}}$ days. Combining these two informations gives us an upper bound for the size of a MACHO:

$$R(M) < 16.51781269\sqrt{M}$$

Using (33) for the radius we find for MACHOs with mass 0.3 and $0.5M_\odot$ radii of

$$R(M = 0.3) = 0.9485753846 a.u.$$

$$R(M = 0.5) = 1.050610170 a.u.$$

both within the upper bounds of 9.047178611 and 11.67985736 a.u. respectively.

Using again that one can choose an arbitrary $n > 0.4$ one finds that for increasing n the radius will decrease up to a particular value:

$$R = 0.034 a.u. \quad \text{and} \quad R = 0.04 a.u.$$

for $M = 0.3$ and $M = 0.5M_\odot$ respectively. It remains to mention that the choice of the upper bound for the radius is not very stringent. A far better one would be the following: Assuming that the matter of the gravastar is such, that light cannot penetrate it, there must exist a maximal radius of the MACHO, such that no light can reach the observer. Similar there will exist a radius of the MACHO, such that light can reach the observer. Of course, the idea of an extended object, which light cannot penetrate, includes that the common mass distribution is to small. Since for a pointmass no light will vanish behind the object. Hence if the object is extended the curvature of the light-paths must be stronger to supply the same increase in brightness and therefore the mass of the object must be higher.
This connection between the extension of the object and it's mass can be used either to determine whether the object can be penetrated by light or if the gravastar is sufficiently small.

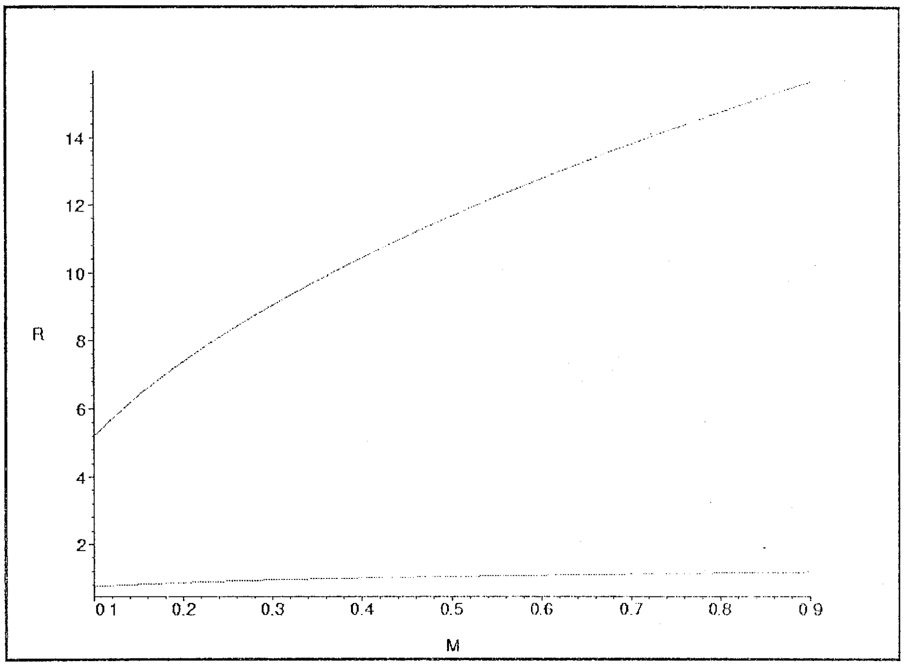

Figure 2: Gravastar (lower line) compared with the upper bound of the Machos, mass in $M_{\odot}$ and R in a.u.

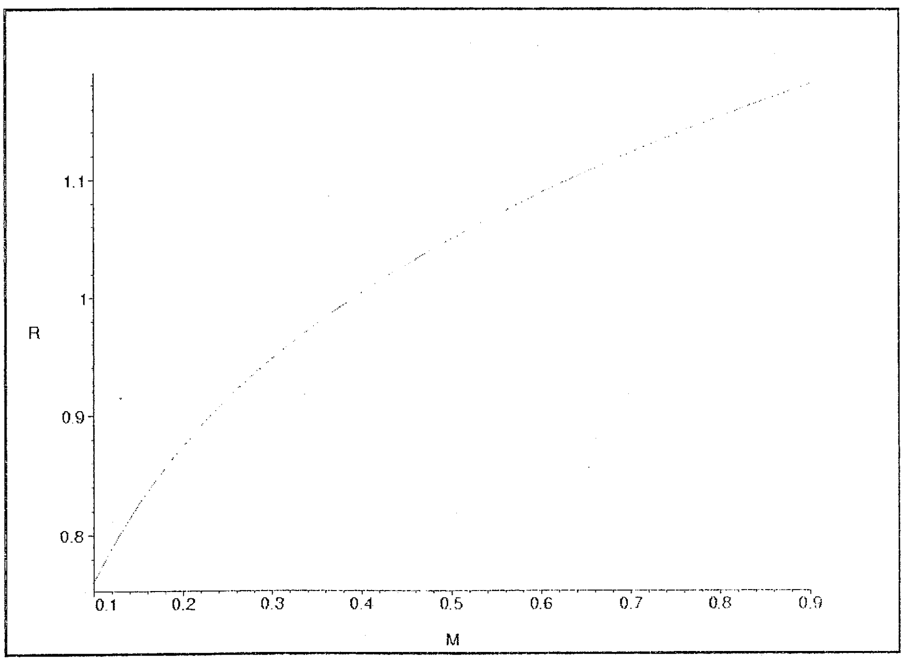

Figure 3: Radius (in a.u.) of the Gravastar with masses between 0.1 and 0.9 $M_{\odot}$

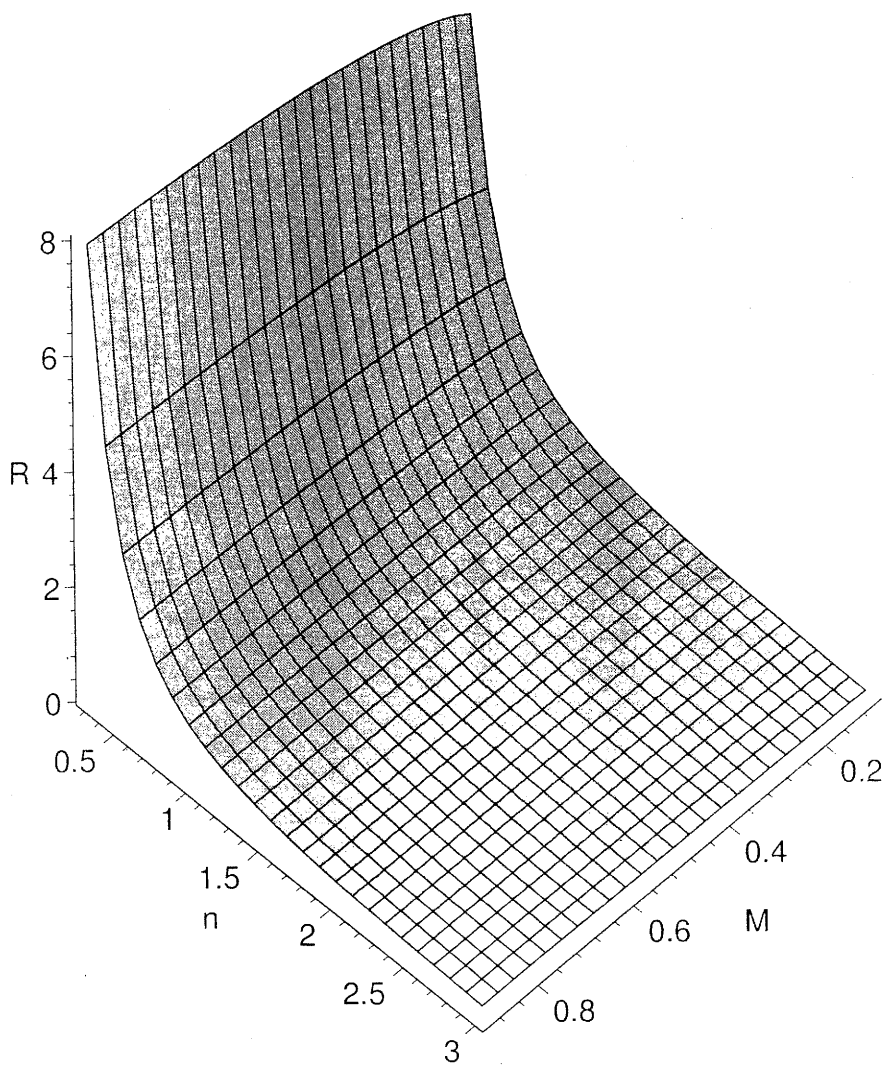

Figure 4: Radius of Gravastar with varying masses and powers n in between 0 and 8 $M_{\odot}$, and $0.4 < n < 3$ respectively

4 Conclusion

For $n > 0.4$ the gravastar can describe Machos in a mass range of 0.1 up to $0.9M_{\odot}$, where we derived the restriction of size by the effect-duration. It can also describe very massive compact dark matter objects such as Sgr A*.
If n is increasing the radius will decrease, converging to a particular value, while it will always remain larger than the event horizon. The Phantom Gravastar model succeeds in explaining compact dark matter object without being a black hole.

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