Kosmology: Cosmology of K-essence with Baryons in an inhomogeneous Universe

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Abstract

We consider the viability of a two-component cosmological model based on a shear-free, spherically symmetric spacetime. It is found that a function of integration arising in the solution of this metric can be interpreted conveniently as a "geometric" baryon representation, and we employ a generic k-essence scalar field as a unified dark matter component. Finally, we construct the Hamiltonian for this model and quantize it to obtain the Wheeler-DeWitt equation for the wave function of the universe.

1 Introduction

This paper will look at an inhomogeneous k-essence dominated cosmology. Key to this scenario is the unification of the dark sector (dark matter and energy) into a single entity.

In the context of general relativity, assumption of the cosmological principle implies a maximally symmetric Friedmann-Robertson Walker model. If one interprets recent astrophysical observations within the framework of this model, one is led to the conclusion that the baryonic matter with which we are familiar constitutes but a tiny fraction of the total energy of the universe. Measurements of the rotation curves of galaxies as well as galaxy cluster dynamics indicate the presence of a large amount of nonbaryonic *dark matter*. Furthermore, the accelerated expansion inferred from observations of the light curves of Ia Supernovae and the Cosmic Microwave Background (CMB) implies in this context the existence of a negative pressure *dark energy* accounting for roughly 70% of the energy density of the universe. Primordial nucleosynthesis calculations constrain the total baryon content to only a few percent ($\Omega_b \approx 0.04$).

The standard "Λ-cold dark matter" (ΛCDM) or "Concordance" model invokes a cosmological constant to account for dark energy, alongside baryons and cold dark matter. Attempts to describe this constant as a quantum field theoretic vacuum energy have failed spectacularly: QFT calculations give a value 120 orders of magnitude too large. The model also suffers from the so-called coincidence problem, that is, the question of why dark matter and dark energy densities should be comparable today. Such difficulties are alleviated to a greater or lesser degree by various dynamical scalar field theories. Conventionally these have been used to model dark energy, but more recently attempts have been made at unifying dark matter and dark energy using nonstandard forms of such fields. K-essence is the name given to a generic class of scalar fields with possible noncanonical kinetic terms.

In light of the aforementioned difficulties with the FRW model, and bearing in mind that we do not, in fact, live in a homogeneous universe, we shall relax slightly the restrictive assumptions employed there. Instead, we shall leave isotropy about a single point, and allow for inhomogeneity subject to this constraint. The metric

$$ds^2 = N^2\, dt^2 - R^2\, d\vec{x}^2 \tag{1}$$

describes a *shear-free* spherically symmetric space-time, where the lapse function N and scale-factor R (analogous to that of the FRW model) are functions of r and t only. The shear-free case is considered for the sake of mathematical simplicity. This criterion is frequently prioritized owing to the otherwise intractable nature of the Einstein field equations. The idea is that with a sufficiently simple mathematical form of the metric one can integrate the field equations analytically. The arbitrary functions of integration that arise can then be chosen in such a way as to obtain a physically reasonable equation of state. Perhaps owing to the particularly simple form of (1), these shear-free spherically symmetric solutions have received a lot of attention and there exists already a considerable literature on the subject. A fairly comprehensive survey is given by Sussman [4].

The first systematic study of the above metric (1) was carried out by Wyman in 1946 [2]. Assuming nonzero pressure and a perfect fluid with barotropic equation of state (i.e.

of the form $P = P(\rho)$), he derived a new set of solutions involving elliptic functions. However, these turn out to be unphysical([5],[6]), as will be shown in section (2.2). We shall later examine a particularly simple analytic solution which we identify as a special case of this solution.

In contrast to the rather mathematical approach described above, we would like some freedom to be able to specify the matter content of the universe, and proceeding from there determine the geometry and dynamics. Now k-essence, not being strictly barotropic, potentially circumvents on its own the pathologies of the Wyman model. Introducing baryons, an important, albeit minor, feature of our actual universe, gives us a two-component matter content which can no longer be described as a perfect fluid, rendering immaterial the objections to the Wyman solution. It is not necessary to engage in the thoroughgoing analysis of Clarkson et al. [7] if one adopts the argument of Raychaudhuri [10] that all peculiar velocities are small, allowing one to establish a common comoving coordinate system. We find somewhat fortuitously that the function of integration arising in the so-called pressure isotropy equation has the interpretation of a "comoving" density of a pressureless dust (baryon) component. This dust however has a *negative* energy density, and it is in this light that we can understand the failure of the original Wyman solution.

A new class of models is obtained then by the inclusion of a *positive* energy density dust component. $R(t, r)$ can be calculated for each time slice from a first-order (non-linear) differential equation - our Friedmann equation analogue.

With a view to quantising the above model, we wish to obtain the action from which the field equations are derived. We have, prima facie, a problem in that the pressureless baryons have a zero Lagrangian density (for a fluid $\mathcal{L} = P$). However, a 'geometric' dust is introduced into the action via the Ricci scalar term using the pressure isotropy equation. With the complete action thus in hand, we proceed to the Hamiltonian formulation via the Arnowitt-Deser-Misner formalism, and quantize the model, whence we can study an inhomogeneous k-essence dominated quantum cosmology.

2 The Spherical Model

2.1 The Friedmann Model

The cosmological principle, or the claim that the universe is homogeneous and isotropic, has historically been taken as a point of departure for the construction of cosmological models. By purely geometric reasoning it leads to the familiar Friedmann-Robertson-Walker (FRW) metric[1]:

$$ds^2 = dt^2 - R(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta dr^2 \right] \tag{2}$$

where the curvature constant k can always be renormalized to $k = +1, 0, -1$ through an appropriate redefinition of the coordinates. If one further assumes Einstein's gravitational

[1]We set $c = 1$ throughout

field equations to hold, this leads to the following equations describing the dynamics of the universe:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \qquad \text{Friedmann equation} \tag{3}$$

and

$$\dot{H} + H^2 = 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = -8\pi G P \tag{4}$$

where the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ measures the expansion rate and $a \equiv R/R_0$ is the scale factor. R_0 is the value of R today. These two equations are equivalent to

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + P) \qquad \text{conservation equation} \tag{5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{6}$$

Equations (3) and (6) are usually taken as the independent dynamical equations governing the model. It would also be useful to be able to specify some conditions on the properties of the matter, without being limited to a particular equation of state. Several standard "energy conditions" have been formulated for this purpose. They are defined with reference to the energy-momentum tensor, but in the case of a perfect fluid reduce to inequalities involving the density and pressure:

(i) The null energy condition: $\rho + P \geq 0$

(ii) The weak energy condition: $\rho \geq 0,\ \ \rho + P \geq 0$

(iii) The strong energy condition: $\rho + P \geq 0,\ \ \rho + 3P \geq 0$

(iv) The dominant energy condition: $\rho \geq |p|$ (Ensures that mass-energy is always subluminal with respect to any observer)

2.2 Generalisation to the Spherical Model

Given that the assumptions of perfect homogeneity and isotropy underlying the above model are rather restrictive, and certainly not realistic beyond a very crude approximation, it is worth looking for a generalization which might afford more flexibility in our description without rendering the equations intractable. As a first step in that direction we consider the (non-static, inhomogeneous) spherical model, a simple generalization of FRW. The most general form of spherically symmetric line element is

$$ds^2 = \mathrm{A}\ dt^2 - \mathrm{B}\ dr^2 - \mathrm{C}\ \left(r^2\ d\theta^2 + r^2 \sin^2\theta\ d\phi^2\right) + \mathrm{D}\ dr\ dt \tag{7}$$

where A, B, C and D are functions of r and t only. With the further assumption of zero shear, via a suitable transformation the metric can be written as follows in "comoving" coordinates:

$$ds^2 = e^{2\nu}\ dt^2 - e^{2\mu}\left(dr^2 + r^2\ d\theta^2 + r^2 \sin^2\theta\ d\phi^2\right) \tag{8}$$

where $\mu = \mu(r,t)$, $\nu = \nu(r,t)$ and $U^\mu = e^{-\nu}\ \delta^\mu_t$.

We start with a general analysis similar to those of Wyman [2] and Kustaanheimo & Qvist[1]. It is interesting to note, as pointed out by Krasínski in his preface to the reprint

of [1], that each of the particular solutions considered by Kustaanheimo and Qvist was later rediscovered, often on numerous occasions (twenty in the case of the conformally flat solution) - an indication of how uncoordinated work on this subject has been.

Denoting $\partial/\partial t$ and $\partial/\partial r$ by "$\cdot$" and "$'$" respectively, the Einstein equations are (Tolman[9] page 252)

$$G^t_t = 3\left(e^{-\nu}\dot{\mu}\right)^2 - e^{-2\mu}\left(2\mu'' + \mu'^2 + 4\mu'/r\right) = 8\pi G\rho \tag{9}$$

$$G^r_r = 3\left(e^{-\nu}\dot{\mu}\right)^2 + 2e^{-\nu}\left(e^{-\nu}\dot{\mu}\right)^{\cdot} - e^{-2\mu}\left(\mu'^2 + 2\mu'\nu' + 2(\mu'+\nu')/r\right) = -8\pi GP \tag{10}$$

$$G^\theta_\theta = G^\phi_\phi = 3\left(e^{-\nu}\dot{\mu}\right)^2 + 2e^{-\nu}\left(e^{-\nu}\dot{\mu}\right)^{\cdot} - e^{-2\mu}\left(\mu'' + \nu'' + \nu'^2 + (\mu'+\nu')/r\right) = -8\pi GP \tag{11}$$

$$G^t_r = -2e^{-\nu}\left(e^{-\nu}\dot{\mu}\right)' = 0 \tag{12}$$

Eliminating P using (10) and (11) we obtain

$$\mu'' + \nu'' + \nu'^2 = \mu'^2 + 2\mu'\nu' + (\mu'+\nu')/r \ . \tag{13}$$

while (12) upon integrating gives

$$e^{-\nu}\,\dot{\mu} = H(t) \tag{14}$$

or

$$\nu = \ln\left(\frac{\dot{\mu}}{H(t)}\right) \tag{15}$$

The stress-energy tensor T^ν_μ has the familiar perfect fluid form $T^0_0 = \rho$, $T^i_i = -P$, $T^i_j(i \neq j) = 0$. From the longitudinal and transverse parts of $T^\nu_{\mu\ ;\nu} = 0$ we obtain the conservation and Euler equations respectively.

The conservation equation is:

$$U^\mu\, T^\nu_{\mu\ ;\nu} = U^\mu\ \rho_{,\mu} + (\rho + P)\ U^\mu_{;\mu} = 0 \tag{16}$$

$$\Rightarrow \quad e^{-\nu}\ \dot{\rho} + 3H\,(\rho + P) \ = \ 0 \tag{17}$$

or

$$\dot{\rho} + 3\dot{\mu}\ (\rho + P) \ = \ 0 \ . \tag{18}$$

Euler's equation

$$U^\nu\ U_{\mu;\nu} \ = \ \left(\delta^\nu_\mu - U_\mu U^\nu\right)\ \frac{P_{,\nu}}{\rho + P} \tag{19}$$

is satisfied trivially except for the $\mu = r$ case, which gives

$$\nu' + \frac{P'}{\rho + P} \ = \ 0 \ . \tag{20}$$

Equations (13) and (15) combine to give

$$\begin{aligned} \mu'^2 + 2\mu'\left(\frac{\dot{\nu}'}{\dot{\mu}}\right) + \frac{1}{r}\left(\nu' + \frac{\dot{\mu}'}{\dot{\mu}}\right) \ &= \ \mu'' + \left(\frac{\dot{\mu}'}{\dot{\mu}}\right)' + \left(\frac{\dot{\mu}'}{\dot{\mu}}\right)^2 \\ &= \ \mu'' + \frac{\dot{\mu}''}{\dot{\mu}} \ . \end{aligned} \tag{21}$$

and multiplying by $\dot{\mu}\, e^{\mu}$, we get

$$\left(e^{\mu}\,\mu'^2\right)^{\cdot} + \left(\frac{1}{r}\,e^{\mu}\,\mu'\right)^{\cdot} = \left(e^{\mu}\,\mu''\right)^{\cdot} \quad . \tag{22}$$

This can be integrated immediately to give the so-called "pressure-isotropy equation"

$$e^{\mu}\left(\mu'^2 + \frac{\mu'}{r} - \mu''\right) = \frac{3}{2}k(r) \tag{23}$$

with arbitrary integration function $k(r)$. Inserting this into the G^t_t field equation (9) then yields a "Friedmann-like" equation:

$$\begin{aligned} G^t_t = 8\pi G\rho &= 3H^2 - e^{-2\mu}\left(2\mu'' + \mu'^2 + \frac{4}{r}\mu'\right) \\ &= 3H^2 - e^{-2\mu}\left(3\mu'^2 + 6\frac{\mu'}{r} - 3k(r)e^{-\mu}\right) \end{aligned}$$

$$\Rightarrow \quad \frac{8\pi G\rho}{3}\rho = H^2 - e^{-2\mu}\left(\mu'^2 + 2\frac{\mu'}{r} - k(r)e^{-\mu}\right) \tag{24}$$

or defining $R \equiv e^{\mu}$ ($\Rightarrow \mu' = R'/R$) we can write

$$H(t)^2 - \frac{1}{R^2}\left[\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right) - \frac{k(r)}{R}\right] = \frac{8\pi G}{3}\rho \tag{25}$$

In principle one can now specify H and ρ (subject to (20)) and for each coordinate time slice solve (25) for the 'scale factor' R. This will be elaborated upon later.

An alternative strategy is the following. Divide 22 by r^2 to obtain

$$\begin{aligned} 0 &= \left(\frac{e^{\mu}\mu'}{r^3} + \frac{e^{\mu}\mu'^2}{r^2} - \frac{e^{\mu}\mu'^2}{r^2}\right)^{\cdot} \\ &= \left(e^{2\mu}\left[\frac{e^{-\mu}\mu'}{r^3} - \frac{1}{r^2}\,(e^{-\mu}\mu')'\right]\right)^{\cdot} \\ &= \left(e^{2\mu}\left[-\frac{1}{r}\,\left\{\frac{1}{r}\,(e^{-\mu}\mu')\right\}'\right]\right)^{\cdot} \\ &= \left(e^{2\mu}\frac{1}{r}\left[\frac{1}{r}\,(e^{-\mu})'\right]'\right)^{\cdot} \end{aligned} \tag{26}$$

Changing to the variable $x \equiv \frac{r^2}{2}$ this becomes

$$e^{-2\mu}\,\frac{\partial^2}{\partial x^2}\,e^{-\mu} = f(x) \tag{27}$$

for arbitrary function of integration $f(x)$.

Here Einstein's equations for a non-static, spherically symmetric shear-free perfect fluid universe have been reduced to a second order linear differential equation in the scale factor, supplemented by equations for ρ and P. Kustaanheimo and Qvist point out that in

general $f(x) = (ax^2 + bx + c)^{-5/2}$ leads to solutions expressible in terms of elementary functions. Several solutions of (27) in terms of elliptic functions, such as those of Wyman [2], are also known. Given an exact solution of (27) for some function $f(x)$, one can solve for ρ using (9) with $e^{-\nu}\dot{\mu} = H(t)$, i.e.

$$G^t_t = 3H^2 - e^{-2\mu}\left(2\mu'' + \mu'^2 + \frac{4}{r}\,\mu'\right) = 8\pi G\rho \ . \tag{28}$$

and then P can be obtained from (18).

Let us consider some examples:

(i) $f(x) = 0$ gives the conformally flat solution. This is the so-called Stephani universe.

$$e^{-\mu} = \frac{1 - k(t)\, r^2/2}{a(t)} \qquad k(t),\ a(t) \text{ arbitrary functions of time} \tag{29}$$

We calculate ρ and P as described above:

$$\begin{aligned} 8\pi G\rho &= 3H^2 - e^{-2\mu}\left[2e^{\mu}\left(\frac{k}{a} + \frac{k^2}{a^2}r^2e^{\mu}\right) + \left(\frac{k}{a}r\right)^2 e^{2\mu} + 4\frac{k}{a}e^{\mu}\right] \\ &= 3H(t)^2 - 6\frac{k(t)}{a(t)^2} \\ \Rightarrow \quad \rho &= \rho(t) = \frac{3}{8\pi G}\left(H(t)^2 - \frac{2k(t)}{a(t)^2}\right) \qquad (30) \\ P &= -3C^2 + 2C\dot{C}\left(\frac{1 - kr^2/2}{a}\right)/\frac{\partial}{\partial t}\left(\frac{1 - kr^2/2}{a}\right) \qquad (31) \end{aligned}$$

where $C = C(t)$ is another arbitrary function of time. Thus the energy density is spatially homogeneous, whereas the pressure turns out to be inhomogeneous ($P = P(r,t)$). The Stephani solution is a simple generalization the FRW model. As in the latter, all spatial sections are homogeneous (disregarding pressure). An interesting feature of the Stephani universe is the possibility for the spatial curvature to change sign as it evolves with time: unlike in the FRW case, a closed universe may "open up", and vice versa.

(ii) $f(x) =$ constant ($6\alpha^2$ here) yields a special case of Wyman's solutions in terms of elliptic functions:

$$e^{-\mu} = \frac{1}{(\sigma(t) + \alpha\, r^2/2)^2} \qquad \sigma(t) \text{ is an arbitrary function of time,} \quad \alpha = \text{constant} \ . \tag{32}$$

Taking $H(t) =$ constant we obtain a barotropic equation of state, but with an unphysical *negative* value of the sound speed squared:

$$\begin{aligned} \mu' &= 2\alpha r e^{-\mu/2} \\ \mu'' &= 2\alpha e^{-\mu/2}\left(1 - \alpha r^2 e^{-\mu/2}\right) \\ 8\pi G\rho &= 3H^2 - e^{-2\mu}\left(2\mu'' + \mu'^2 + \frac{4}{r}\,u'\right) \\ &= 3H^2 - e^{-2\mu}\left(4\alpha e^{-\mu/2}\left(1 - \alpha r^2 e^{-\mu/2}\right) + 4\alpha^2 r^2 e^{-\mu} + 8\alpha e^{-\mu/2}\right) \end{aligned}$$

$$\rho = \frac{\left(3H^2 - 12\alpha e^{-5\mu/2}\right)}{8\pi G} \tag{33}$$

$$\begin{aligned}\dot{\rho} &= \frac{3\alpha}{2\pi G}\left(\frac{5\dot{\mu}}{2}\right)e^{-5\mu/2} \\ &= \frac{15\alpha}{4\pi G}\dot{\mu}e^{-5\mu/2}\end{aligned}$$

$$\begin{aligned}P &= -\frac{\dot{\rho}}{3\dot{\mu}} - \rho \\ &= -\frac{5\alpha}{4\pi G}e^{-5\mu/2} - \frac{3H^2 - 12\alpha e^{-5\mu/2}}{8\pi G} \\ &= \frac{\alpha}{4\pi G}e^{-5\mu/2} - \frac{3H^2}{8\pi G}\end{aligned} \tag{34}$$

The corresponding equation of state is then

$$P = \frac{1}{6}\left(\frac{3H^2}{8\pi G} - \rho\right) - \frac{3H^2}{8\pi G} \tag{35}$$

$$= -\frac{\rho}{6} - \frac{5H^2}{16\pi G} \tag{36}$$

and the adiabatic sound speed

$$c_s^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_{s/n} = -\frac{1}{6} \tag{37}$$

3 Dark Matter, Dark Energy and K-essence

Since the initial surprising discovery of an apparent accelerated expansion of the universe, further corroborating evidence from Type Ia supernovae and the cosmic microwave background has been mounting. In the Friedmann context, this implies through equation (6)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{38}$$

the existence of some form of "dark energy" which violates the strong energy condition ($\rho + 3P < 0$), giving rise to a positive acceleration. As it stands the data can still be accommodated through the inclusion of a cosmological constant in the field equations. As was mentioned in the introduction, however, the most obvious source of such a term, namely the quantum vacuum energy, is predicted to be many orders of magnitude too large. Observational evidence for dark energy currently says nothing about its evolution, so the cosmological constant need not, in fact, be constant. Scalar field theories arising out of, for example, string theory and particle physics, have been proposed as candidates for dark energy. The so-called quintessence models involve a (possibly inhomogeneous) dynamical scalar field minimally coupled to gravity. A general quintessence Lagrangian comprises a standard kinetic term together with potential, i.e.

$$\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - V(\phi) \tag{39}$$

Such models treat dark matter and dark energy separately, as in the ΛCDM model. A more recent proposal is that of Unified Dark Matter (UDM), also referred to as *quartessence*, which treats them as different manifestations of a single entity. An example is the Chaplygin Gas, an exotic fluid with equation of state

$$P = -\frac{A}{\rho} \ . \tag{40}$$

A key feature is that it interpolates between a decelerating dust-dominated universe and an accelerating cosmological constant dominated one. It also admits an equivalent scalar field formulation with the following Born-Infeld type Lagrangian

$$\mathcal{L} = -\sqrt{A}\,\sqrt{1 - X^2}, \qquad X^2 \equiv g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} \tag{41}$$

Interestingly, this can be interpreted as the effect of the immersion of a (3+1) dimensional brane (our space-time) in a (4+1) dimensional bulk. The Chaplygin gas does however suffer from the problem of frustrated structure formation. This is a challenge common to all models that seek to unify dark matter and energy - they must 'compete' with the so-called Jeans instability to evolve primordial perturbations into what we observe today as dark matter. This particular candidate is ruled out on these grounds; a "generalized Chaplygin gas" ($P = -A/\rho^\alpha,\ 0 \leq \alpha \leq 1$) has also been investigated, but is found to require severe fine-tuning in order to obtain agreement with the data.

Tachyon models constitute an immediate generalisation of the Chaplygin gas, in that the Lagrangian is simply that of equation (41) with a potential $V(\phi)$ in place of $\sqrt{A}$. These arise in the context of string theory, and are of interest in relation to the UDM scenario. In fact they are a special case of k-essence, a scalar field with non-canonical kinetic energy terms. Originally introduced as a model for inflation, k-essence is now being investigated as a candidate for UDM.

The most general k-essence action involves a scalar field ϕ and its first derivatives, so we can write

$$S = \int d^4x\ \sqrt{-g}\ \left[-\frac{R}{16\pi G} + \mathcal{L}(\phi, X)\right] \tag{42}$$

where $X^2 \equiv g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu}$. The associated field equation is then

$$\begin{aligned} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \mathcal{L}_\phi &= 0 \\ \frac{\partial X}{\partial \phi_{,\mu}} &= \frac{g^{\mu\nu}\phi_{,\nu}}{X} \\ \left(\mathcal{L}_X\ \frac{g^{\mu\nu}\phi_{,\nu}}{X}\right)_{;\mu} &= \mathcal{L}_\phi \end{aligned} \tag{43}$$

The energy-momentum tensor is defined as

$$T_{\mu\nu} \ = \ 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L} \tag{44}$$

leading to

$$T_{\mu\nu} \ = \ 2\,\mathcal{L}_X\ \frac{\partial \sqrt{g^{\mu\nu}\ \phi_{,\mu}\ \phi_{,\nu}}}{\partial g^{\mu\nu}} - g^{\mu\nu}\mathcal{L} \tag{45}$$

$$= \ \mathcal{L}_X\ \frac{\phi_{,\mu}\phi_{,\nu}}{X} - g_{\mu\nu}\ \mathcal{L} \tag{46}$$

This can be cast in perfect fluid form by defining the following hydrodynamic quantities:

$$\begin{array}{lll} \text{4-velocity} & U_\mu = \frac{\phi_{,\mu}}{X} & \\ \text{pressure} & P = \mathcal{L} & \\ \text{energy density} & \rho = X\,\mathcal{L}_X - \mathcal{L} & \end{array} \tag{47}$$

$$\Rightarrow \qquad \mathcal{L} = (\rho + P)\, U_\mu\, U_\nu - g_{\mu\nu}\, P \tag{48}$$

There is a correspondence between purely kinetic k-essence ($\mathcal{L} = \mathcal{L}(X)$) and the existence of a barotropic equation of state. For assume $\mathcal{L}$ to be a function of X only; then according to (47) we have

$$P = \mathcal{L} = P(X) \tag{49}$$

$$\rho = X\mathcal{L}_X - \mathcal{L} = \rho(X) \tag{50}$$

This defines a parametric equation of state, with the parameter X relating P and ρ. We therefore have an (implicit) barotropic equation of state. This we can demonstrate with the special Wyman solution discussed earlier. The equation of state was found to be

$$P = -\frac{1}{6}\rho - C \tag{51}$$

where $C = \frac{5H^2}{16\pi G}$. Using the relations (47)

$$\begin{aligned} P &= \mathcal{L} \\ \rho &= X\,\mathcal{L}_X - \mathcal{L} \end{aligned}$$

equation (51) becomes

$$\begin{aligned} \mathcal{L} &= -\frac{1}{6}(X\,\mathcal{L}_X - \mathcal{L}) - C \\ X\,\mathcal{L}_X &= -5\,\mathcal{L} - 6\,C \end{aligned}$$

An ansatz of the form $\mathcal{L} = a\,X^n + b$ leads to the somewhat exotic-looking Lagrangian

$$\mathcal{L} = A\,X^{-5} + K\,, \qquad X = \sqrt{g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu}} \tag{52}$$

with A arbitrary and $K = \frac{3H^2}{8\pi G}$. In this picture the sound speed is calculated as[11]:

$$\begin{aligned} c_s^2 &= \left.\frac{\partial P}{\partial \rho}\right|_\phi \\ &= \frac{\partial p/\partial X}{\partial \rho/\partial X} \\ &= \frac{\mathcal{L}_X}{X\mathcal{L}_{XX}} \end{aligned}$$

giving as before

$$\rho = -\frac{1}{6} \tag{53}$$

4 Model building

Consider a two-component universe containing baryons (b) and a k-essence field (ϕ):

$$\begin{aligned} P &= P_b + P_\phi = P_\phi \\ \rho &= \rho_b + \rho_\phi \end{aligned}$$

The conservation equation applies separately to each component. For baryons

$$\begin{aligned} \dot{\rho}_b + 3\,\dot{\mu}\,\rho_b &= 0 \\ \frac{\dot{\rho}}{\rho} &= -3\,\dot{\mu} \\ \rho_b &= \frac{F(r)}{e^{3\mu}} \end{aligned} \tag{54}$$

for arbitrary function $F(r)$. Comparing this with equation (25), note that by redefining $k(r) = -\frac{8\pi G}{3}B(r)$ we can move this term across to the right hand side and identify it as a "geometric dust":

$$H(t)^2 - \frac{1}{R^2}\left[\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right)\right] = \frac{8\pi G}{3}\rho + \frac{8\pi G}{3}\frac{B(r)}{R^3} = \frac{8\pi G}{3}\rho_{eff} \tag{55}$$

This sheds some light on the special Wyman solution. The pressure isotropy equation there becomes

$$\begin{aligned} k(r) &= \frac{2}{3}\,e^{\mu}\left(\mu'^2 + \frac{\mu'}{r} - \mu''\right) \\ &= \frac{2}{3}\,e^{\mu}\left(4\alpha^2 r^2 e^{-\mu} + 2\alpha e^{-\mu/2} - 2\alpha e^{-\mu/2} + 2\alpha^2 r^2 e^{-\mu}\right) \\ &= 4\alpha^2 r^2 \\ &> 0 \end{aligned}$$

In light of equation (55) it is now clear why this solution is unphysical: the geometric dust has a negative energy density, in flagrant contravention of the null energy condition.

To rectify this, one can incorporate a dust component into ρ_1, chosen such that the total baryon energy density is positive. Taking then as the remaining matter content a k-essence scalar field, one has a two-component, inhomogeneous, k-essence dominated cosmological model. Note that the expansion can be specified freely via $H(t)$ in (55). Thus at the centre one could choose $H(t)$ to agree with ΛCDM, and with $\sigma(t) = \sqrt{(a)}$, reproduce the Friedmann metric at $r = 0$.

5 Towards K-essence Quantum Cosmology

5.1 The Wheeler-DeWitt Equation

Quantum cosmology is a semiclassical approach to quantizing gravity. Since any classical theory necessarily breaks down as the initial singularity is approached, a quantum treatment of gravity is required in any investigation of the very early universe ($t\ t_{Planck}$).

(Quantum cosmology actually allows for the avoidance of an initial singularity altogether: a universe (of nonzero radius) could quantum tunnel into existence out of nothing). It does not claim to be a fundamental theory of quantum gravity - the idea is that it should agree with a more fundamental theory in the classical limit.

Essentially, quantum cosmology amounts to the canonical quantization of the gravitational degrees of freedom in the Hamiltonian formulation of general relativity, leading to an equation for the "wave function of the universe." In contrast to the standard Lagrangian formulation of general relativity, the Hamiltonian formulation is *not* spacetime covariant. In passing to this formulation, one must "foliate" the spacetime into a sequence of spacelike hypersurfaces, parametrized by a global time t. The Arnowitt, Deser, Misner (ADM) decomposition achieves this (3+1) split by expressing the metric as

$$ds^2 = N^2\, dt^2 - h_{ij}\,(dx^i + N^i dt)(dx^j + N^j dt) \tag{56}$$

where $N(x^\mu)$ is the lapse function, relating coordinate time to proper time, $N^i(x^\mu)$ the shift vector, measuring 'distortion' of the manifold from one hypersurface to the next, and $h_{ij}(x^\mu)$ the induced spatial metric on the hypersurfaces.

The *extrinsic curvature* K_{ij} describes the curvature of the spatial hypersurfaces with respect to the embedding spacetime:

$$K_{ij} = \frac{1}{2N}\left[N_{i|j} + N_{j|i} - \frac{\partial h_{ij}}{\partial t}\right] \tag{57}$$

where "|" signifies covariant differentiation with respect to h_{ij}. The Ricci scalar can be expressed as

$$\mathcal{R} = K^2 - K_{ij}K^{ij} - {}^{(3)}\mathcal{R} \tag{58}$$

With this, the gravitational Lagrangian density becomes

$$\mathcal{L} = -\frac{\sqrt{-g}}{16\pi G}\,\mathcal{R} = -\frac{N\sqrt{h}}{16\pi G}\left[K^2 - K_{ij}K^{ij} - {}^{(3)}\mathcal{R}\right] \tag{59}$$

and the total action is

$$S = -\frac{1}{16\pi G}\int d^4x\, N\sqrt{h}\left[K^2 - K_{ij}K^{ij} - {}^{(3)}\mathcal{R}\right] + \int d^4x\, N\sqrt{h}\,\mathcal{L}_m(\phi, X) \tag{60}$$

where we are assuming for now a typical matter field with $\mathcal{L}_m = \frac{1}{2}g^{\mu\nu}\,\phi_{,\mu}\,\phi_{,\nu} - V(\phi)$. In general there is also a so-called "Gibbons-Hawking" surface term:

$$S_{GH} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\,\sqrt{h}\,K \tag{61}$$

One can now write down the canonical momenta conjugate to N, N^i, h_{ij} and ϕ :

$$\pi^0 \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}} = 0 \tag{62}$$

$$\pi^i \equiv \frac{\delta\mathcal{L}}{\delta\dot{N}^i} = 0 \tag{63}$$

$$\pi^{ij} \equiv \frac{\delta\mathcal{L}}{\delta\dot{h}_{ij}} = \frac{\sqrt{h}}{16\pi G}\left(K^{ij} - h^{ij}K\right) \tag{64}$$

$$\pi^\phi \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = \frac{\sqrt{h}}{N}\left(\dot{\phi} - N^i\phi_{,i}\right) \tag{65}$$

The vanishing of π and π^i gives a pair of constraints. Evidently N and N^i are not dynamical variables: they function as Lagrange multipliers.

We are now in a position to construct the Hamiltonian via the canonical Legendre transformation

$$H = \int d^3x \, (\pi_a \dot{q}^a - \mathcal{L}) \tag{66}$$

$$= \int d^3x \left[\pi^0 \dot{N} + \pi^i \dot{N}_i + \pi^{ij} \dot{h}_{ij} + \pi^\phi \dot{\phi} - \mathcal{L}\right] \tag{67}$$

Defining

$$\mathcal{H} \equiv \frac{\sqrt{h}}{16\pi G}\left(K_{ij}K^{ij} - K^2 - {}^{(3)}\mathcal{R}\right) - \sqrt{h}\,\mathcal{L}_m \tag{68}$$

$$= 16\pi G\, \mathcal{G}_{ijkl}\, \pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{16\pi G}\,{}^{(3)}\mathcal{R} + \sqrt{h}\left[\frac{(\pi^\phi)^2}{2h} + \frac{1}{2}h^{ij}\,\phi_{,i}\,\phi_{,j} + V(\phi)\right] \tag{69}$$

(where $\mathcal{G}_{ijkl} \equiv (h_{ij}h_{jl} + h_{jl}h_{jk} - h_{ij}h_{kl})\pi^{ij}\pi^{kl}$)

and

$$\mathcal{H}^i \equiv -\frac{1}{8\pi G}\pi^{ij}_{|j} + h^{ij}\phi_{,j}\pi^\phi \tag{70}$$

the Hamiltonian becomes

$$H = \int d^3x \left(N\mathcal{H} + N_i\mathcal{H}^i\right) \tag{71}$$

Now $\pi^0 = \pi^i = 0$ implies $\dot{\pi}^0 = \dot{\pi}^i = 0$, but in terms of Poisson brackets we have

$$\dot{\pi}^0 \equiv \{\pi, H\} = \frac{\partial H}{\partial N} \tag{72}$$

$$\dot{\pi}^i \equiv \{\pi^i, H\} = \frac{\partial H}{\partial N^i} \tag{73}$$

thus yielding the dynamical constraints

$$\mathcal{H} = 0 \tag{74}$$

$$\mathcal{H}^i = 0 \tag{75}$$

In the canonical quantization, the momenta in the constraint equations are replaced as follows:

$$\pi^{ij} \longrightarrow -i\frac{\delta}{\delta h_{ij}} \qquad \pi^0 \longrightarrow -i\frac{\delta}{\delta h_{ij}}$$

$$\pi^\phi \longrightarrow -i\frac{\delta}{\delta \phi} \qquad \pi^i \longrightarrow -i\frac{\delta}{\delta N_i}$$

Equation (74) is identified as the zero energy Schrödinger equation, where $\hat{\mathcal{H}}$ is now an operator acting on a wave function (more correctly, wave *functional*)

$$\hat{\mathcal{H}}\,\Psi[h_{ij}, \phi] = 0 \tag{76}$$

Or in full:

$$\left[-16\pi G\, \mathcal{G}_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}} - \frac{\sqrt{h}}{16\pi G}\,{}^{(3)}\mathcal{R} - \frac{1}{2\sqrt{h}}\frac{\delta^2}{\delta\phi^2} + \frac{1}{2}h^{ij}\,\phi_{,i}\phi_{,j} + V(\phi)\right]\Psi[h_{ij}, \phi] = 0 \tag{77}$$

This is known as the Wheeler-DeWitt equation for the wave function of the universe. $\Psi[h_{ij}, \phi]$ is a functional on *superspace*, an infinite-dimensional space of all three-geometries with all possible ϕ field configurations. In practice, it is not feasible to work with so complex a system in full generality. Instead, symmetry considerations are usually employed to reduce the class of geometries to a finite-dimensional subspace. In a closed FRW model, for example, each possible three-geometry is completely characterized by the value of the scale factor; in that case we have a one-dimensional so-called *minisuperspace*.

In this context, spacetime itself only emerges in the classical approximation. There is no such thing as a "trajectory" through superspace. We can only infer from a wave function likely correlations between observables.

5.2 Quantizing the Two-Component Spherical Model

We now apply the procedure outlined above to the model introduced in section (4), namely an inhomogeneous, spherical, k-essence dominated model with baryons. The canonical scalar field that was used above for illustration is now replaced with a k-essence field. The analysis is greatly simplified by the fact that for our shear-free metric we can set $N_i = 0$. As mentioned in the introduction, inclusion of baryons poses a problem as the corresponding Lagrangian density is just the fluid pressure, which is zero. This problem is overcome by means of a geometric representation of the baryons in the action, as will be seen below.

Consider first the geometric part of the action, namely

$$\begin{aligned} S_{E-H} &= -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\mathcal{R} \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,G^\mu_\mu \end{aligned}$$

where we use the fact that the Ricci scalar $\mathcal{R} = R^\mu_\mu = -G^\mu_\mu$. We ignore for now the k-essence Lagrangian and Gibbons-Hawking term.

G^μ_μ is obtained from the field equations (9), (10) and (11):

$$\begin{aligned} G^\mu_\mu &= G^t_t + G^r_r + 2\,G^\theta_\theta \\ &= 12\left(e^{-\nu}\dot{\mu}\right)^2 + 6e^{-\nu}\left(e^{-\nu}\dot{\mu}\right)^{\cdot} \\ &\quad -e^{-2\mu}\left[4\mu'' + 2\mu'^2 + 2\nu'' + 2\nu'^2 + 2\mu'\nu' + 8\frac{\mu'}{r} + 4\frac{\nu'}{r}\right] \end{aligned}$$

and of course $\sqrt{-g} = e^{\nu+3\mu}$. Putting this into S_{E-H} and integrating out the angular part leads to

$$\begin{aligned} S_{E-H} &= \frac{1}{2G}\int dt\int dr\,r^2\left[6e^{3\mu-\nu}\dot{\mu}^2 + 3e^{3\mu}(e^{-\nu}\dot{\mu})^{\cdot}\right. \\ &\quad \left.-e^{\mu+\nu}\left(2\mu'' + \mu'^2 + \nu'' + \nu'^2 + \mu'\nu' + 4\frac{\mu'}{r} + 2\frac{\nu'}{r}\right)\right] \end{aligned} \tag{78}$$

Second r-derivatives of ν as well as second time derivatives are next eliminated by partial integration:

$$\begin{aligned}
4\pi\int dt\int dr\, r^2\left[3e^{3\mu}\left(e^{-\nu}\dot{\mu}\right)^{\cdot}-e^{\mu+\nu}\nu''\right] &= \\
&= \int dt\int dr\, r^2\Big[-3\left(3\dot{\mu}\right)e^{3\mu}\left(e^{-\nu}\dot{\mu}\right)+\left(\mu'+\nu'+\frac{2}{r}\right)e^{\mu+\nu}\nu'\Big]+ \\
&\qquad +\underbrace{\int dt\int dr\left[3\left(r^2e^{3\mu-\nu}\dot{\mu}\right)^{\cdot}-\left(r^2e^{\mu+\nu}\nu'\right)'\right]}_{\text{surface terms}}
\end{aligned} \tag{79}$$

Putting this together, the surface terms cancel with the Gibbons-Hawking term and we are left with

$$S_{E-H}=\frac{1}{2G}\int dt\int dr\, r^2\left[-3e^{3\mu-\nu}\dot{\mu}^2-e^{\mu+\nu}\left(2\mu''+\mu'^2+4\frac{\mu'}{r}\right)\right] \tag{80}$$

The pressure isotropy equation rearranged gives

$$\mu''=\mu'^2+\frac{\mu'}{r}-+4\pi GB(r)e^{-\mu} \tag{81}$$

The geometric representation of the baryons is achieved by substituting this into the action, eliminating μ'' :

$$\begin{aligned}
S_{E-H} &= \frac{1}{2G}\int dt\int dr\, r^2\left[-3e^{3\mu-\nu}\dot{\mu}^2-e^{\mu+\nu}\left(3\mu'^2+6\frac{\mu'}{r}+8\pi GB(r)e^{-\mu}\right)\right] \\
&= -\frac{3}{2G}\int dt\int dr\, r^2\left[R^3e^{-\nu}\left(\frac{\dot{R}}{R}\right)^2+Re^{\nu}\left\{\left(\frac{R'}{R}\right)^2+\frac{2}{r}\left(\frac{R'}{R}\right)+\frac{8\pi G}{3}\frac{B(r)}{R}\right\}\right] \\
&= -\frac{3}{2G}\int dt\int dr\, r^2R\left[\dot{R}^2e^{-\nu}+e^{\nu}\left\{\left(\frac{R'}{R}\right)^2+\frac{2}{r}\left(\frac{R'}{R}\right)+\frac{8\pi G}{3}\frac{B(r)}{R}\right\}\right] \\
&= \int dt\int 4\pi r^2\, dr\mathcal{L}_g
\end{aligned}$$

where $\mathcal{L}_g$ is the (geometric) Lagrangian density. Thus

$$\mathcal{L}_g=-\frac{3}{8\pi G}R\left[\dot{R}^2e^{-\nu}+e^{\nu}\left\{\left(\frac{R'}{R}\right)^2+\frac{2}{r}\left(\frac{R'}{R}\right)+\frac{8\pi G}{3}\frac{B(r)}{R}\right\}\right] \tag{82}$$

The only dynamical variable is the scale factor R. We construct the canonical momentum density associated with it

$$\mathcal{P}_R \equiv \frac{\partial\mathcal{L}_g}{\partial\dot{R}} = -\frac{3}{4\pi G}R\dot{R}e^{-\nu} \tag{83}$$

$$\Rightarrow \qquad \dot{R} = -\frac{4\pi G}{3}\frac{\mathcal{P}_R}{R}e^{\nu} \tag{84}$$

from which we obtain the Hamiltonian density

$$\mathcal{H} \equiv \mathcal{P}_a\dot{\mathcal{Q}}^a-\mathcal{L} \tag{85}$$

$$
\begin{aligned}
\mathcal{H}_g &= -\frac{\mathcal{P}_R^2}{6R}e^{\nu} + \frac{3R}{4\pi G}\left[\left(\frac{4\pi G}{3}\frac{\mathcal{P}_R}{R}e^{\nu}\right)^2 e^{-\nu} + e^{\nu}\left\{\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right) - \frac{k(r)}{R}\right\}\right] \\
&= e^{\nu}\left[-\frac{2\pi G}{3}\frac{\mathcal{P}_R^2}{R} + \frac{3R}{8\pi G}\left\{\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right) + \frac{8\pi G}{3}\frac{B(r)}{R}\right\}\right] \\
&= e^{\nu}\left[-\frac{2\pi G}{3}\frac{\mathcal{P}_R^2}{R} + B(r) + \frac{3R}{8\pi G}\left\{\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right)\right\}\right] \qquad (86)
\end{aligned}
$$

The Hamiltonian density for a k-essence field with Lagrangian density $\mathcal{L}_{ke}(\phi, X)$ is

$$\mathcal{H}_{ke}(\phi, \mathcal{P}_\phi) = X\,\mathcal{L}_X - \mathcal{L} \qquad (87)$$

where $\mathcal{P}_\phi = \mathcal{L}_X$. The total Hamiltonian density $\mathcal{H}$ is just

$$\mathcal{H}(R, \phi, \mathcal{P}_\phi, \mathcal{P}_R) = \mathcal{H}_g + \mathcal{H}_{ke} \qquad (88)$$

We now apply the canonical quantization procedure to this Hamiltonian,

$$
\begin{array}{rcrcl}
\mathcal{P}_R & \longrightarrow & \hat{\mathcal{P}}_R & = & -i\frac{\delta}{\delta R} \\
\mathcal{P}_\phi & \longrightarrow & \hat{\mathcal{P}}_\phi & = & -i\frac{\delta}{\delta \phi}
\end{array} \qquad (89)
$$

$$\mathcal{H}(R, \phi, \mathcal{P}_\phi, \mathcal{P}_R) \longrightarrow \hat{\mathcal{H}}(R, \phi, \hat{\mathcal{P}}_\phi, \hat{\mathcal{P}}_R) \qquad (90)$$

$$\hat{\mathcal{H}}\,\Psi(R, \phi) = 0 \qquad (91)$$

leading to the Wheeler-DeWitt equation

$$\left[\frac{\delta^2}{\delta R^2} + \frac{9R^2}{16\pi^2 G^2}\left(\left(\frac{R'}{R}\right)^2 + \frac{2}{r}\left(\frac{R'}{R}\right)\right) + \frac{3R}{2\pi G}\left(B(r) + \mathcal{H}_{ke}\left(\phi, -i\frac{\delta}{\delta\phi}\right)\right)\right]\Psi(R, \phi) = 0 \qquad (92)$$

where $\Psi(R, \phi)$ is the wavefunction of the universe.

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