

# Third order relativistic dissipative fluid dynamics

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- Relativistic second order dissipative fluid dynamics is a very important scientific achievement of the last decades, and has inspired many subsequent authors to apply its methodology to lots of other possible applications. See U. Heinz's Talk. In short it furnishes equations which are closed by imposing the relativity principle and the entropy principle up to second order, with respect to equilibrium.
- Most authors have refrained from exploiting subsequent orders because that requires long and cumbersome representations.
- However, the exploitation of subsequent orders with respect to equilibrium is desirable, for the following two reasons
  - (i) a second order approach is necessary to link more closely the relativistic case with the classical one
  - (ii) it is necessary to answer the following questions: By imposing our conditions at a given order  $N$ , we obtain some equations for the determination of the  $N$ -th order expressions of the constitutive functions and these equations depend also on the lower order terms; so it might happen that the existing condition of the solution imposes further conditions on the lower order terms. And it may also happen that the further conditions affect the equilibrium expressions which, however, are already known.

## Basics & Formalism

The objective of relativistic dissipative fluid dynamics for one component fluid is the determination of the 14 fields of

$$\begin{array}{ll} N^\mu(x^\beta) & \text{net charge density — net charge flux vector} \\ T^{\mu\nu}(x^\beta) & \text{stress — energy — momentum tensor} \end{array}$$

$T^{\mu\nu}$  is assumed symmetric so that it has 10 independent components.

The 14 fields are determined from the field equations (fluid dynamical equations)

$$\begin{array}{ll} \partial_\mu N^\mu = 0 & \text{net charge (e.g., baryon, strangeness, etc) conservation} \\ \partial_\nu T^{\mu\nu} = 0 & \text{energy – momentum conservation} \\ \partial_\lambda F^{\mu\nu\lambda} = P^{\mu\nu} & \text{balance law of fluxes} \end{array}$$

$F^{\mu\nu\lambda}$  is completely symmetric tensor of fluxes and  $P^{\mu\nu}$  is its production density such that

$$F^{\mu\nu}{}_\nu = m^2 N^\mu \quad \text{and} \quad P^\nu{}_\nu = 0$$

We then have a set of **14** independent equations ( **net charge conservation (1)**; **energy-momentum conservation (4)**; **balance of fluxes (9)**)

However, the dynamic equations cannot serve as the field equations for the thermodynamic fields  $N^\mu$  and  $T^{\mu\nu}$ . Because the additional fields  $F^{\mu\nu\lambda}$  and  $P^{\mu\nu}$  have appeared.

Restriction on the general form of the constitutive functions  $F^{\mu\nu\lambda}(N^\alpha, T^{\alpha\beta})$  and  $P^{\mu\nu}(N^\alpha, T^{\alpha\beta})$  is imposed by

- **principle of relativity** — field equations have the same form in all frames  $\implies$  **the constitutive functions be invariant under the change of frames**
- **entropy principle** —the entropy density–entropy flux vector  $S^\mu(N^\alpha, T^{\alpha\beta})$  is a constitutive quantity which obeys the inequality

$$\partial_\mu S^\mu = \Xi \geq 0 \quad \text{for all thermodynamic process}$$

- **requirement of hyperbolicity** — ensures that Cauchy problems of our field equations are well-posed and all wave speeds are finite  $\implies$  **our set of field equations should be symmetric hyperbolic**

Net charge 4-current

$$N^\mu = nu^\mu$$

$n \equiv \sqrt{N^\mu N_\mu} = u_\mu N^\mu$  net charge density in fluid rest frame,

$u^\mu \equiv \frac{N^\mu}{\sqrt{N^\nu N_\nu}}$  the fluid 4-velocity,

$u^\nu u_\nu = 1 \implies u^\mu$  has 3 independent components

**Stress–energy–momentum tensor**  $T^{\mu\nu} = \varepsilon u^\mu u^\nu - (p + \Pi) \Delta^{\mu\nu} + 2q^{(\mu} u^{\nu)} + \pi^{\langle\mu\nu\rangle}$

$\varepsilon \equiv u_\mu u_\nu T^{\mu\nu}$     **energy density** in fluid rest frame,

$p \equiv p(\varepsilon, n)$     **pressure** in fluid rest frame,

$\Pi$     **bulk viscous pressure**,     $(p + \Pi) \equiv -\frac{1}{3} \Delta_{\mu\nu} T^{\mu\nu}$

$\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$     **projection tensor onto 3-space**,     $\Delta^{\mu\nu} u_\nu = \Delta^{\mu\nu} u_\mu = 0$

$g^{\mu\nu} \equiv \text{diag}(+1, -1, -1, -1)$     **metric tensor**

$q^\mu \equiv \Delta^\mu_\alpha u_\beta T^{\alpha\beta}$     **heat flux 4-current**,

$q^\mu u_\mu = 0 \implies q^\mu$  has **3 independent components**

$\pi^{\langle\mu\nu\rangle} \equiv T^{\langle\mu\nu\rangle}$     **shear stress tensor**

$\pi^{\langle\mu\nu\rangle} u_\mu = \pi^{\langle\mu\nu\rangle} u_\nu = 0$ ,     $\pi^{\langle\nu\rangle} = 0 \implies \pi^{\langle\mu\nu\rangle}$  has **5 independent components**

**Production densities tensor**  $P^{\mu\nu} = \mathcal{P}_\Pi \Pi (\Delta^{\mu\nu} - 3u^\mu u^\nu) + 2\mathcal{P}_q q^{(\mu} u^{\nu)} + \mathcal{P}_\pi \pi^{\langle\mu\nu\rangle}$

The functions  $\mathcal{P}_\Pi$ ,  $\mathcal{P}_q$ ,  $\mathcal{P}_\pi$  are related to the bulk viscosity, heat conductivity and shear viscosity and thus may be determined from measurements of these coefficients



Tensor of fluxes (up to 2nd order)

$$\begin{aligned}
 F^{\mu\nu\lambda} = & \frac{1}{2} \mathcal{F}_1^0 g^{(\mu\nu} u^{\lambda)} + \frac{1}{2} \mathcal{F}_2^0 \left( g^{(\mu\nu} u^{\lambda)} - 2u^\mu u^\nu u^\lambda \right) \\
 & + \mathcal{F}_1^1 \Pi \left( \Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda \right) + \mathcal{F}_2^1 \left( \Delta^{(\mu\nu} q^{\lambda)} - 5u^{(\mu} u^\nu q^{\lambda)} \right) \\
 & + \mathcal{F}_3^1 \pi^{(\langle\mu\nu\rangle} u^{\lambda)} \\
 & + \mathcal{F}_1^2 \Pi^2 \left( \Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda \right) + \mathcal{F}_2^2 \left( -q^\nu q_\nu \Delta^{(\mu\nu} u^{\lambda)} - 3u^{(\mu} q^\nu q^{\lambda)} \right) \\
 & - \mathcal{F}_3^2 q^\alpha q_\alpha \left( \Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda \right) + \mathcal{F}_4^2 \left( 3u^\mu \pi^{2(\nu\lambda)} - \pi^{2\langle\alpha\alpha\rangle} u^\mu u^\nu u^\lambda \right) \\
 & + \mathcal{F}_5^2 \pi^{2\langle\alpha\alpha\rangle} \left( \Delta^{(\mu\nu} u^{\lambda)} - u^\mu u^\nu u^\lambda \right) + \mathcal{F}_6^2 \left( q^{(\mu} \pi^{\langle\nu\lambda\rangle)} - 2u^{(\mu} u^\nu \pi^{\langle\lambda\rangle\nu)} q_\nu \right) \\
 & + \mathcal{F}_7^2 \left( \Delta^{(\mu\nu} \pi^{\langle\lambda\rangle\alpha)} q_\alpha - 5u^\mu u^\nu \pi^{\langle\lambda\rangle\alpha} q_\alpha \right) + \mathcal{F}_8^2 \Pi u^{(\mu} \pi^{\langle\nu\lambda\rangle)} \\
 & + \mathcal{F}_9^2 \Pi \left( \Delta^{(\mu\nu} q^{\lambda)} - 5q^{(\mu} u^\nu u^{\lambda)} \right)
 \end{aligned}$$

Zeroth order (Equilibrium) + First order + Second order

Entropy 4-current (up to 3rd order)

$$\begin{aligned}
 S^\mu = & \mathcal{S}_1^0 u^\mu \\
 & + \mathcal{S}_1^1 \Pi u^\mu + \mathcal{S}_2^1 q^\mu \\
 & + \left( \mathcal{S}_1^2 \Pi^2 - \mathcal{S}_2^2 q^\alpha q_\alpha + \mathcal{S}_3^2 \pi^{2\langle\alpha\alpha\rangle} \right) u^\mu \\
 & + \mathcal{S}_4^2 \Pi q^\mu + \mathcal{S}_5^2 \pi^{\langle\mu\alpha\rangle} q_\alpha \\
 & + \left( \mathcal{S}_1^3 \Pi^3 - \mathcal{S}_2^3 \Pi q_\alpha q^\alpha + \mathcal{S}_3^3 \Pi \pi^{2\langle\alpha\alpha\rangle} + \mathcal{S}_4^3 q_\alpha q_\beta \pi^{\langle\alpha\beta\rangle} + \mathcal{S}_5^3 \pi^{3\langle\alpha\alpha\rangle} \right) u^\mu \\
 & + \left( \mathcal{S}_6^3 \Pi^2 - \mathcal{S}_7^3 q_\alpha q^\alpha + \mathcal{S}_8^3 \pi^{2\langle\alpha\alpha\rangle} \right) q^\mu + \mathcal{S}_9^3 \Pi \pi^{\langle\mu\alpha\rangle} q_\alpha + \mathcal{S}_{10}^3 \pi^{2\langle\mu\alpha\rangle} q_\alpha
 \end{aligned}$$

Zeroth order (Equilibrium) + First order + Second order + Third order

Parentheses around some indices denote symmetrization, while angular brackets around two indices denote skew-symmetrization

$$a^{(\mu\nu)} \equiv \frac{1}{2} (a^{\mu\nu} + a^{\nu\mu})$$

$$a^{\langle\mu\nu\rangle} \equiv \left( \Delta_{\alpha}^{(\mu} \Delta_{\beta}^{\nu)} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) a^{\alpha\beta}$$

The space-time derivative will be split into time and spatial components as follows

$$\partial_{\mu} \equiv u_{\mu} D + \nabla_{\mu}$$

with  $D \equiv u^{\alpha} \partial_{\alpha}$  **convective (comoving) time derivative**  
and  $\nabla_{\mu} \equiv \Delta^{\mu\nu} \partial_{\nu}$  **spatial gradient**

$$\dot{a}^{\dots} \equiv D a^{\dots} = u^{\mu} \partial_{\mu} a^{\dots} \quad \text{convective (comoving) time derivative of } a^{\dots}$$

$$\theta = \nabla_{\mu} u^{\mu} = \partial_{\mu} u^{\mu} \quad \text{expansion scalar (divergence of 4-velocity)}$$

Equilibrium is defined as a process in which production densities vanish and/or the entropy production vanishes

$$\left. \begin{aligned} P_{Eq.}^{\mu\nu} &= 0 \\ \Xi_{Eq.} &= 0 \end{aligned} \right\} \implies \Pi_{Eq.} = 0, \quad q_{Eq.}^{\mu} = 0, \quad \pi_{Eq.}^{\langle\mu\nu\rangle} = 0$$

$$\begin{aligned} F_{Eq.}^{\mu\nu\lambda} &= \frac{1}{2} \mathcal{F}_1^0 g^{(\mu\nu} u^{\lambda)} + \frac{1}{2} \mathcal{F}_2^0 (g^{(\mu\nu} u^{\lambda)} - 2u^{\mu} u^{\nu} u^{\lambda}) \\ S_{Eq.}^{\mu} &= s(\varepsilon, n) u^{\mu} \end{aligned}$$

The energy-momentum tensor reduces to

$$T_{Eq.}^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - p \Delta^{\mu\nu}$$

In the ideal “perfect” fluid limit one has **5 independent fields** ( $p(n, e)$ (2),  $u^{\mu}$ (3)) and **5 field equations**

# 14-Fields Theory of Relativistic Dissipative Fluid Dynamics :

In dissipative(non-ideal) fluid dynamics one needs 9 additional equations for the dissipative fluxes. The 14 fields  $p(n, \varepsilon), \Pi, u^\alpha, q^\alpha, \pi^{(\alpha\beta)}$  are governed by the following fields equations

$$\begin{aligned}\partial_\mu N^\mu &= 0 \\ \Delta_{\alpha\mu} \partial_\nu T^{\mu\nu} &= 0 \\ u_\mu \partial_\nu T^{\mu\nu} &= 0 \\ u_\mu u_\nu \partial_\lambda F^{\mu\nu\lambda} &= -\mathcal{P}_\Pi \Pi \\ \Delta_\alpha^\mu u_\nu \partial_\lambda F^{\alpha\nu\lambda} &= \mathcal{P}_q q^\mu \\ \left( \Delta_\alpha^{(\mu} \Delta_\beta^{\nu)} - \frac{1}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta} \right) \partial_\lambda F^{\alpha\beta\lambda} &= \mathcal{P}_\pi \pi^{(\mu\nu)}\end{aligned}$$

For all thermodynamic processes the entropy principle holds

$$\partial_\mu S^\mu = \Xi \geq 0$$

$$\begin{aligned}\Pi &= \Pi_{Eq.} = 0 \\ q^\alpha &= q_{Eq.}^\alpha = 0 \\ \pi^{\langle\alpha\beta\rangle} &= \pi_{Eq.}^{\langle\alpha\beta\rangle} = 0\end{aligned}$$

$$\begin{aligned}\Pi^{(1)} &= \Pi_E = -\zeta \nabla_\alpha u^\alpha \\ q^\alpha{}^{(1)} &= q_E^\alpha = \kappa T \Delta^{\alpha\mu} \left( \frac{\nabla_\alpha T}{T} - \dot{u}_\alpha \right) \\ \pi^{\langle\alpha\beta\rangle}{}^{(1)} &= \pi_E^{\langle\alpha\beta\rangle} = 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \nabla_{\langle\alpha} u_{\beta\rangle}\end{aligned}$$

Relativistic versions of the laws of Navier-Stokes and Fourier  
first derived by Eckart, Landau-Lifshitz.

$\zeta$  is the bulk viscosity,  $\kappa$  is the thermal conductivity,  $\eta$  is the shear viscosity

- simple algebraic expressions of dissipative fluxes
- but leads to acausal and unstable equations of motion

## Dissipative fluxes: Second order

Müller-Israel-Stewart (MIS) equations:  $F^{\mu\nu\lambda}$  linear (first-order) in dissipative fluxes and  $S^\mu$  quadratic (second-order) in dissipative fluxes  
 Resulting equations causal and hyperbolic

$$\begin{aligned}
 \Pi^{(2)} &= \Pi_{MIS} = -\zeta \left[ 2S_1^2 \dot{\Pi} + S_4^2 \nabla_\alpha q^\alpha \right] \\
 &\quad -\zeta \left[ \Pi (\dot{S}_1^2 + S_1^2 \nabla_\alpha u^\alpha) + q^\alpha (\nabla_\alpha S_4^2 - S_4^2 \dot{u}_\alpha) \right] \\
 q^\mu{}^{(2)} &= q_{MIS}^\mu = \kappa T \Delta^{\alpha\mu} \left[ 2S_2^2 \dot{q}_\alpha + S_4^2 \nabla_\alpha \Pi + S_5^2 \nabla^\beta \pi_{\langle\alpha\beta\rangle} \right] \\
 &\quad + \kappa T \Delta^{\alpha\mu} \left[ q_\alpha (\dot{S}_2^2 + S_2^2 \nabla_\nu u^\nu) + \Pi (\nabla_\alpha S_4^2 - S_4^2 \dot{u}_\alpha) \right. \\
 &\quad \left. + \pi_{\langle\alpha\beta\rangle} (\nabla^\beta S_5^2 - S_5^2 \dot{u}^\beta) \right] \\
 \pi^{\langle\mu\nu\rangle}{}^{(2)} &= \pi_{MIS}^{\langle\mu\nu\rangle} = 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \left[ 2S_3^2 \dot{\pi}_{\langle\alpha\beta\rangle} + S_5^2 \nabla_{\langle\alpha} q_{\beta\rangle} \right] \\
 &\quad + 2\eta \Delta^{\alpha\mu} \Delta^{\beta\nu} \left[ \pi_{\langle\alpha\beta\rangle} (\dot{S}_3^2 + S_3^2 \nabla_\lambda u^\lambda) \right. \\
 &\quad \left. + q_{\langle\alpha} (\nabla_{\beta\rangle} S_5^2 - S_5^2 \dot{u}_{\beta\rangle}) \right]
 \end{aligned}$$



- The terms in **red** are neglected in the original MIS formulation. Terms of the general form  $\Pi \partial_\nu u^\mu$ ,  $\Pi \partial_\lambda n$ ,  $\Pi \partial_\lambda \varepsilon$ ,  $q_\alpha \partial_\nu u^\mu$ ,  $q_\alpha \partial_\lambda n$ ,  $q_\alpha \partial_\lambda \varepsilon$ ,  $\pi_{\langle \alpha \beta \rangle} \partial_\nu u^\mu$ ,  $\pi_{\langle \alpha \beta \rangle} \partial_\lambda n$ ,  $\pi_{\langle \alpha \beta \rangle} \partial_\lambda \varepsilon$  have been considered non-linear and thus ignored. These terms have been shown to be important in heavy ion collisions. They will be even more important at low energies and high densities.
- Derivations of the equations from kinetic theory reveals terms that are not explicit from phenomenological considerations (e.g., **vorticity terms**)  
**To be published**

We derive the third order entropy 4-current as well the non-classical coefficients by going beyond Israel-Stewart entropy 4-current expression in kinetic theory. The kinetic expression for entropy, can be written as

$$S^\mu = - \int dwp^\mu \psi[f(x, p)] ,$$

where

$$\psi[f(x, p)] = f(x, p) \left\{ \ln[A_0^{-1} f(x, p)] - 1 \right\} ,$$

and  $f(x, p)$  is the out of the equilibrium distribution function. Expanding  $\psi(f)$  around  $\psi(f^{eq})$  up to third order we get,

$$\begin{aligned} \psi(f) &= \psi(f^{eq}) + \psi'(f^{eq})(f - f^{eq}) + \frac{1}{2}\psi''(f^{eq})(f - f^{eq})^2 \\ &\quad + \frac{1}{6}\psi'''(f^{eq})(f - f^{eq})^3 + \dots , \end{aligned}$$

$$S^{\mu(1)} = \frac{q^\mu}{T},$$

$$S^{\mu(2)} = \frac{1}{2}\beta u^\mu \left[ S_1^2 \Pi^2 - S_2^2 q^\alpha q_\alpha + S_3^2 \pi^{\nu\alpha} \pi_{\nu\alpha} \right] + \beta \left[ S_4^2 q^\mu \Pi + S_5^2 q_\alpha \pi^{\mu\alpha} \right],$$

$$\begin{aligned} S^{\mu(3)} = & \frac{1}{6}\beta u^\mu \left\{ S_1^3 \Pi^3 + S_2^3 \Pi q^\alpha q_\alpha + S_3^3 \Pi \pi^{\nu\alpha} \pi_{\nu\alpha} + S_4^3 q_\nu q_\alpha \pi^{\nu\alpha} + S_5^3 \pi_{\nu\alpha} \pi_\beta^\nu \pi^{\alpha\beta} \right\} \\ & - \frac{1}{6}\beta q^\mu \left\{ S_6^3 \Pi^2 + S_7^3 q^\alpha q_\alpha - S_8^3 \pi^{\nu\alpha} \pi_{\nu\alpha} \right\} - \beta S_9^3 \Pi q_\alpha \pi^{\mu\alpha} \\ & + \frac{1}{2}\beta S_{10}^3 q_\alpha \pi^{\nu\alpha} \pi_\nu^\mu, \end{aligned}$$

- There are three mechanisms for entropy production. One is related to the dynamic pressure, one due to heat flux and one due to shear stress.
- For the entropy production to be non-negative the coefficients related to bulk viscosity, shear viscosity and heat flux some inequality relations.
- This work increases the appreciation for previous work - because the conditions imposed in the previous work are satisfied also up to higher orders in expansion around thermodynamic equilibrium.
- The many identities we encounter in deriving the equations implies that perhaps it is possible to find an explicit and exact solution for the conditions
- The equations presented here are the same whether ones used divergence methodology or kinetic theory.