JIMWLK evolution in the Gaussian approximation

Edmond Iancu
Institut de Physique Théorique de Saclay

with D.N. Triantafyllopoulos, arXiv:1109.0302, 1112.1104 [hep-ph]

Introduction

- Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

Introduction

- Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

- ... it appeared to be so complicated that any solution to it seemed to be out of reach !

Introduction

- Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

- ... it appeared to be so complicated that any solution to it seemed to be out of reach !
- Yet, only a few years later (following *Blaizot, Iancu, Weigert, 02*), Rummukainen and Weigert presented the first numerical solution (03).
- Nowadays, we have several 'codes' available: *Lappi, Schenke & collabs*

Introduction

- Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

- ... it appeared to be so complicated that any solution to it seemed to be out of reach !
- Yet, only a few years later (following *Blaizot, Iancu, Weigert, 02*), Rummukainen and Weigert presented the first numerical solution (03).
- Nowadays, we have several 'codes' available: *Lappi, Schenke & collabs*
- This has been completed by various 'mean field studies'
    - solutions to the Balitksy–Kovchegov (BK) equation (large N_c)
    - Gaussian Ansatz for the CGC weight function

Iancu, Itakura, McLerran (02), Kovchegov, Kuokkanen, Rummukainen, Weigert (09)

Introduction

- Once that the B–JIMWLK equation/hierarchy has been finally established, following a strenuous and heroic, collective work ...

Balitsky (96), Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner (97-00)

- ... it appeared to be so complicated that any solution to it seemed to be out of reach !
- Yet, only a few years later (following *Blaizot, Iancu, Weigert, 02*), Rummukainen and Weigert presented the first numerical solution (03).
- Nowadays, we have several 'codes' available: *Lappi, Schenke & collabs*
- This has been completed by various 'mean field studies'
  - solutions to the Balitksy–Kovchegov (BK) equation (large N_c)
  - Gaussian Ansatz for the CGC weight function

Iancu, Itakura, McLerran (02), Kovchegov, Kuokkanen, Rummukainen, Weigert (09)

- For quite some time, these efforts were restricted to the dipole amplitude (a 2–point function generalizing the gluon distribution)

Introduction (2)

- Directly relevant to the phenomenology ...
  - deep inelastic scattering
  - single inclusive particle production in p+A
- ... and also easier to compute.

Introduction (2)

- Directly relevant to the phenomenology ...
  - deep inelastic scattering
  - single inclusive particle production in p+A
- ... and also easier to compute.
- Recently, phenomenology started to be more demanding: multi–particle correlations at RHIC
- ... thus requiring the study of higher n–point correlations $(n \geq 4)$.

Introduction (2)

- Directly relevant to the phenomenology ...
  - deep inelastic scattering
  - single inclusive particle production in p+A
- ... and also easier to compute.
- Recently, phenomenology started to be more demanding: multi–particle correlations at RHIC
- ... thus requiring the study of higher n–point correlations $(n \geq 4)$.
- The first numerical calculation of 4-p and 6-p functions for special configurations *(Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan, 11)*
- ... came with a big surprise: numerics is very well reproduced by high–energy extrapolations of the McLerran–Venugopalan model !

Introduction (2)

- Directly relevant to the phenomenology ...
  - deep inelastic scattering
  - single inclusive particle production in p+A
- ... and also easier to compute.
- Recently, phenomenology started to be more demanding: multi–particle correlations at RHIC
- ... thus requiring the study of higher n–point correlations ($n \geq 4$).
- The first numerical calculation of 4-p and 6-p functions for special configurations *(Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan, 11)*
- ... came with a big surprise: numerics is very well reproduced by high–energy extrapolations of the McLerran–Venugopalan model !
- A Gaussian approximation : information only about the 2–p function !

Introduction (3)

- Is that just numerical coincidence restricted to special configurations?

Introduction (3)

- Is that just numerical coincidence restricted to special configurations?
- Or rather is a generic feature of the B–JIMWLK evolution that one could further exploit ?

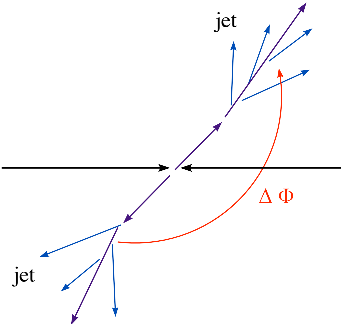
Introduction (3)

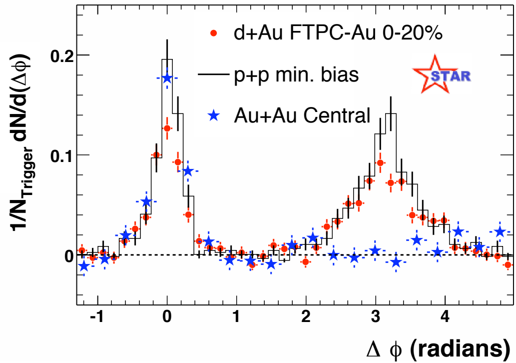
- Is that just numerical coincidence restricted to special configurations?
- Or rather is a generic feature of the B–JIMWLK evolution that one could further exploit ?
- Previous studies of the Gaussian approximation did not address its validity for higher n–point correlations
- No *a priori* reason to expect it should work !
  - complicated, non–linear, evolution
  - infinite hierarchy of equations coupling n–p functions with arbitrary n

Introduction (3)

- Is that just numerical coincidence restricted to special configurations?
- Or rather is a generic feature of the B–JIMWLK evolution that one could further exploit ?
- Previous studies of the Gaussian approximation did not address its validity for higher n–point correlations
- No *a priori* reason to expect it should work !
  - complicated, non–linear, evolution
  - infinite hierarchy of equations coupling n–p functions with arbitrary n
- And yet it works ! *(E.I., Triantafyllopoulos, 2011)*
  - a meaningful piecewise approximation, which is correct both in the dilute (BFKL) and the dense (saturation) regimes
  - smooth interpolation between the two limiting regimes
  - good agreement with numerics ... whenever the latter exists
- Analytic solutions which should greatly facilitate phenomenology

Di–hadron azimuthal correlations

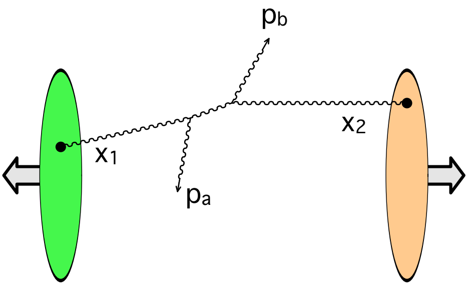




[Nucl.Phys.A783:249-260,2007]

- Typical final state: a pair of jets back–to–back in the transverse plane
- Particle distribution as a function of the azimuthal angle:
  a peak at $\Delta\Phi = 180°$

Particle production in hadron–hadron collisions

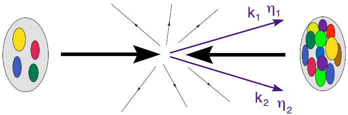


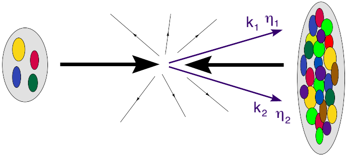
- The colliding partons carry longitudinal momentum fractions

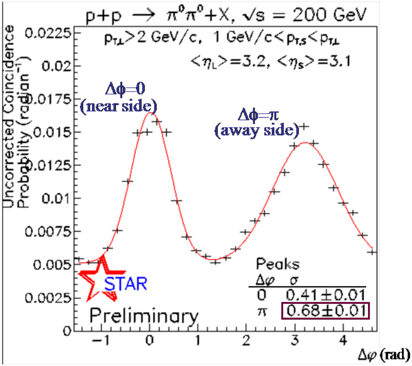
$$x_1 = \frac{|\boldsymbol{p}_a|\, e^{y_a} + |\boldsymbol{p}_b|\, e^{y_b}}{\sqrt{s}}, \qquad x_2 = \frac{|\boldsymbol{p}_a|\, e^{-y_a} + |\boldsymbol{p}_b|\, e^{-y_b}}{\sqrt{s}}$$

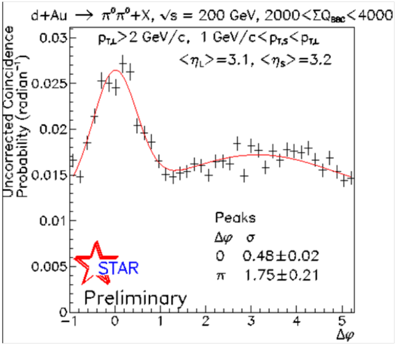
- Forward rapidities : $y_a \sim y_b$ are both positive and large
  $\Longrightarrow x_1 \sim \mathcal{O}(1)$ and $x_2 \ll 1$ ('dense–dilute scattering')
- One may be able to probe saturation effects in the target
- These effects are enhanced for a nuclear target

Di–hadron correlations at RHIC: p+p vs. d+Au



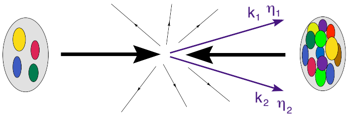


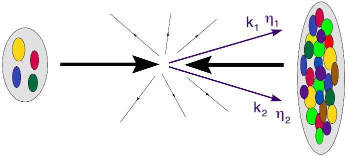


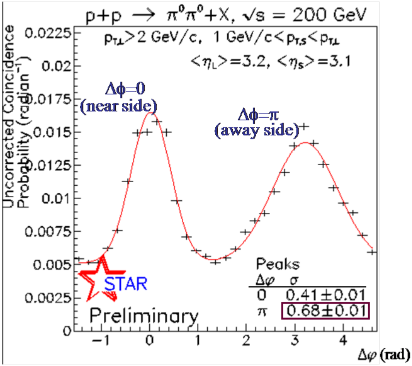


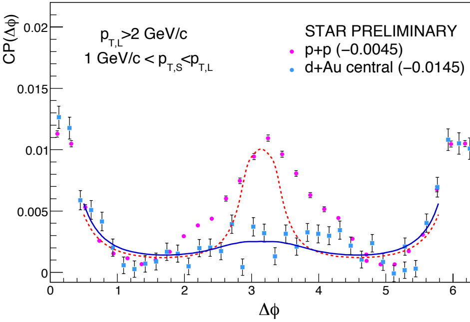
- d+Au : the ‘away jet’ gets smeared out $\Longrightarrow$ saturation in Au

Di–hadron correlations at RHIC: p+p vs. d+Au





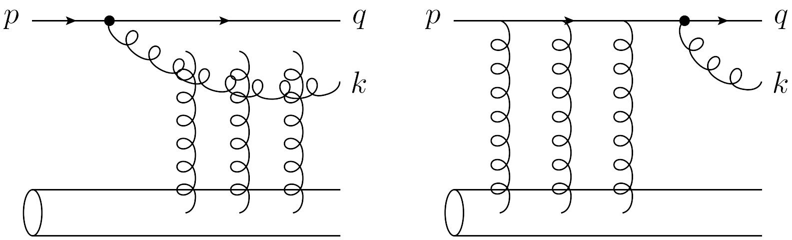




(Albacete and Marquet, 2010, PRL)

- d+Au : the 'away jet' gets smeared out ⟹ saturation in Au

Multiple scattering & Wilson line



- The produced quark and gluon undergo multiple scattering
- Broadening of their transverse momentum distribution: important if $p_\perp \sim Q_s(x_2, A)$ ... in agreement with the data !
- Eikonal approximation $\Longrightarrow$ Wilson lines :

$$V_{\boldsymbol{x}}^\dagger \equiv \mathrm{P}\exp\left[ig\int \mathrm{d}x^- A_a^+(x^-,\boldsymbol{x})T^a\right]$$

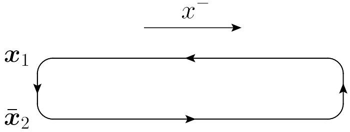
$\Rightarrow$ two WL's per parton (direct amplitude + the c.c. amplitude)

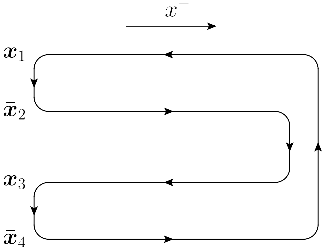
Higher–point correlations of the Wilson lines

- Quark–gluon pair production: the color trace of a product of 4 Wilson lines (2 fundamental, 2 adjoint)
- Equivalently (after using Fierz identity): 6 fundamental Wilson lines

$$\left\langle \frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^\dagger_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})\, \frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_4} V_{\boldsymbol{x}_3}) \right\rangle_Y \equiv \left\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \hat{S}_{\boldsymbol{x}_4\boldsymbol{x}_3} \right\rangle_Y$$

- Expectation value of a 2–trace operator: quadrupole × dipole





Higher–point correlations of the Wilson lines

- Quark–gluon pair production: the color trace of a product of 4 Wilson lines (2 fundamental, 2 adjoint)
- Equivalently (after using Fierz identity): 6 fundamental Wilson lines

$$\left\langle \frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^\dagger_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})\,\frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_4} V_{\boldsymbol{x}_3})\right\rangle_Y \equiv \left\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\hat{S}_{\boldsymbol{x}_4\boldsymbol{x}_3}\right\rangle_Y$$

- Expectation value of a 2–trace operator: quadrupole $\times$ dipole
- The target dynamics is encoded in the CGC average :

$$\langle\hat{\mathcal{O}}\rangle_Y \equiv \int \mathcal{D}\alpha\,\mathcal{O}[\alpha]\,W_Y[\alpha]\,, \qquad \alpha_a \equiv \mathcal{A}^+_a(x^-,\boldsymbol{x})\,, \quad Y \equiv \ln\frac{1}{x_2}$$

- The CGC weight function $W_Y[\alpha]$ obeys JIMWLK equation

  high–energy evolution [leading log $\ln(1/x)$] of the multigluon correlations for the case of a dense target

JIMWLK Hamiltonian

- Renormalization group equation for the CGC weight function $W_Y[\alpha]$:

$$\frac{\partial}{\partial Y} W_Y[\alpha] = H W_Y[\alpha]$$

$$H = -\frac{1}{16\pi^3} \int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} \left(1 + \widetilde{V}_{\boldsymbol{u}}^{\dagger} \widetilde{V}_{\boldsymbol{v}} - \widetilde{V}_{\boldsymbol{u}}^{\dagger} \widetilde{V}_{\boldsymbol{z}} - \widetilde{V}_{\boldsymbol{z}}^{\dagger} \widetilde{V}_{\boldsymbol{v}}\right)^{ab} \frac{\delta}{\delta \alpha_{\boldsymbol{u}}^{a}} \frac{\delta}{\delta \alpha_{\boldsymbol{v}}^{b}}$$

  - dipole kernel: $\mathcal{M}_{\boldsymbol{uvz}} \equiv \frac{(\boldsymbol{u}-\boldsymbol{v})^2}{(\boldsymbol{u}-\boldsymbol{z})^2 (\boldsymbol{z}-\boldsymbol{v})^2}$
  - functional derivatives: 'creation operators' for the emission of a new gluon at small x
  - (adjoint) Wilson lines: multiple scattering between the newly emitted gluon and the color field created by the previous ones with $x' \gg x$

- N.B. : The first 2 terms within H ('virtual') and the last 2 ones ('real') will play different roles in what follows

Balitsky–JIMWLK hierarchy

- Infinite hierarchy of coupled evolution equations for the n–point functions of the Wilson lines (Balitsky, 1996)

$$\frac{\partial \langle \hat{\mathcal{O}} \rangle_Y}{\partial Y} = \int \mathcal{D}\alpha \, \mathcal{O}[\alpha] \, \frac{\partial}{\partial Y} W_Y[\alpha] = \langle H \hat{\mathcal{O}} \rangle_Y$$

- Functional derivatives act on the color field at the largest value of x^- :

$$\frac{\delta}{\delta \alpha^a_{\boldsymbol{u}}} V^\dagger_{\boldsymbol{x}} = \mathrm{i} g \delta_{\boldsymbol{x}\boldsymbol{u}} \, t^a V^\dagger_{\boldsymbol{x}}$$

... i.e. at the end point of the Wilson lines

- Generators of color rotations 'on the left' (or 'left Lie derivatives'): each evolution step adds a new layer of field at a larger value of x^- :

$$V^\dagger_n(\boldsymbol{x}) \;\to\; V^\dagger_{n+1}(\boldsymbol{x}) = \exp[\mathrm{i} g \epsilon \alpha_{n+1}(\boldsymbol{x})] \, V^\dagger_n(\boldsymbol{x})$$

- We shall later return to this point (longitudinal structure of the target)

Dipole evolution (1)

- Observables involving $2n$ Wilson lines are coupled to those with $2n+2$
- Dipole S–matrix: $\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} = \frac{1}{N_c}\,\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2})$

$$H_{\rm virt}\,\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} = -\frac{\bar{\alpha}}{2\pi}\left(1-\frac{1}{N_c^2}\right)\int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}}\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}$$

$$H_{\rm real}\,\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} = \frac{\bar{\alpha}}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}}\Big(\hat{S}_{\boldsymbol{x}_1\boldsymbol{z}}\hat{S}_{\boldsymbol{z}\boldsymbol{x}_2} - \frac{1}{N_c^2}\,\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\Big)$$

- The $1/N_c^2$ corrections cancel between 'real' and 'virtual' contributions

$$\frac{\partial\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}}\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{z}}\hat{S}_{\boldsymbol{z}\boldsymbol{x}_2} - \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$$

- Physical interpretation: projectile (dipole) evolution

Dipole evolution (2)

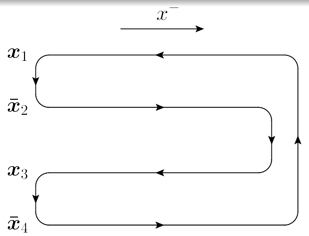
- Use the rapidity increment ($Y \to Y + \mathrm{d}Y$) to boost the dipole
- The dipole 'evolves' by emitting a small–x gluon
- 'Real' term: quark-antiquark-gluon system interacts with the target

- At large N_c, this system looks like two dipoles.
- 'Virtual' term: the emitted gluon does not interact with the target

- The probability for the dipole not to evolve.

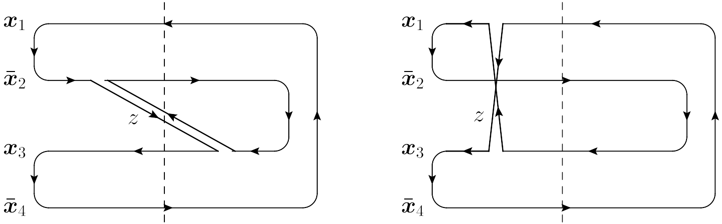
Quadrupole evolution (1)

$$\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} = \frac{1}{N_c}\,\mathrm{tr}(V^{\dagger}_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} V^{\dagger}_{\boldsymbol{x}_3} V_{\boldsymbol{x}_4})$$



$$\frac{\partial\langle\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{4\pi}\int_{\boldsymbol{z}}\Big[(\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_4\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_4\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{z}}\hat{Q}_{\boldsymbol{z}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$$
$$+ \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{z}\boldsymbol{x}_2}\hat{Q}_{\boldsymbol{x}_1\boldsymbol{z}\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$$
$$+ (\mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_3\boldsymbol{x}_4\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_4\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{x}_3\boldsymbol{z}}\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}\boldsymbol{x}_4}\rangle_Y$$
$$+ (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_4\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_3\boldsymbol{x}_4\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{z}\boldsymbol{x}_4}\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{z}}\rangle_Y$$
$$- (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_3\boldsymbol{x}_4\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_4\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{z}})\langle\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$$
$$- (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_3\boldsymbol{x}_4\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_4\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$$
$$- (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_4\boldsymbol{z}} + \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_3\boldsymbol{z}} - \mathcal{M}_{\boldsymbol{x}_2\boldsymbol{x}_4\boldsymbol{z}})\langle\hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_2}\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_4}\rangle_Y\Big]$$

- More complicated, but the same structural properties as for the dipole:
  - Real terms ($2n+2=6$ WL's) : $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}} \hat{Q}_{\boldsymbol{z}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$
  - Virtual terms ($2n=4$ WL's) : $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$, $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_4} \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_2} \rangle_Y$



  - $1/N_c^2$ corrections have cancelled between 'real' and 'virtual'
  - Single–trace couples to double–trace under the evolution

The limit of a large number of colors: $N_c \to \infty$

- Multi–trace expectation values of WL's factorize into single–trace ones

$$\left\langle \frac{1}{N_c}\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} \dots)\, \frac{1}{N_c}\mathrm{tr}(V^\dagger_{\boldsymbol{y}_1} V_{\boldsymbol{y}_2})\right\rangle_Y \simeq \left\langle \frac{1}{N_c}\mathrm{tr}(V^\dagger_{\boldsymbol{x}_1} V_{\boldsymbol{x}_2} \dots)\right\rangle_Y \left\langle \frac{1}{N_c}\mathrm{tr}(V^\dagger_{\boldsymbol{y}_1} V_{\boldsymbol{y}_2})\right\rangle_Y$$

- B–JIMWLK hierarchy boils down to closed equations
- Dipole: $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}} \hat{S}_{\boldsymbol{z}\boldsymbol{x}_2}\rangle \simeq \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}}\rangle \langle \hat{S}_{\boldsymbol{z}\boldsymbol{x}_2}\rangle \Longrightarrow$ Balitsky–Kovchegov (BK)

$$\frac{\partial \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} \left[\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}}\rangle_Y \langle \hat{S}_{\boldsymbol{z}\boldsymbol{x}_2}\rangle_Y - \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y\right]$$

- Closed, non–linear equation for $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$, studied at length.
- Saturation momentum : unitarity limit for the dipole scattering

$$\langle \hat{S}(r)\rangle_Y \sim \mathcal{O}(1) \quad \text{when} \quad 1/r \sim Q_s(Y) \propto \mathrm{e}^{\lambda Y}$$

The limit of a large number of colors: $N_c \to \infty$

- Quadrupole: $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}} \hat{Q}_{\boldsymbol{z}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y \simeq \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}} \rangle_Y \langle \hat{Q}_{\boldsymbol{z}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$

$$\frac{\partial \langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{4\pi} \int_{\boldsymbol{z}} \Big[(\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \cdots) \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{z}} \rangle_Y \langle \hat{Q}_{\boldsymbol{z}\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$$
$$+ \ldots\ldots$$
$$- (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \cdots) \langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$$
$$- (\mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}} + \cdots) \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y \langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y \Big].$$

- An equation for $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$ with $\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y$ acting as a source.
- Numerical solution still complicated (due to real terms)
  - non–linear terms
  - transverse non–locality (integral over $\boldsymbol{z}$)
- In practice it is easier to solve the full JIMWLK equation (finite N_c) using its reformulation as a (functional) Langevin equation

  (Blaizot, E.I., Weigert, 2002) cf. talk by T. Lappi

Towards a Gaussian approximation

- The prototype for it: the McLerran–Venugopalan model

$$W_{\rm MV}[\rho] \;=\; \exp\left[-\frac{1}{2}\int {\rm d}x^- \int_{\boldsymbol{x}} \frac{\rho^a(x^-,\boldsymbol{x})\rho^a(x^-,\boldsymbol{x})}{\lambda(x^-)}\right]$$

- Large nucleus ($A \gg 1$), not so small x :
  'color sources' = independent valence quarks
- $\rho_a(x^-, \boldsymbol{x})$ color charge density : $-\nabla_\perp^2 \alpha_a = \rho_a$
- Often used as an initial condition for JIMWLK at $Y_0 \sim 4$
- Could a Gaussian be a reasonable approximation also at $Y \gg Y_0$?
  - high energy evolution introduces correlations among the color sources
  - non–linear effects ⇒ coupled equations for n–point functions of WL's
- Yet... there is impressive agreement between numerical solutions to JIMWLK and simple extrapolations of the MV model !

(Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan 2011)

Some encouraging arguments (1)

- In the dilute regime $(k_\perp \gg Q_s(Y)$ or $|\boldsymbol{x}_i - \boldsymbol{x}_j| \ll 1/Q_s(Y))$, the correlations refer to the BFKL evolution of the 2–point function :

$$\langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y \simeq 1 - \frac{g^2}{4N_c}\langle(\alpha^a_{\boldsymbol{x}_1} - \alpha^a_{\boldsymbol{x}_2})^2\rangle_Y \equiv 1 - \langle \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$$

$$1 - \langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y \simeq \langle \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2} - \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_3} + \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_4} + \hat{T}_{\boldsymbol{x}_2\boldsymbol{x}_3} - \hat{T}_{\boldsymbol{x}_2\boldsymbol{x}_4} + \hat{T}_{\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$$

- $\langle \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$ (dipole scattering amplitude) obeys the BFKL equation :

$$\frac{\partial \langle \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{z}}\,\langle \hat{T}_{\boldsymbol{x}_1\boldsymbol{z}} + \hat{T}_{\boldsymbol{z}\boldsymbol{x}_2} - \hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$$

- A 2–point function can always be encoded in a Gaussian !

Some encouraging arguments (2)

- Saturation regime : $k_\perp \ll Q_s(Y)$ or $|\boldsymbol{x}_i - \boldsymbol{x}_j| \gg 1/Q_s(Y)$

  $\longrightarrow$ 'keep only the first term (no WL's) in $H_{\rm JIMWLK}$'

$$H = -\frac{1}{16\pi^3}\int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} \left(1 + \widetilde{V}^\dagger_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}} - \widetilde{V}^\dagger_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{z}} - \widetilde{V}^\dagger_{\boldsymbol{z}}\widetilde{V}_{\boldsymbol{v}}\right)^{ab} \frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

'Random phase approximation' *(E.I. & McLerran, 2001)*

$$H_{\rm RPA} \simeq -\frac{1}{8\pi^2}\int_{\boldsymbol{uv}} \ln\left[(\boldsymbol{u}-\boldsymbol{v})^2 Q_s^2(Y)\right] \frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^a_{\boldsymbol{v}}}$$

- Free diffusion ... obviously consistent with a Gaussian weight function !
- Qualitatively right, but a bit naive though !
- The first two terms within $H_{\rm JIMWLK}$ act on the same footing !

  together, they generate the 'virtual' terms in the B-JIMWLK equations

On the importance of the virtual terms

$$H_{\rm virt} = -\frac{1}{16\pi^3}\int_{\boldsymbol{uvz}} \mathcal{M}_{\boldsymbol{uvz}} \left(1 + \widetilde{V}^{\dagger}_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}}\right)^{ab} \frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

- The virtual terms dominate the evolution deeply at saturation
  - surprising at the first sight: the non–linear effects are encoded precisely in the real terms
  - even less obvious at finite N_c : real and virtual term seem to receive $1/N_c^2$ corrections of the same order
- One can promote $H_{\rm virt}$ into a mean field approximation to $H_{\rm JIMWLK}$ which is valid both in the dense and the dilute regimes !
- Is this consistent with a Gaussian weight function $W_Y[\alpha]$?

  $H_{\rm virt}$ is still non–linear to all orders in the field α_a ...

Virtual terms dominate deeply at saturation

- They control the approach towards the 'black disk limit': $\langle \hat{S} \rangle_Y \to 0$, $\langle \hat{Q} \rangle_Y \to 0$, etc.
- Easier to understand at large N_c ; e.g. for the dipole (BK equation)

$$\frac{\partial \langle \hat{S}_{\boldsymbol{x}_1 \boldsymbol{x}_2} \rangle_Y}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{x}_1 \boldsymbol{x}_2 \boldsymbol{z}} \left[\langle \hat{S}_{\boldsymbol{x}_1 \boldsymbol{z}} \rangle_Y \langle \hat{S}_{\boldsymbol{z} \boldsymbol{x}_2} \rangle_Y - \langle \hat{S}_{\boldsymbol{x}_1 \boldsymbol{x}_2} \rangle_Y \right]$$

- Deeply at saturation: $\langle \hat{S} \rangle_Y \langle \hat{S} \rangle_Y \ll \langle \hat{S} \rangle_Y \ll 1$

$$\frac{\partial \langle \hat{S}(r) \rangle_Y}{\partial Y} \simeq -\bar{\alpha} \ln[r^2 Q_s^2(Y)] \, \langle \hat{S}(r) \rangle_Y$$

- A Sudakov factor : the probability for the dipole not to evolve.
- The conclusion persists at finite N_c, for the same physical reason:

  the dipole (quadrupole, etc) has more chances to survive its scattering off the CGC if it remains simple !

Virtual terms can encode BFKL too...

- ... provided one generalizes the kernel in the Hamiltonian:

$$H_{\rm MFA} = -\frac{1}{2}\int_{\boldsymbol{uv}} \gamma_Y(\boldsymbol{u},\boldsymbol{v})\big(1+\widetilde{V}^\dagger_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}}\big)^{ab}\frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

- Mean–field evolution of the dipole :

$$\frac{\partial\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y}{\partial Y} = \langle H_{\rm MFA}\,\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y = -2g^2C_F\,\gamma_Y(\boldsymbol{x}_1,\boldsymbol{x}_2)\langle\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y$$

- Weak scattering (BFKL): $\langle\hat{S}\rangle_Y = 1-\langle\hat{T}\rangle_Y$ with $\langle\hat{T}\rangle_Y \ll 1$

$$\frac{\partial\langle\hat{T}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y}{\partial Y} = 2g^2C_F\,\gamma_Y(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

- Use this equation, with the l.h.s. estimated at the BFKL level, as the definition of $\gamma_Y(\boldsymbol{x}_1,\boldsymbol{x}_2)$ for $|\boldsymbol{x}_1-\boldsymbol{x}_2| \ll 1/Q_s(Y)$

The Mean Field Approximation

- ... is defined by the following Hamiltonian:

$$H_{\mathrm{MFA}} = -\frac{1}{2}\int_{\boldsymbol{uv}} \gamma_Y(\boldsymbol{u},\boldsymbol{v})\big(1+\widetilde{V}^{\dagger}_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}}\big)^{ab}\frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

- ... where the kernel $\gamma_Y(\boldsymbol{u},\boldsymbol{v})$ is uniquely defined
  - in the dilute regime at $|\boldsymbol{u}-\boldsymbol{v}| \ll 1/Q_s(Y)$ (BFKL)
  - in the dense regime at $|\boldsymbol{u}-\boldsymbol{v}| \gg 1/Q_s(Y)$
- The transition region around $|\boldsymbol{u}-\boldsymbol{v}| \sim 1/Q_s(Y)$ goes beyond the accuracy of the MFA $\Rightarrow$ any smooth interpolation is equally good
- In practice: trade the kernel for the dipole S–matrix :

$$\gamma_Y(\boldsymbol{u},\boldsymbol{v}) \;=\; -\frac{1}{2g^2C_F}\frac{\partial\ln\langle\hat{S}_{\boldsymbol{uv}}\rangle_Y}{\partial Y}$$

The Mean Field Approximation

- ... is defined by the following Hamiltonian:

$$H_{\rm MFA} = -\frac{1}{2}\int_{\boldsymbol{uv}} \gamma_Y(\boldsymbol{u},\boldsymbol{v})\big(1+\widetilde{V}^\dagger_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}}\big)^{ab}\frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\,\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

- ... where the kernel $\gamma_Y(\boldsymbol{u},\boldsymbol{v})$ is uniquely defined
  - in the dilute regime at $|\boldsymbol{u}-\boldsymbol{v}| \ll 1/Q_s(Y)$ (BFKL)
  - in the dense regime at $|\boldsymbol{u}-\boldsymbol{v}| \gg 1/Q_s(Y)$
- The transition region around $|\boldsymbol{u}-\boldsymbol{v}| \sim 1/Q_s(Y)$ goes beyond the accuracy of the MFA $\Rightarrow$ any smooth interpolation is equally good
- The kernel is independent of N_c $\Rightarrow$ can be inferred from the solution to the BK equation (large N_c) ... and then used at finite N_c :

$$\gamma_Y(\boldsymbol{u},\boldsymbol{v}) \;=\; -\frac{1}{g^2N_c}\frac{\partial \ln\langle \hat{S}^{\rm BK}_{\boldsymbol{uv}}\rangle_Y}{\partial Y}$$

N.B. this yields the same kernel as Heribert's 'Gaussian truncation'

Evolution equations in the MFA

- Obtained by keeping only the virtual terms in the respective B–JIMWLK equations and replacing the kernel according to

$$\frac{1}{8\pi^3}\int_{\boldsymbol{z}} \mathcal{M}_{\boldsymbol{uvz}} \to \gamma_Y(\boldsymbol{u},\boldsymbol{v})$$

- Considerably simpler than the original equations :
  - linear
  - local in transverse coordinates
  - coupled, but closed, systems: they couple only n–point functions with the same value of n (e.g. $\langle \hat{Q} \rangle_Y$ with $\langle \hat{S}\hat{S} \rangle_Y$)
- The equations can be solved analytically.
- The solutions becomes especially simple if
  - the kernel is separable: $\gamma_Y(\boldsymbol{u},\boldsymbol{v}) = h_1(Y)\, g(\boldsymbol{u},\boldsymbol{v}) + h_2(Y)$
  - at large N_c (any kernel)
  - for special configurations of the external points in the transverse space

The MV model strikes back

- The mean–field equations allow one to compute the n–point functions of the WL's with $n \geq 4$ in terms of the dipole S matrix $\langle \hat{S} \rangle_Y$ $(n = 2)$
- For a separable kernel, the Y–dependence in the final results enters exclusively via $\langle \hat{S} \rangle_Y$

  $\triangleright$ separability is a good approximation, in both dense and dilute limits
- In that case, the functional form of the solutions is formally the same as in the MV model !
- This is rewarding: it explains the numerical findings in arXiv:1108.4764 *(Dumitru, Jalilian-Marian, Lappi, Schenke, Venugopalan 2011)*
- ... but it also rises a puzzle: it strongly suggests that the mean field approximation has an underlying Gaussian structure
- How is that possible ?

The Gaussian CGC weight function

$$H_{\rm MFA} = -\frac{1}{2}\int_{\boldsymbol{uv}} \gamma_Y(\boldsymbol{u},\boldsymbol{v})\big(1+\widetilde{V}^\dagger_{\boldsymbol{u}}\widetilde{V}_{\boldsymbol{v}}\big)^{ab}\frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{v}}}$$

- The functional derivatives act as generators of color rotations:

$$\frac{\delta}{\delta\alpha^a_{\boldsymbol{u}}}V^\dagger_{\boldsymbol{x}} = \mathrm{i}g\delta_{\boldsymbol{xu}}\,t^a V^\dagger_{\boldsymbol{x}} \qquad \widetilde{V}^{ab}_{\boldsymbol{u}}\frac{\delta}{\delta\alpha^b_{\boldsymbol{u}}}V^\dagger_{\boldsymbol{x}} = \mathrm{i}g\delta_{\boldsymbol{xu}}\,V^\dagger_{\boldsymbol{x}}t^a,$$

- ... both on the left and on the right

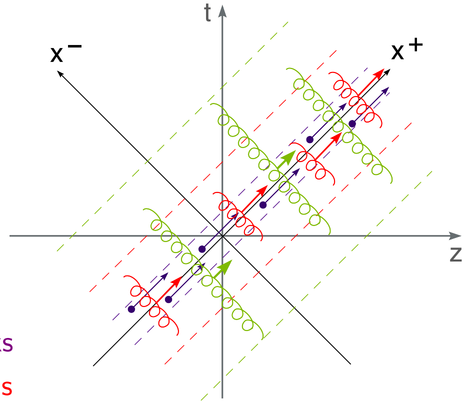
$$H_{\rm MFA} = -\frac{1}{2}\int_{\boldsymbol{uv}} \gamma_Y(\boldsymbol{u},\boldsymbol{v})\left(\frac{\delta}{\delta\alpha^a_{L\boldsymbol{u}}}\frac{\delta}{\delta\alpha^a_{L\boldsymbol{v}}}+\frac{\delta}{\delta\alpha^a_{R\boldsymbol{u}}}\frac{\delta}{\delta\alpha^a_{R\boldsymbol{v}}}\right)$$

- This is free diffusion ... but simultaneously 'towards the left' (increasing x^-) and 'towards the right' (decreasing x^-)
- With increasing Y, the target color field expands symmetrically in x^- around the light–cone $(x^- = 0)$
- The CGC weight function in the MFA is a Gaussian symmetric in x^-

Longitudinal structure of the CGC

$$W_Y[\alpha] = \mathcal{N}_Y \exp\left\{-\frac{1}{2}\int_{-x_M^-(Y)}^{x_M^-(Y)} \mathrm{d}x^- \int_{\boldsymbol{x}_1\boldsymbol{x}_2} \frac{\alpha_a(x^-,\boldsymbol{x}_1)\alpha_a(x^-,\boldsymbol{x}_2)}{\gamma(x^-,\boldsymbol{x}_1,\boldsymbol{x}_2)}\right\}$$

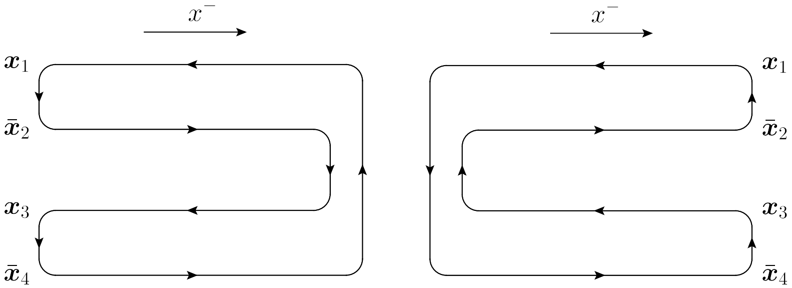
- $x_M^-(Y) = x_0^- \exp(Y - Y_0)$



- valence quarks
- small x gluons
- even smaller x gluons

The mirror symmetry

- This has observable consequences: $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y = \langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_4\boldsymbol{x}_3\boldsymbol{x}_2} \rangle_Y$

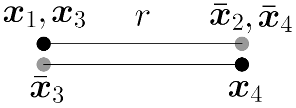


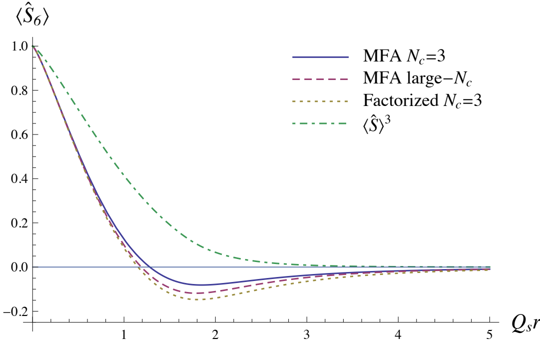
- Time reversal symmetry for the projectile (with 'time' $= x^-$).
- Similar identities hold for the higher n–point functions.
- An exact symmetry of the JIMWLK equation.

Applications to special configurations

- Di–hadron correlations: quadrupole × dipole — line configuration

$$\hat{S}_{6\,\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} = \frac{N_c^2}{N_c^2-1}\,\hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\hat{S}_{\boldsymbol{x}_4\boldsymbol{x}_3} - \frac{1}{N_c^2-1}\,\hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}$$

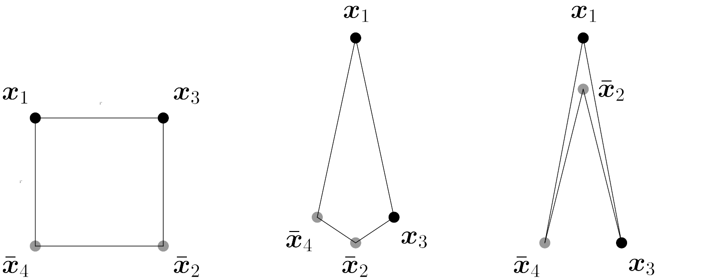




- Our full MFA result cannot be distinguished from the numerical solution to JIMWLK *(Dumitru et al, 2011)*

A versatile configuration

- $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$ with $r_{13} = r_{14}$ and $r_{23} = r_{24}$ & arbitrary r_{12} and r_{34}

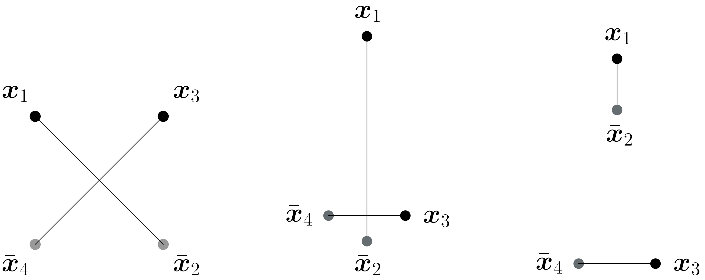


- One finds exact factorization: $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y = \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y \langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y$
- Natural when $r_{12}, r_{34} \ll r_{14}, r_{23}$ ... but remarkable in general.

$$\langle \hat{S}_{6\,\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y = \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y \Big[\langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y \Big]^{\frac{2N_c^2}{N_c^2-1}} \simeq \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2} \rangle_Y \Big[\langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4} \rangle_Y \Big]^2$$

A versatile configuration

- $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$ with $r_{13} = r_{14}$ and $r_{23} = r_{24}$ & arbitrary r_{12} and r_{34}



- One finds exact factorization: $\langle \hat{Q}_{\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y = \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y \langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y$
- Natural when $r_{12}, r_{34} \ll r_{14}, r_{23}$ ... but remarkable in general.

$$\langle \hat{S}_{6\,\boldsymbol{x}_1\boldsymbol{x}_2\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y = \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y \left[\langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y\right]^{\frac{2N_c^2}{N_c^2-1}} \simeq \langle \hat{S}_{\boldsymbol{x}_1\boldsymbol{x}_2}\rangle_Y \left[\langle \hat{S}_{\boldsymbol{x}_3\boldsymbol{x}_4}\rangle_Y\right]^2$$

THANK YOU !