6-1   L11, 21.2.-11 ⟼   17.2.-11

QFT on the Light cone (LC)

In the Heisenberg picture of QM, the operators are time dependent. Their quantum nature is expressed in commutation relations of operators at equal time t

usual QM: $[A(t), B(t')]\big|_{t=t'} = \ldots$

Since the theory is Lorentz-invariant, one can boost to any coordinate system moving at velocity $v < 1$.

However, one can not boost to exactly $v = 1$.

Thus there is a separate way to define QM / QFT, taking as a time coordinate the trajectory of a particle moving at $v = 1$

LC time x^+, $[A(x^+), B(y^+)]\big|_{x^+=y^+} = \ldots$

The Hamiltonian is the operator that generates translations in time, i.e. now

LC Hamiltonian p^- ← Commutes with the maximal number of Poincaré generators!

~~One way to define~~

We will now canonically quantize field theory on the light cone.

Scalar: $\phi(x) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3}\sqrt{2p^+}} \left[e^{ip\cdot x} a^+(\bar{p}) + e^{-ip\cdot x} a(\bar{p}) \right]$

p^- energy
p^+ momentum!

- $\bar{p} = (p^+, \underset{\sim}{p})$!
- on shell $p^- = \frac{\underset{\sim}{p}^2 + m^2}{2p^+}$ → if p^+ constant (conserved), just like 2d nonrelativistic (Problematic if p^+ not = const.)

Funny normalization with $\sqrt{2p^+}$, to stay consistent with references

Commutation relations

$$[a(\bar{p}), a^+(\bar{q})] = \delta^3(\bar{p} - \bar{q})$$

$$\Leftrightarrow \underbrace{[\partial_+ \phi(x), \phi(y)]}\big|_{x^+=y^+} = -i\delta^3(\bar{x} - \bar{y}) = -i\delta(x^- - y^-)\delta^2(\underset{\sim}{x} - \underset{\sim}{y})$$

$\bar{x} = (x^-, \underset{\sim}{x})$

$$= \frac{\delta}{\delta(\partial_+\phi)} \left[\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi = \partial_-\phi\,\partial_+\phi - \tfrac{1}{2}(\underset{\sim}{\nabla}\phi)^2 \right] \to \partial_-\phi \text{ conjugate to } \phi$$

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Fermion:

$$\psi(x) = \sum_s \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2p^+}} \left[b_s(\bar{p}) u_s(p) e^{-ip\cdot x} + d_s^+ v_s(p) e^{ip\cdot x} \right]$$

annihilates fermions — creates antifermions

The fermionic operators have ~~com~~ anti commutation relations $\{ b(\bar{p}), b^+(\bar{q}) \} = \delta^3(\bar{p}-\bar{q})$ $\quad (\{A,B\} \equiv AB+BA)$

Vector:

$$A_\mu(x) = \sum_\lambda \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2p^+}} \left[a^{(\lambda)}(\bar{p}) \epsilon_\mu^{(\lambda)}(p) e^{-ip\cdot x} + a^{+(\lambda)}_{(\lambda)} \epsilon_\mu^{(\lambda)*}(p) e^{ip\cdot x} \right]$$

commutation relation $[a(\bar{p}), a^+(\bar{q})] = \delta^3(\bar{p}-\bar{q})$ like for scalar.

To derive the equations of motion etc. one starts from the Lagrangean, in this case for QED:

$$\mathcal{L} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \right] = \int d^4x \, \mathcal{L}$$

The usual procedure is then to derive from this

$\hookrightarrow$ Hamiltonian $\quad \hat{P}^- = \dfrac{\delta L}{\delta(\partial_+ \psi)} \partial_+ \psi + \dfrac{\delta L}{\delta(\partial_+ A^\mu)} \partial_+ A^\mu - \int d^3\bar{x} \, \mathcal{L}$

This is then promoted to an operator in terms of the field operators etc. This gives the quantum Hamiltonian operator that defines the theory. Trying to do this one encounters a few problems:

- Problem 1: $F_{\mu\nu} = -F_{\nu\mu} \to$ no $\partial_+ A_+ = \partial_+ A^- \to A^-$ not dynamical

  This problem is the same in ordinary canonical quantization; A_0 does not have a canonical momentum conjugate. It is solved in a similar way.

  But in the LC case we encounter a similar problem also in the fermion sector.

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Problem 2:

$$1 = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1+\gamma^0\gamma^3) + \frac{1}{\sqrt{2}}(1-\gamma^0\gamma^3)\right)$$

$$= \frac{\gamma^0}{\sqrt{2}}(\gamma^+ + \gamma^-) = \mathbb{P}^+ + \mathbb{P}^- \quad , \mathbb{P}^+\mathbb{P}^- = 0$$

Thus $\mathbb{P}^+$ and $\mathbb{P}^-$ are projection operators that divide the vector space of the ψ:s into two ~~(different)~~ separate subspaces. $\psi = \psi^- + \psi^+$
$= \mathbb{P}^-\psi + \mathbb{P}^+\psi$

$$\partial\!\!\!/ = \gamma^+\partial_+ + \gamma^-\partial_- - \underset{\sim}{\gamma}\cdot\underset{\sim}{\partial}$$

$$\sim \partial_+\psi^+ + \partial_-\psi^- \quad \rightarrow \text{no } \partial_+\psi^-$$

$$\rightarrow \psi^- \text{ not dynamical}$$

In other words, one does not get an equation of motion for ψ^- (i.e. an equation for its time development) but only a constraint expressing ψ^- in terms of ψ^+.

Both of these problems can, however, be solved. This makes the quantization procedure a bit more involved formally.

⋮

$$\hat{P}^- = \int d^3\bar{x}\Big[\;\bar{\psi}\,\frac{m^2+(i\underset{\sim}{\partial})^2}{2i\partial_-}\gamma^+\psi + \frac{1}{2}A^\mu(i\underset{\sim}{\partial})^2A_\mu \Big\}\;\hat{P}_0^-$$

$$\hat{P}^-_{int}\left\{\begin{array}{l} + e\,\bar{\psi}A\!\!\!/\psi \qquad \leftarrow \text{normal } \gamma f\bar{f} \text{ vertex} \\[2ex] + e^2\bar{\psi}\gamma^+\psi\frac{1}{(i\partial_-)^2}\bar{\psi}\gamma^+\psi + e^2\bar{\psi}A\!\!\!/\frac{\gamma^+}{i\partial_-}A\!\!\!/\psi \;\Big] \end{array}\right.$$

funny instantaneous $f\bar{f}f\bar{f}$ and $\gamma\gamma f\bar{f}$ like Coulomb interaction in normal QFT.

Note: here instantaneous is not acausal, since light can propagate along the x^--axis and connect separate points at equal LC time x^+.

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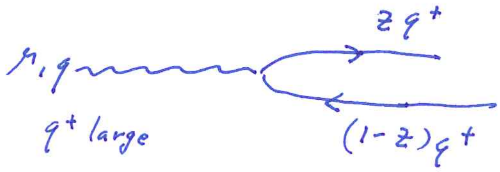
Disadvantages of LC QFT:

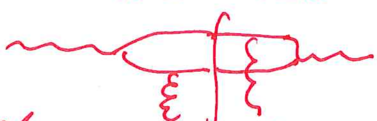
- not manifestly Lorentz-covariant
- more complicated interactions
- problematic zero-modes $p^+ = 0$

Advantages

- no $\sqrt{\ }$ in dispersion relation ($\sqrt{\vec p^2 + m^2}$ vs. $\frac{p_\perp^2 + m^2}{2p^+}$)
- Fock space vacuum simple; no chiral condensate
- especially useful for bound state properties

γ^* wave function



We ~~often~~ remember that in ~~the~~ (~) target rest frame picture DIS at high W can be ~~..~~ factorized into a) γ^* splitting into $q\bar q$ dipole and b) $q\bar q$ dipole scattering eikonally off the target. We shall now calculate part a) in LC perturbation theory. The same result could, of course, be obtained in ordinary covariant scattering theory from diagrams like  etc, but there one needs to specify some interaction with the target. The calculation we do now better reflects the physical picture: splitting $\gamma^* \to q\bar q$ is independent of scattering $q\bar q$ + target.

$|\gamma^*(q)\rangle_0 = a^\dagger_{(\lambda)}(\vec q)\,|0\rangle$ free γ^*, $\hat P_0^- |\gamma^*\rangle_0 = q^- |\gamma^*\rangle_0$

$|q(k)\bar q(k')\rangle_0 = b^\dagger(k)\, d^\dagger(k')\,|0\rangle$ free $q\bar q$: $\hat P_0^- |q\bar q\rangle_0 = (k^- + k'^-)|q\bar q\rangle_0$

$|\gamma^*\rangle$ = interacting γ^*, $(\hat P_0^- + \hat P_{int}^-)|\gamma^*\rangle = q^- |\gamma^*\rangle$

We do ordinary ~~..~~ quantum mechanical perturbation theory to ~~..~~ find an expression for $|\gamma^*\rangle$ in terms of the eigenstates of the free Hamiltonian.

$$|\gamma^*\rangle = |\gamma^*\rangle_0 + \int d^3\bar{l}\, d^3\bar{l}'\, \delta^3(\bar{q}-\bar{l}-\bar{l}')\, \underbrace{\psi(\bar{l})}_{\sim e}\, |q(l)\bar{q}(l')\rangle_0 + \mathcal{O}(e^2)$$

- omit spin, helicity, momentum labels for now.

1) ${}_0\langle q(k)\bar{q}(k')|$

${}_0\langle \bar{q}q|\gamma^0\rangle_0 = 0$; ${}_0\langle \bar{q}(k')q(k)|q(l)\bar{q}(l')\rangle_0 = \delta^3(\bar{k}-\bar{l})\delta^3(\bar{k}'-\bar{l}')$

$${}_0\langle q(k)\bar{q}(k')|\gamma^*\rangle = 0 + \delta^3(\bar{q}-\bar{k}-\bar{k}')\,\psi(\bar{k})$$

2) ${}_0\langle q(k)\bar{q}(k')|(P_0^- + P_{int}^-)$

$\psi\, {}_0\langle q\bar{q}|P_{int}^-|q\bar{q}\rangle_0$ goes here

$$q^-\, {}_0\langle q(k)\bar{q}(k')|\gamma^*\rangle = q^-\cdot 0 + {}_0\langle q(k)\bar{q}(k')|P_{int}^-|\gamma^*\rangle_0 + (k^- + k'^-)\,\delta^3(\bar{q}-\bar{k}-\bar{k}')\,\psi(\bar{k}) + \mathcal{O}(e^2)$$

$$\delta^3(\bar{q}-\bar{k}-\bar{k}')\,\psi(\bar{k}) = \frac{{}_0\langle q(k)\bar{q}(k')|P_{int}^-|\gamma^*\rangle_0}{q^- - k^- - k'^-}$$

← usual energy denominator in QM perturbation theory.

- a, a^+, b, b^+, d, d^+, (anti) commutators

$$\Rightarrow \ldots \Rightarrow \psi(\bar{k}) = \underbrace{\frac{e\,\bar{u}(k)}{\sqrt{(2\pi)^3 2k^+}}\;\frac{\varepsilon(q)}{\sqrt{(2\pi)^3 2q^+}}\;\frac{v(k')}{\sqrt{(2\pi)^3 2k'^+}}}_{\text{external lines}}\;\underbrace{\frac{(2\pi)^3}{q^- - k^- - k'^-}}_{\text{energy denominator}}$$

→ from δ-function

LC Feynman rules:

1. $k^+ = zq^+$, $k'^+ = (1-z)q^+$; $\int dk^+ |\psi(\bar{k})|^2 = \int dz\, |\psi(z)|^2$

$$\Rightarrow \psi(z) = \sqrt{q^+}\,\psi(\bar{k})$$

2. $$q^- - k^- - k'^- = -\frac{Q^2}{2q^+} - \frac{m^2+\underline{k}^2}{2zq^+} - \frac{m^2+\underline{k}^2}{2(1-z)q^+} = -\frac{\overbrace{Q^2 z(1-z)+m^2}^{\equiv \varepsilon^2}+\underline{k}^2}{2q^+ z(1-z)}$$

3. $$\psi(\underline{r},z) = \int \frac{d^2\underline{k}}{\sqrt{(2\pi)^2}}\, e^{i\underline{k}\cdot\underline{r}}\,\psi(z,\underline{k})$$

$\underline{r}$ = size of dipole

$$\left(e^{i\underline{k}\cdot\underline{x}}\, e^{i\underline{k}'\cdot\underline{y}} = e^{i\underbrace{\frac{\underline{k}-\underline{k}'}{2}}_{=\underline{k}}(\underline{x}-\underline{y}) + \underbrace{(\underline{k}+\underline{k}')}_{=0}\cdot\frac{\underline{x}+\underline{y}}{2}} \right)$$

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Now we also need the matrix element $\bar u \chi v$.
This depends on the spin and polarizations of γ^* and $q\bar q$.
Calculate, for example, the longitudinal γ^*:

$$\varepsilon_L^\mu(q) = \left(\frac{q^+}{Q}, \frac{Q}{2q^+}, \mathbf{0}\right) \xrightarrow{-\frac{q^\mu}{Q}} \left(0, \frac{Q}{q^+}, \mathbf{0}\right)$$

cov. gauge — LC gauge, $A^+=0$

Now a) with explicit γ-matrix representation or
b) Pauli, hep-ph 10103106

$$\frac{\bar u_s(k)\gamma^+ v(k')}{\sqrt{2k^+ 2k'^+}} = \delta_{s,-s'}$$

$$\Psi_L(z, \mathbf{r}) = \frac{-e}{\sqrt{(2\pi)^3}} \frac{\sqrt{q^+}}{\sqrt{2q^+}} \frac{Q}{q^+} \delta_{s,-s'} \underbrace{\int \frac{d^2k}{2\pi} \frac{2q^+ z(1-z)}{\varepsilon^2 + \mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{r}}}_{2q^+ z(1-z) K_0(\varepsilon r)}$$

$$= \frac{-e}{2\pi\sqrt{\pi}} \delta_{s,-s'} Q z(1-z) K_0(\varepsilon r)$$

$$\sum_{s,s'} |\Psi_L|^2 = \frac{2\alpha}{\pi^2} Q^2 z^2(1-z)^2 K_0^2(\varepsilon r)$$

$K_0(x) \sim e^{-x}$
$\Rightarrow \frac{1}{r} \sim \varepsilon \sim Q$

correct:
- avg color
- charge $\frac{1}{3}$ (d), $\frac{2}{3}$ (u)

T: K_0, K_1

Now the virtual photon-target cross section comes to

$$\sigma^{\gamma^* p}_{L,T} = \underbrace{\int d^2\mathbf{r} \int_0^1 dz \, |\Psi_{L,T}(\mathbf{r}, z)|^2}_{\gamma^* \to \text{ dipole } \mathbf{r}} \cdot \underbrace{2\, \mathrm{Im}\, A_{q\bar q p}(\mathbf{r}, x)}_{\text{dipole cross section}}$$

This is why the ep cross section ~~[illegible]~~ can really be thought of as a hadron-hadron cross-section; the photon is effectively a $q\bar q$ pair. The same form can be derived in usual pQCD, with the result $A_{q\bar q p} \sim x g(x, Q^2)$, but this admits a more general scattering amplitude.