Lecture notes for NITheP

and

CPTEIC

1) Kinematics

2) Introduction to DIS, parton model

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Kinematics

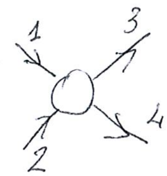
Consider a generic scattering process

$$1+2 \longrightarrow 3+4+\cdots+N$$

$\Rightarrow$ the number of total Lorentz-invariant variable is $3N-10$
(impose conservation of 4-momenta and mass-shell conditions)
+ arbitrariness in fixing a 4-dimensional reference frame: 6 constraints

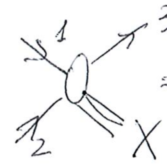
We consider 2 type of process

1) 2-body exclusive scattering : $1+2 \to 3+4$



$\Rightarrow$ 2-indep. variable

2) single-particle inclusive scattering: $1+2 \to 3+X$



$\Rightarrow$ 3-indep. variable

X is an unresolved system of particles

Elastic scattering is denoted by $1+2 \to 1'+2'$ (particular case of 1))

Single diffractive dissociation $1+2 \to 1'+X_2$ (particular case of 2))

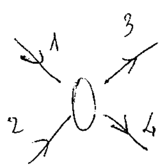
X_2 carries the same quantum of particle 2

Mandelstam variables

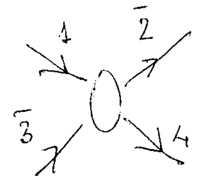
The reaction $1+2 \to 3+4$ (s-channell) is described by 2 variables and usually these are chosen among the 3 Mandelstam invariants

$$\left.\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 \\ p_1+p_2 &= p_3+p_4 \\ p_i^2 &= m_i^2 \quad i=1,\dots,4 \end{aligned}\right\} \Rightarrow s+t+u = \sum_{i=1}^{4} m_i^2$$

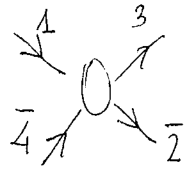
Usually we choose s and t.



s-channell $\Leftrightarrow$ [$s=(p_1+p_2)^2$ represents the CM (Center of Mass) energy and t is the square momentum transfer]



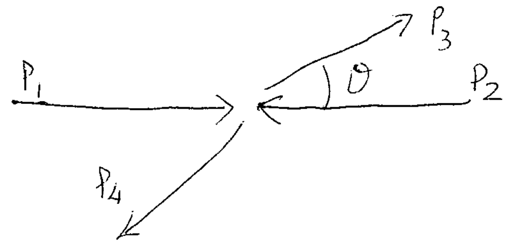
$1+\bar{3} \to \bar{2}+4$ (t-channell) $\Leftrightarrow$ t CM energy



$1+\bar{4} \to \bar{2}+3$ (u-channell) $\Leftrightarrow$ u CM energy

where, for example, $\bar{3}$ represents the antiparticle of 3: all the additive quantum number have changed sign and have opposite momentum

The center of mass reference frame



Consider the s-channell in CM $\Rightarrow$ $\vec{p}_1 + \vec{p}_2 = 0$

We assume p_1 and p_2 along z $\Rightarrow$

$$p_1 = (E_1, \vec{p}) = (E_1, 0, 0, p_z)$$
$$p_2 = (E_2, -\vec{p}) = (E_2, 0, 0, -p_z)$$
$$p_3 = (E_3, \vec{p}\,') = (E_3, \vec{p}_\perp, p'_z)$$
$$p_4 = (E_4, -\vec{p}\,') = (E_4, -\vec{p}_\perp, -p'_z)$$

Note: $(0, p_\perp, 0) \equiv$
$\equiv (0, p^1, p^2, 0)$

Remember: only 2-independent variable $\Rightarrow$ chose $|\vec{p}| = p_z$ and the scattering angle θ

where θ is defined by
$$p'_z = |\vec{p}\,'| \cos\theta$$
$$|\vec{p}_\perp| = |\vec{p}\,'| \sin\theta$$

Now we can express E_1, E_2, E_3, E_4 in terms of s

Use
$$s = (p_1+p_2)^2 = m_1^2 + m_2^2 + 2(E_1E_2 + p_z^2)$$
$$= m_1^2 + m_2^2 + 2\sqrt{s}\,E_1 - 2E_1^2 + 2p_z^2$$
$$= m_1^2 + m_2^2 + 2E_1\sqrt{s} - 2(m_1^2 + p_z^2) + 2p_z^2$$

$\longleftarrow$ use $p_1^2 = m_1^2 = E_1^2 - p_z^2$
$\Downarrow$
$E_1^2 = m_1^2 + p_z^2$

$$\Rightarrow E_1 = \frac{1}{2\sqrt{s}}\left(s + m_1^2 - m_2^2\right)$$

similarly we get

$$E_2 = \frac{1}{2\sqrt{s}}\left(s + m_2^2 - m_1^2\right)$$

$$E_3 = \frac{1}{2\sqrt{s}}\left(s + m_3^2 - m_4^2\right)$$

$$E_4 = \frac{1}{2\sqrt{s}}\left(s + m_4^2 - m_3^2\right)$$

Using the mass shell condition we get the following relation

$$|p|^2 = p_z^2 = E_1^2 - m_1^2 = \frac{1}{4s}\left(s + m_1^2 - m_2^2\right)^2 - m_1^2$$

$$= \frac{1}{4s}\left[s - (m_1+m_2)^2\right]\left[s - (m_1-m_2)^2\right]$$

$$= \frac{1}{4s}\lambda\left(s, m_1^2, m_2^2\right)$$

where we define $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$

we can also express $|p'|$ in terms of s

$$|p'|^2 = p_\perp^2 + p_z'^2 = E_3^2 - m_3^2 = \frac{1}{4s}\left(s + m_3^2 - m_4^2\right)^2 - m_3^2$$

$$= \frac{1}{4s}\left[s - (m_3 + m_4)^2\right]\left[s - (m_3 - m_4)^2\right]$$

$$= \frac{1}{4s}\lambda\left(s, m_3^2, m_4^2\right)$$

let's consider the limit of high CM energy $s \to \infty$

$$\Rightarrow \quad E_1 = E_2 = E_3 = E_4 \underset{s\to\infty}{\simeq} \frac{\sqrt{s}}{2}$$

and $|\vec{p}| = |\vec{p}'| \underset{s\to\infty}{\simeq} \frac{\sqrt{s}}{2}$

So we may write

$$P_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1)$$

$$P_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1)$$

$$P_3 = \left(\frac{\sqrt{s}}{2}, P_\perp, \frac{\sqrt{s}}{2}\cos\theta\right)$$

let's now consider the Mandelstam invariant t

$$t = (P_1 - P_3)^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2|\vec{P}||\vec{P}'|\cos\theta \qquad *$$

we now can express $\cos\theta$ in terms of the CM variables using $*$

we get $$\cos\theta = \frac{s^2 + s\left(2t - \sum_{i=1}^{4} m_i^2\right) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda^{\frac{1}{2}}(s, m_1^2, m_2^2)\,\lambda^{\frac{1}{2}}(s, m_3^2, m_4^2)}$$

for the case of equal mass: $m_1 = m_2 = m_3 = m_4 = m$

we have $|\vec{P}| = \frac{1}{2}\sqrt{s - 4m^2}$; $\cos\theta = 1 + \frac{2t}{s - 4m^2}$

the inverse relation are $s = 4(|\vec{P}|^2 + m^2)$; $t = -2|\vec{P}|^2(1 - \cos\theta)$

and $s + t + u = 4m$, and $u = -2|\vec{P}|^2(1 + \cos\theta)$

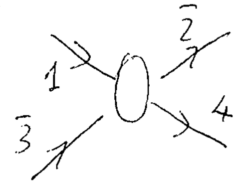
<u>Case of negligible masses</u> $(\equiv s \to \infty)$

$$\cos\theta \simeq 1 + \frac{2t}{s}$$

and using $|P_\perp|^2 = |P'|^2\sin^2\theta = \frac{s}{2}(1 - \cos^2\theta) \simeq \frac{s}{4}\left(1 - \left(1 + \frac{2t}{s}\right)^2\right) \simeq \frac{s}{4}\left(1 - \left(1 + \frac{4t}{s}\right)\right) \simeq -t$

$$\Rightarrow \quad t \underset{s\to\infty}{\simeq} -|P_\perp|^2$$

In the t-channell we have that the CM energy is $t = (p_1 + (-p_3))^2$ whyle the momentum transfer is $s = (p_1 - (-p_2))^2$

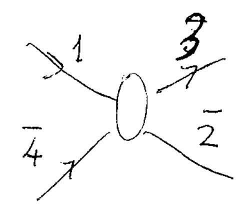


$\Rightarrow$ to find the angle θ_t in the t-channell we just make the replacement $s \leftrightarrow t$ and $m_2 \leftrightarrow m_3$ and get

$$\cos\theta_t = \frac{t^2 + t\left(2s - \sum_i^4 m_i^2\right) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{\lambda^{\frac{1}{2}}(t, m_1^2, m_3^2)\ \lambda^{\frac{1}{2}}(t, m_2^2, m_4^2)}$$

and for equal masses $\cos\theta_t = 1 + \dfrac{2s}{t - 4m^2}$

Similarly in the u-channell, the CM energy, is $u = (p_1 + (-p_4))^2$ whyle the momentum transfer is $t = (p_1 - p_3)^2$

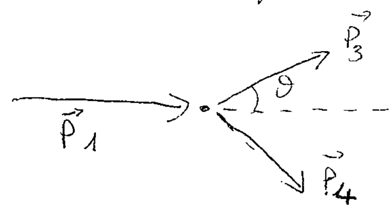


$\Rightarrow$ make $s \Leftrightarrow u$ and $m_4 \Leftrightarrow m_2$ we get

$$\cos\theta_u = \frac{u^2 + u\left(2t - \sum_{i=1}^4 m_i^2\right) + (m_1^2 - m_4^2)(m_3^2 - m_2^2)}{\lambda^{\frac{1}{2}}(u, m_1^2, m_4^2)\ \lambda^{\frac{1}{2}}(t, m_3^2, m_2^2)}$$

equal masses case $\cos\theta_u = 1 + \dfrac{2t}{u - 4m^2}$

The Laboratory Reference frame



s-channel

assume $\vec{p}_2 = 0$

$$p_1 = (E_L, 0, 0, p_L)$$
$$p_2 = (m_2, 0, 0, 0)$$

p_L : Laboratory total momentum

For the outgoing particle we have

$$p_3 = (E_3, \vec{p}_3)$$
$$p_4 = (E_4, \vec{p}_4)$$

Consider the Mandelstam Variable

$$\left.\begin{aligned} s &= (p_1+p_2)^2 = m_1^2 + m_2^2 + 2E_L m_2 \\ t &= (p_2-p_4)^2 = m_2^2 + m_4^2 - 2m_2 E_4 \\ u &= (p_2-p_3)^2 = m_2^2 + m_3^2 - 2m_2 E_3 \end{aligned}\right\} \Rightarrow \begin{aligned} E_L &= \frac{1}{2m_2}(s - m_1^2 - m_2^2) \\ E_4 &= \frac{1}{2m_2}(m_2^2 + m_4^2 - t) \\ E_3 &= \frac{1}{2m_2}(m_2^2 + m_3^2 - u) \end{aligned}$$

and

$$p_L^2 = E_L^2 - m_1^2 = \frac{1}{4m_2^2}(s - m_1^2 - m_2^2)^2 - m_1^2 = \frac{1}{4m_2^2}\lambda(s, m_1^2, m_2^2)$$

$$|\vec{p}_4| = E_4^2 - m_4^2 = \frac{1}{4m_2^2}\lambda(t, m_2^2, m_4^2)$$

$$|\vec{p}_3| = E_3^2 - m_3^2 = \frac{1}{4m_2^2}\lambda(u, m_2^2, m_3^2)$$

at $s \to \infty$ $m \sim 0$ $\Rightarrow$
$$s \simeq 2m_2 E_L \simeq 2m_2 p_L$$
$$t \simeq -2m_2 E_4$$
$$u \simeq -2m_2 E_3$$

Instead of E_3 or E_4 we may use as final state variable the lab scattering angle θ_L: angle formed by the direction of the outgoing particle 3 with the collision axis

$$t = (P_1 - P_3)^2 = P_1^2 + P_3^2 - 2 P_1 \cdot P_3 = m_1^2 + m_3^2 - 2 E_L E_3 + 2 P_L |\vec{P}_3| \cos\theta_L$$

Now, using the expression of E_L, E_3, $|\vec{P}_3|$ we derived in the previous pag. and using also $s + t + u = \sum_{i=1}^{4} m_i^2$, we can express θ_L as function of s and t (or s and u)

We consider only the equal mass case (for semplicity)

and we get

$$\cos\theta_L = \frac{s(s+t-4m^2)}{\lambda^{\frac{1}{2}}(s, m^2, m^2)\ \lambda^{\frac{1}{2}}(s+t, m^2, m^2)}$$

Physical domain of s, u, t channel

Use the kinematics limits : $P \geq 0$ and $-1 \leq \cos\theta \leq 1$

Consider the equal masses case

$\Rightarrow \quad s = 4(|P|^2 + m^2) \quad , \quad t = -2|\vec{P}|(1-\cos\theta) \quad , \quad u = -2|\vec{P}|(1+\cos\theta)$

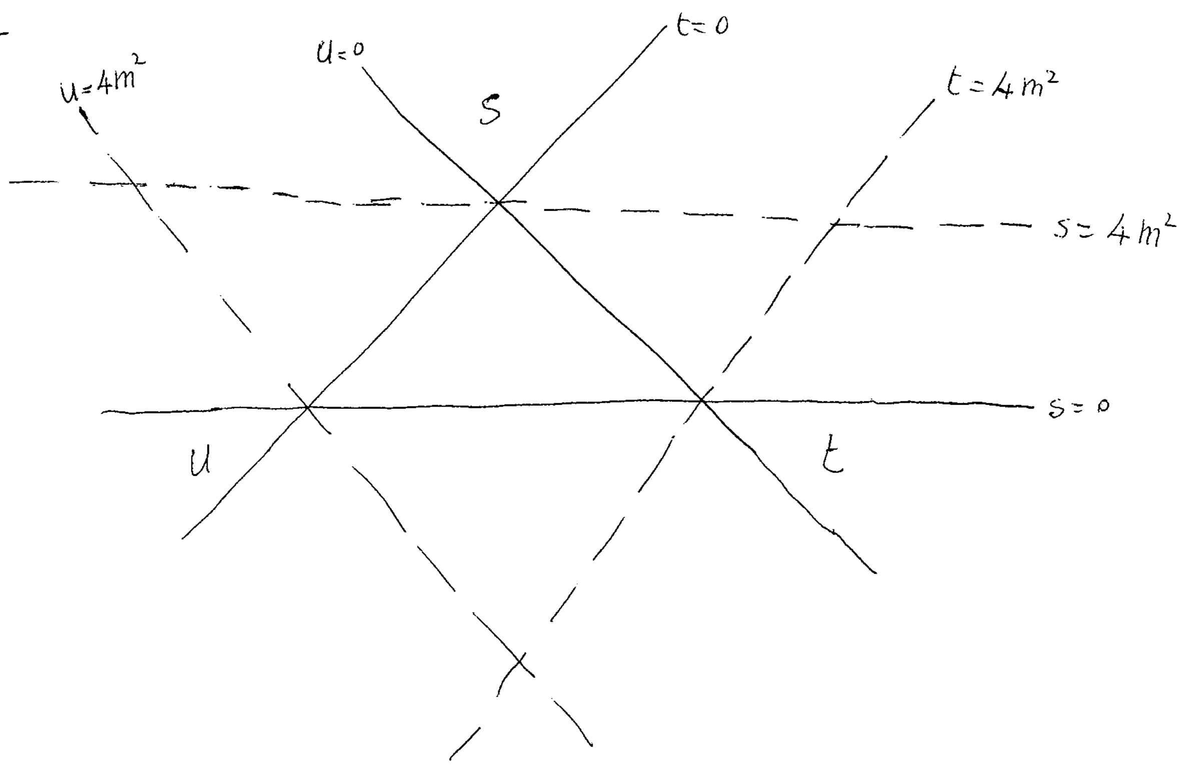
for s-channel we have

$$s \geq 4m^2 \quad , \quad t \leq 0 \quad , \quad u \leq 0$$

$\Rightarrow$ s has a treshold value corresponding to the production of 2 particles of mass m

similarly
$$\begin{cases} t\text{-ch.} & t \geq 4m^2 , \quad s \leq 0 \quad , \quad u \leq 0 \\ u\text{-ch.} & u \geq 4m^2 , \quad s \leq 0 \quad , \quad t \leq 0 \end{cases}$$

Diagram



Single inclusive processes

The single inclusive processes are $1+2 \longrightarrow 3+X$ and they are described by 3 independent variables usually s, t and $M^2 = (P_1+P_2-P_3)^2$ which is the invariant mass of the system X (missing mass)

Note: M is not a fixed variable $\Leftarrow$ X state is not on mass shell

The Mandelstam invariant are defined as usual: Just replace P_4 with P_X

In the CM system we have
$$\begin{cases} P_1 = (E_1, \vec{P}) = (E_1, 0, 0, P_z) \\ P_2 = (E_2, -\vec{P}) = (E_2, 0, 0, -P_z) \\ P_3 = (E_3, \vec{P}') = (E_3, \vec{P}'_\perp, P'_z) \end{cases}$$

we can derive similarly to the exclusive process case, the relation between CM variables $\Rightarrow$ Just replace $m_4 \rightarrow M$

for $s \gg m_1^2, m_2^2$ and $s, M^2 \gg m_3^2$ we get easly

$$|\vec{P}| = P_z \simeq \frac{\sqrt{s}}{2}, \quad E_1, E_2 \simeq \frac{\sqrt{s}}{2} \quad \text{and}$$

$$|\vec{P}'| \simeq \frac{s-M^2}{2\sqrt{s}}, \quad E_3 \simeq \frac{s-M^2}{2\sqrt{s}}$$

To get above expression Just use result we derived for CM system pag 3-6

Besides, from $t = (p_1 - p_3)^2 = m_1^2 + m_2^2 - 2E_1E_3 + 2|\vec{p}|\,|\vec{p}'|\cos\theta$

we get for $s, M^2 \gg m_1^2, m_3^2$ (use expression of $E_1, E_3, |\vec{p}|$ and $|\vec{p}'|$ we derived previus page)

$$t \simeq -\frac{2\sqrt{s}}{2}\,\frac{s-M^2}{2\sqrt{s}} + \frac{2\sqrt{s}}{2}\,\frac{s-M^2}{2\sqrt{s}}\cos\theta \Rightarrow \cos\theta \simeq 1 + \frac{2t}{s-M^2}$$

$$\vec{p}_\perp^{\,2} = |p'|^2 \sin^2\theta \simeq \frac{(s-M^2)^2}{4(\sqrt{s})^2}\left(1-\cos^2\theta\right) \simeq \frac{(s-M)^2}{4s}\left(1 - \left(1 + \frac{2t}{s-M^2}\right)^2\right) \simeq$$

$$\simeq \frac{(s-M)^2}{4s}\left(1 - \left(1 + \frac{4t}{s-M^2}\right)\right) = -t\left(1 - \frac{M^2}{s}\right)$$

So
$$\begin{cases} \cos\theta \simeq 1 + \dfrac{2t}{s-M^2} \\ \vec{p}_\perp^{\,2} = -t\left(1 - \dfrac{M^2}{s}\right) \end{cases}$$

compare these with those in pag 5 (bottom of page)

Feynman's X_F variable

$$X_F \equiv \frac{|P'_z|}{P_z}$$

Note that at high energy $|P_\perp|$ is small $\cdot$ $|P_\perp| \lesssim 0.5\,\text{GeV}$

$$\Rightarrow \quad |\vec{P}'_z| \simeq |\vec{P}'| \quad \Rightarrow \quad |\vec{P}'_z| \simeq |P'| \simeq \frac{s - M^2}{2\sqrt{s}}$$

we also saw that for $s, M \gg m_i^2 \quad \Rightarrow \quad |\vec{P}| = P_z \simeq \frac{\sqrt{s}}{2}$

$$\Rightarrow \text{for} \quad s, M^2 \gg m_i^2,\ |\vec{P}_\perp|^2 \quad \Rightarrow \quad X_F \simeq \frac{s - M^2}{2\sqrt{s}} \cdot \frac{2}{\sqrt{s}} = 1 - \frac{M^2}{s}$$

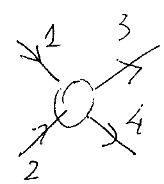
$$\text{So} \quad \boxed{X_F \simeq 1 - \frac{M^2}{s}}$$

Note that in the limit $M^2 \simeq s$, $X_F = 0 \Leftrightarrow |\vec{P}'_z| \simeq 0$ <u>Central region</u>
while in the limit $M^2 = m_4^2$ on-shell particle with $m_4^2 \ll s$ we have the <u>2-body exclusive scattering</u> and $X_F = 1$

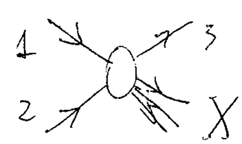
$\Rightarrow X_F \in [0, 1]$ i.e. X_F is in the interval $0 \leq X_F \leq 1$

Recall

1) <u>2-body exclusive scattering</u> $\cdot$ $1 + 2 \to 3 + 4$



2) single-particle inclusive scattering : $1 + 2 \to 3 + X$



So, X_F tells us how far or close a process is from process type 1) or type 2)

<u>Note</u>: In alternative to the variables s, t, M^2, other used variable to describe scattering processes are s, X_F, $|\vec{P}_\perp|$

Using result we derived before i.e. $|\vec{P}_\perp| \simeq -t\left(1 - \frac{M^2}{s}\right) \Rightarrow$

$$t \simeq -\frac{|\vec{P}_\perp|^2}{X_F}$$

While $|\vec{P}_\perp|$ is generally small, P'_z is in the interval $-P_z \leq |\vec{P}'_z| \leq P_z$

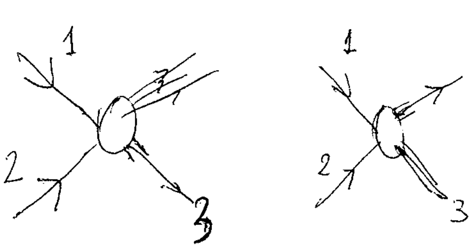
If particle 3 is produced as a fragment of particle 1 $\Rightarrow |\vec{P}'_z| = P_z$

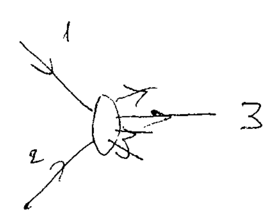
If particle 3 is produced as a fragment of particle 2 $\Rightarrow |\vec{P}'_z| = -P_z$

<u>Summary</u>: $|P'_z| \simeq 0 \Rightarrow X_F \simeq 0$ <u>Central region</u>

$|P'_z| \simeq P_z \Rightarrow X_F = 1$ <u>Fragmentation region</u>

} In the CM system





<u>Fragmentation region</u>

<u>Central region</u>

the 2-body exclusive scattering is a limiting case

Light-cone variable

$$A^+ = \frac{1}{\sqrt{2}}(A^0 + A^3) \qquad A^- = \frac{1}{\sqrt{2}}(A^0 - A^3)$$

$\vec{A}_\perp = (A^1, A^2)$ 2-dimensional vector

$\Rightarrow$ a generic vector is $A^\mu = (A^+, A^-, \vec{A}_\perp)$

Now compare this expression with $A^\mu = (A^0, \vec{A}_\perp, A^3)$

$\Rightarrow$ when we use $\vec{A}_\perp$ in the far right we are using light-cone variables

Note: $A^\mu A_\mu = (A^0)^2 - |\vec{A}|^2 = 2A^+A^- - \vec{A}_\perp^2$

$$B^\mu A_\mu = A\cdot B = A^0B^0 - \vec{A}\cdot\vec{B} = A^+B^- + A^-B^+ - \vec{A}_\perp\cdot\vec{B}_\perp$$

The need of the Light cone variables will be clear later on

Sudakov Parametrization

Introduce 2 light-like vector p^μ, n^μ

$$p^\mu = \frac{1}{\sqrt{2}}(\Lambda, 0, 0, \Lambda) \qquad n^\mu = \frac{1}{\sqrt{2}}(\Lambda^{-1}, 0, 0, -\Lambda^{-1})$$

Λ is an arbitrary parameter. $p^2 = n^2 = 0$; $p\cdot n = 1$; $n^+ = p^- = 0$

In light-cone components we have $\begin{cases} p^\mu = (\Lambda, 0, \vec{0}_\perp) \\ n^\mu = (0, \Lambda^{-1}, \vec{0}_\perp) \end{cases}$

$$\qquad\qquad + \quad - \quad \perp$$

Sudakov decomposition for a generic vector

$$A^\mu = \alpha P^\mu + \beta n^\mu + A^\mu_\perp = (A\cdot n) P^\mu + (A\cdot P) n^\mu + A^\mu_\perp$$

with $A^\mu_\perp = (0, \vec{A}_\perp, 0) = (0, A^1, A^2, 0)$

Note that $A^\mu_\perp$ is a in a 4-dimensional space
while $\vec{A}_\perp$ is in a 2-dimensional vector

$$A^2 = 2\alpha\beta - \vec{A}_\perp^{\,2}$$

<u>Exercise</u>: Show that $g^{\mu\nu}_\perp = g^{\mu\nu} - (P^\mu n^\nu + P^\nu n^\mu)$

$g^{\mu\nu}_\perp$ projects onto the plane perpendicular to P, n^μ

Let's now go back to Feynman's variable x_F.

$$P_1 = (E_1, 0, 0, P_z) \Rightarrow P_1 = \left(\overset{+}{P_1^+}, \overset{-}{\frac{m_1^2}{2P_1^+}}, \overset{\perp}{0_\perp}\right) \qquad P_1^2 = 2P_1^+ P_1^- = m_1^2$$

$$P_3 = (E_3, \vec{P}_\perp, P_z') \Rightarrow P_3 = \left(P_3^+, \frac{P_\perp^2 + m_3^2}{2P_3^+}, \vec{P}_\perp\right) \qquad P_3^2 = 2P_3^+ P_3^- - \vec{P}_\perp^{\,2} = m_3^2$$

Now, assuming that particle 3 is produced as a fragment of particle 1 (i.e. $P_z' \geq 0$) and using that at high-energy we have $|\vec{P}| = P_z \simeq \frac{\sqrt{s}}{2}$ and $|\vec{P}'| \simeq \frac{s - \Pi^2}{2\sqrt{s}}$ and

$$E_1, E_2 \simeq \frac{\sqrt{s}}{2} \quad \text{and} \quad E_3 \simeq \frac{s-\Pi^2}{2\sqrt{s}} \qquad \text{for } s, \Pi^2 \gg m_3^2$$

and recalling that $x_F \simeq 1 - \frac{M^2}{s}$

we get that $\frac{p_3^+}{p_1^+} = 1 - \frac{M^2}{s}$

So $$\boxed{x_F \simeq \frac{p_3^+}{p_1^+}}$$

where we also used that $|\vec{p}_\perp| \lesssim 0.5$ GeV and so $|\vec{p}_z\,'| \simeq |\vec{p}\,'|$

(Note: In the Lab system we talk of <u>target fragmentation region</u> if $p_z' \simeq p_z$ and <u>beam fragmentation region</u> if $p_z' \simeq -p_z$)

So, the 4-momentum of the outgoing particle can be written as

$$p_3 = \Big(\underset{+}{x_F p_1^+} , \underset{-}{\frac{p_\perp^2 + m_3^2}{2 x_F p_1^+}} , \underset{\perp}{\vec{p}_\perp} \Big)$$

Rapidity

Consider the Lorentz boost of a light-cone vector in the z-direction

define $y \equiv \frac{1}{2} \ln \frac{1+v}{1-v}$

then, one can easily see that the light cone components of a vector gets boosted

$(V')^+ = V^+ e^{y}$, $(V')^- = V^- e^{-y}$, $\vec{V}'_\perp = V_\perp$

we note the if we make an infinite boost, say $v \to \infty$ $\Rightarrow$ one of the components of the light-cone vector, in this case V_1^- is negligible i.e. goes to 0 as $v \to \infty$.

Thus the advantage of light cone vectors is that at high energy only 1 component "survive" the boost!

<u>Exercise</u>: show that if we make 2 consecutive boosts

$$y_1 = \frac{1}{2} \ln \frac{1+v_1}{1-v_1} \quad \text{and} \quad y_2 = \frac{1}{2} \ln \frac{1+v_2}{1-v_2}$$

Then $(V'')^+ = e^{y_1 + y_2} V^+$ and $(V'')^- = e^{-(y_1+y_2)} V^-$.

We see that for $y_1, y_2 > 0 \Rightarrow V^+$ is enhanced while V^- is suppressed.

<u>Note</u>: The advantage of rapidity (and light cone vectors) is that rapidity act additively under 2 consecutive boosts, so it behaves as velocity under a Galilean boost!

Let's consider a particle at rest $\Rightarrow p^\mu = (m, 0, 0, 0) \Rightarrow$

in light-cone components $p^\mu = \left(\frac{m}{\sqrt{2}}, \frac{m}{\sqrt{2}}, 0_\perp\right) \xrightarrow[\text{boost}]{} \left(\frac{m}{\sqrt{2}} e^{\psi}, \frac{m}{\sqrt{2}} e^{-\psi}, 0_\perp\right)$

if we now consider the ratio $\frac{p^+}{p^-} = \frac{\frac{m}{\sqrt{2}} e^{\psi}}{\frac{m}{\sqrt{2}} e^{-\psi}} = e^{2\psi}$ it gives a mesure

of the boost from the rest frame $\Rightarrow$ define rapidity Y as

$$Y = \frac{1}{2} \ln \frac{p^+}{p^-} \qquad \left(p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3), \quad p^- = \frac{1}{\sqrt{2}}(p^0 - p^3)\right)$$

Exercise: show that Y and ψ are the same

Suppose now we start with $p^\mu = (p^0, p_\perp, 0)$ with $p^2 = m^2$

$$\Rightarrow p^\mu = \left(\sqrt{m^2 + p_\perp^2}, p_\perp, 0\right) = \left(\underset{+}{\frac{\sqrt{m^2+p_\perp^2}}{\sqrt{2}}}, \underset{-}{\frac{\sqrt{m^2+p_\perp^2}}{\sqrt{2}}}, \underset{\perp}{\vec{p}_\perp}\right)$$

after the boost we get

$$(p')^+ = \sqrt{\frac{m^2+p_\perp^2}{2}}\, e^{Y}$$

$$(p')^- = \sqrt{\frac{m^2+p_\perp^2}{2}}\, e^{-Y}$$

So, $p^\mu \xrightarrow[\text{boost}]{} \left(\sqrt{\frac{m^2+p_\perp^2}{2}}\, e^{Y}, \sqrt{\frac{m^2+p_\perp^2}{2}}\, e^{-Y}, 0_\perp\right)$

We define $\sqrt{m^2 + p_\perp^2}$ transverse mass

Pseudo rapidity

Pseudo rapidity is defined for massless particles ~~or~~ when the mass of the particle is negligible.

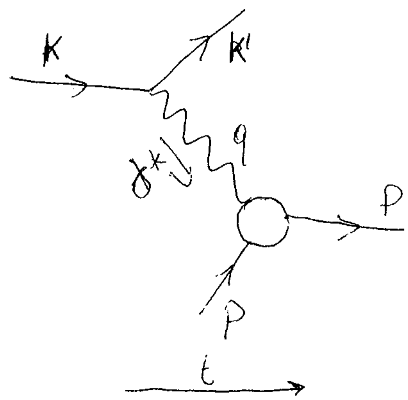
Suppose that ϑ is the angle of the 3-momentum of the particle relative to the z-axis, then $\boxed{\eta = -\ln \tan \frac{\vartheta}{2}}$ is pseudo rapidity

Exercise show that $y = \frac{1}{2} \ln \frac{p^+}{p^-} \xrightarrow[m \to 0]{} \eta$

Convince yourself that $y < \eta$ always.

Electron proton process: elastic case

Let's consider the electron-proton scattering process



$$\ell(k) + P(p) \longrightarrow \ell(k') + P(p')$$

the electron fluctuates in a virtual photon, which we denote γ^* (gamma star), and it (the γ^*) scatters off the Target which, in this case, is the proton.

If we assume the proton as a point-like object, then in Quantum Electrodynamics (QED) the vertex virtual photon-proton is the usual QED vertex i.e.

$\bar{u}(p')\gamma^\mu u(p)$ where $u(p)$ denote the spinor of the proton (omitting the index for spin).

So using the QED Feynman rules, the amplitude of the $ep \to e'p'$ process is simply

$$\bar{u}(p')\,\gamma^\mu\,u(p)\,\frac{g^{\mu\nu}}{q^2}\,\bar{u}(k')\,\gamma^\nu\,u(k)$$

Since we know that the proton is not really a point-like object like the electron $\Rightarrow$ we cannot assume anymore that the γ^*-proton vertex is the same as the electron-γ^* vertex!

This suggests us that the γ^*-proton vertex must be a more complicated object.

Let's us write down the most generic object we can think of using the 16-dimensional $\{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\}$ independent set of (hermitian) 4×4 matrices; we can expand the γ^*-proton vertex as it were a vector ~~and~~ and the 16-matrices ~~are~~ represents a base:

$$\bar{u}(p')\,\Gamma^\mu\,u(p) = \bar{u}(p')\left(A\gamma^\mu + B\,p'_\mu + C\,p_\mu + iD\,p'^\nu\sigma_{\mu\nu} + iE\,p^\nu\sigma_{\mu\nu}\right)u(p)$$

You can convince your self that this is the most generic object you can write down using the 16-matrices and the ~~[illegible]~~ momenta p^μ and p'^μ. Note also that we did not use γ^5 because it violates parity. γ^5 is used in the case of neutrino-proton scattering. (We did not use also $\varepsilon_{\mu\nu\alpha\beta}$ tensor).

Note also that $\sigma^{\mu\nu} \equiv -\frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$

the ~~cross~~ cross section in QED for the electron-proton scattering is

$$\frac{d^2\sigma}{d\Omega\, dE'} = \frac{\alpha_{EM}^2}{Q^4} \frac{E'}{E} L_{\mu\nu} W^{\mu\nu}$$

where Ω is the solid angle where the electron is diffused after the scattering. E and E' are the energy of the incoming and outgoing electron respectively.
We have also neglected the mass of the electron.
Note also that we wrote the cross section in a factorized form $L^{\mu\nu}$ is the leptonic tensor, while ~~the~~ $W^{\mu\nu}$ is the hadronic tensor (hadronic is referred to the proton in this case)

The coefficients we introduced in the expansion of the γ^*-proton vertex namely A, B, D, E depend only on Lorentz invariants

<u>Exercise</u>: Convinse yourself that this is the case i.e. the coefficients can only depend on Lorentz invariants quantity.

We can express these invariant quantities using M_N^2 and q^2 where M_N^2 is the mass of the nucleon (proton in our case) and q is the momentum carried by the virtual photon. $q = p' - p$

the invariants are

$$P\cdot P = P'\cdot P' = M_N^2$$

$$P\cdot P' = -\frac{1}{2}(P-P')^2 + M_N^2 = -\frac{1}{2}q^2 + M_N^2$$

$$P\cdot q = P\cdot P' - P^2 = -\frac{1}{2}q^2$$

$$P'\cdot q = P'^2 - P'\cdot P = \frac{1}{2}q^2$$

$\Rightarrow \quad A = A(q^2),\ B(q^2)$ etc.

We now find some constraints on the coefficients by imposing

1) Gauge invariance: $\quad q^\mu\, \bar{U}(P')\,\Gamma^\mu\, U(P) = 0$

$\Rightarrow \quad D = -E \quad$ and $\quad C = B$

$$\Rightarrow \bar{u}(P')\,\Gamma^\mu\, U(P) = \bar{u}(P')\left(A(q^2)\,\gamma^\mu + B(q^2)\,(P'+P)^\mu + i D(q^2)\,(P'-P)_\nu\,\sigma^{\mu\nu}\right) u(P)$$

2) the current must be Hermitian $\Rightarrow$ the current $u(P')\,\Gamma^\mu\, u(P)$ must be invariant for the following transformation

$$\left[\bar{u}(P')\,\Gamma^\mu\, U(P)\right]^+_{P_\mu \to P'_\mu} = \bar{u}(P')\,\Gamma^\mu\, u(P)$$

This imply that A, B and D must be real

3) Use Gordon decomposition $\quad \bar{u}(P')\,\gamma^\mu\, u(P) = \frac{1}{2M_N}\,\bar{u}(P')\left[(P+P')^\mu + \sigma^{\mu\nu}(P'-P)_\nu\right] u(P)$

<u>Exercise</u> prove Gordon identity.

Hint: use eq. of motion and start from right expression and show that it reduces to the $\bar{u}(P')\,\gamma^\mu\, u(P)$.

So using Gordon identity we get

$$\bar{u}(p')\Gamma^\mu u(p) = \bar{u}(p')[A(q^2)\gamma^\mu + iB(q^2)q_\nu\sigma^{\mu\nu}]u(p)$$

where $q^\mu = p'^\mu - p^\mu$ virtual photon momentum.

the hadronic tensor is

$$W_{\mu\nu} \sum_{spin} [\bar{u}(p')\Gamma_\mu u(p)]^* [\bar{u}(p')\Gamma_\nu u(p)] =$$

$$= Tr\{(A(q^2)\gamma_\mu - iB(q^2)q^\lambda\sigma_{\mu\lambda})(\not{p}'+M_N)(A\gamma_\nu + iB(q^2)q^\rho\sigma_{\nu\rho})(\not{p}+M_N)\}$$

$$\Rightarrow W_{\mu\nu} = -2(A+2M_N B)^2 q^2 \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right) +$$

$$+ (2A^2 - 2B^2q^2)4\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)$$

Note: $\not{P} = \gamma^\mu P_\mu$

let's now introduce 2 new functions W_1 W_2 and define $Q^2 \equiv -q^2 > 0$

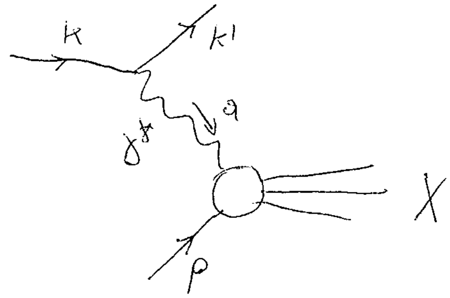
$$\Rightarrow W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1(Q^2) + \left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)\frac{W_2(Q^2)}{M_N^2}$$

with $W_1 \equiv -2(A+2M_N B)^2 q^2$

$W_2 \equiv 4(2A^2 - 2B^2q^2)$

Deep inelastic electron-proton scattering DIS

Let's us now consider the process of electron-proton scattering where in the final state we have the diffused electron and a set of other particles which are produced by the fragmentation of the proton.



Let's then calculate the hadronic tensor in the inclusive case i.e. we sum over ~~of~~ all possible final states.

P_X: final state momentum and note $P_X^2 \neq M_N^2 \Rightarrow$

$P \cdot q$ and $P' \cdot q$ are ~~...~~ independent ~~...~~

$W_{\mu\nu}$ is given by unknown matrix elements of the electromagnetic current J^μ_{em} between the final state on the proton state

$$W_{\mu\nu} = \frac{1}{4M_N} \sum_\sigma \sum_n \int \prod_i^n \left[\frac{d^3 P_i}{(2\pi)^3 2P_i^0}\right] \langle P, \sigma | J^{+em}_\mu(0) | n \rangle \langle n | J^{em}_\nu(0) | P, \sigma \rangle \cdot$$

$$\cdot (2\pi)^3 \delta^{(4)}(P_n - P - q)$$

$|P, \sigma\rangle$ proton state of momentum P and spin σ

$|n\rangle$ generic final state (one of the infinite possible final states)

P_n momentum of the generic final state

Since the $W^{\mu\nu}$ tensor cannot be calculated from first principle it will be parametrized as we did for the elastic case

$\Rightarrow$

$$W_{\mu\nu}^{\text{inclusive}} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(Q^2, \nu) +$$

$$+ \left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right) \frac{W_2(Q^2,\nu)}{M_N^2}$$

where we have now introduced the dependence of the function W_1 and W_2 on the parameter $\nu \equiv \frac{P\cdot q}{M_N}$ which is a mesure of the anelasticity of the process.

To get the cross section we multiply the hadronic tensor with the leptonic tensor

$$L_{\mu\nu} = \frac{1}{2} \sum_{s,s'} \bar{u}(k',s') \gamma_\mu u(k,s) \; \bar{u}(k,s) \gamma_\nu u(k',s')$$

$$= \frac{1}{2} \operatorname{tr}\{ (\not{k} + m) \gamma_\mu (\not{k}' + m) \gamma_\nu \} =$$

$$= 2\left(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu} k\cdot k' + g_{\mu\nu} m^2\right)$$

m = mass of the electron

using again $\frac{d^2\sigma}{dE'd\Omega} = \frac{E'}{E} \frac{\alpha_{em}^2}{Q^4} L^{\mu\nu} W_{\mu\nu}^{incl.}$ we get (neglicting the mass of the electron)

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{E'\alpha^2}{E\,Q^4}\,2\left(k_\mu k_\nu - g_{\mu\nu}\,k\cdot k'\right)\cdot\left[\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right)W_1(Q^2,\nu) + \right.$$

$$\left. + \left(P_\mu - q_\nu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)\right]\frac{W_2(Q^2,\nu)}{M_N^2}$$

now observe that $-Q^2 = q^2 \simeq -2\,k\cdot k'$

$$2\,k\cdot q = 2\,k\cdot(k-k') = 2m^2 - 2k\cdot k' \simeq -Q^2$$

$$\Rightarrow \frac{d^2\sigma}{dE'd\Omega} = \frac{E'}{E}\frac{2\alpha_{em}^2}{Q^4}\left[Q^2 W_1(Q^2,\nu) + 2\left(P\cdot k - \frac{1}{2}P\cdot k\right)^2\frac{W_2(Q^2,\nu)}{M_N^2} - \frac{Q^2}{2}\left(M_N^2 - \frac{(P\cdot q)^2}{q^2}\right)\frac{W_2(Q^2,\nu)}{M_N^2}\right]$$

Let's now consider the process in the laboratory reference frame where the proton is at rest $\Rightarrow$ $P_\mu = (M, 0)$

$$\nu = \frac{P\cdot q}{M_N} = \frac{P\cdot(k-k')}{M_N} = \frac{M_N(E-E')}{M_N} = E - E'$$

$$\Rightarrow \quad \nu = E - E' \quad \text{in the Lab. rest frame}$$

Let θ be the angle of the diffused electron in the Lab rest frame $\Rightarrow$

$$k\cdot k' = |k|\,|k'|\cos\theta = E\,E'\cos\theta$$

$$Q^2 \simeq 2\,k\cdot k' = 2(EE' - \vec{k}\cdot\vec{k}') = 4\,EE'\sin^2\frac{\theta}{2}$$

So in the laboratory reference frame the cross section is

$$\frac{d^2\sigma}{dE'd\Omega} = 4E'\frac{\alpha^2}{Q^2}\left[2\sin^2\left(\frac{\theta}{2}\right)W_1(Q^2,\nu) + \cos^2\left(\frac{\theta}{2}\right)W_2(Q^2,\nu)\right]$$

this is the inclusive cross section for electron-proton (nucleon) in the non-polarized case.

The function $W_1(Q^2,\nu)$ and $W_2(Q^2,\nu)$ are called <u>structure functions</u> which can be calculated in Quantum Chromodynamics (QCD) in the perturbative regime.

Let's now show that the hadronic tensor reduces to

$$W_{\mu\nu} = \frac{1}{2\pi}\int d^4x\, e^{iq\cdot x}\,\frac{1}{2}\sum_{\text{Polarization}}\langle N(P)|\,J_\mu(x)\,J_\nu(0)\,|N(P)\rangle$$

where $|N(P)\rangle$ represents the nucleon (proton) state with momentum P

let $|X(P_X)\rangle$ an arbitrary final state with momentum P_X

our starting point is

$$W_{\mu\nu} = \frac{1}{2}\sum_{\text{Polar.}}\sum_{X,P_X}\langle N(P)|\,\hat{J}_\mu(0)\,|X(P_X)\rangle\langle X(P_X)|\,\hat{J}_\nu(0)\,|N(P)\rangle \times$$

$$\times (2\pi)^3\,\delta^4(P_X-P-q)$$

note that in page 6 we used $|n\rangle$ instead of $|X(P_X)\rangle$ and we used $\sum_{X,P_X}$ (for brevity) to indicate $\sum_n\int\prod_i^n\left[\frac{d^3P_i}{(2\pi)^3 2P_i^0}\right]$

Now use $\delta^{(4)}(P_X - P - q) = \int \frac{d^4 y}{(2\pi)^4}\, e^{-i(P_X - P - q)\cdot y}$

use $\langle X(P_X)|\, \hat{J}_\mu(x)\, |N(P)\rangle = \langle X(P_X)|\, \hat{J}_\mu(0)\, |N(P)\rangle\, e^{-i(P-P_X)\cdot x}$

we get

$$W_{\mu\nu} = \frac{1}{4\pi} \sum_{Pol.} \sum_{X, P_X} \int d^4 y\, \langle N(P)|\, \hat{J}_\mu(y)\, |X(P_X)\rangle \langle X(P_X)|\, \hat{J}_\nu(0)\, |N(P)\rangle\, e^{iq\cdot y}$$

Now we use the completeness relation

$$\sum_{X, P_X} |X(P_X)\rangle\langle X(P_X)| = \mathbb{1}$$

and we get

$$\boxed{W_{\mu\nu} = \frac{1}{2\pi} \int d^4 x\, e^{iq\cdot x}\, \frac{1}{2} \sum_{Polar.} \langle N(P)|\, J_\mu(x) J_\nu(0)\, |N(P)\rangle}$$

Let's now introduce a new variable the <u>Bjorken x_B</u>

$$x_B \equiv \frac{Q^2}{2P\cdot q}$$

if we are in the rest frame $\quad x = \dfrac{Q^2}{2\nu M_N}$ $\quad$ with $\nu = E - E'$ only in the rest frame (lab frame)

<u>Elastic case</u> $\quad x_B = \dfrac{Q^2}{2\nu M_N} = \dfrac{-q^2}{2P\cdot q} \underset{\uparrow}{=} \dfrac{-q^2}{-q^2} = 1$

look pag 4

So for <u>elastic case</u>

$$\boxed{x_B = 1}$$

the invariant mass of the final state is

$$M_X = \sqrt{(P+q)^2} \geq M_N$$

<u>Exercise</u>: Explain why $M_X \geq M_N$; at most we can have

$$M_X = M_N \quad \text{in the elastic case.}$$

Using $(P+q)^2 - P^2 - q^2 = 2P\cdot q$ we get

$$x_B = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{(P+q)^2 - P^2 + Q^2} = \frac{Q^2}{M_X^2 - M_N^2 + Q^2} = \frac{1}{1 + \frac{M_X^2 - M_N^2}{Q^2}}$$

$$\Rightarrow \quad x_B = \frac{1}{1 + \frac{M_X^2 - M_N^2}{Q^2}} \quad \Rightarrow \quad \boxed{x_B < 1} \quad \text{inelastic case}$$

$\Rightarrow$ we conclude that $\boxed{0 < x_B \leq 1}$

Let's now <u>redefine</u> our structure functions

$$\boxed{\begin{aligned} M_N W_1(Q^2,\nu) &= F_1(Q^2, x) \\ \nu W_2(Q^2,\nu) &= F_2(Q^2, x) \end{aligned}}$$

it is convenient to introduce a new variable $\boxed{y \equiv \frac{P\cdot q}{P\cdot k}}$

in the rest frame $\boxed{y = \frac{\nu}{E} = 1 - \frac{E'}{E}}$ it measures the inelasticity of the process.

y indicates the fraction of energy that the electron transferred to the proton in the laboratory rest frame.

We can now rewrite the cross section in terms of x and y

note first that
$$x = \frac{Q^2}{2(E-E')M_N} = \frac{2EE'}{(E-E')M_N}\sin^2\frac{\theta}{2}$$

Using the cylindrical symmetry of the process we have

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{d^2\sigma}{dE'\,2\pi\sin\theta\,d\theta}$$

$$\frac{d^2\sigma}{dE'd\Omega} = \frac{1}{2\pi\sin\theta}\left|\frac{\partial(x,y)}{\partial(E',\theta)}\right|\frac{d^2\sigma}{dx\,dy} = \frac{E'}{2\pi M_N E y}\frac{d^2\sigma}{dx\,dy}$$

Exercise: Calculate the Jacobian $\left|\dfrac{\partial(x,y)}{\partial(E',\theta)}\right|$.

Therefore, we may now write the differential cross section in terms of x and y and we get

$$\frac{d^2\sigma}{dx\,dy} = \frac{8\pi M_N y E E'\alpha_{em}^2}{Q^4}\left[\frac{2}{M_N}\sin^2\frac{\theta}{2}F_1(Q^2,x) + \frac{\cos^2\theta/2}{yE}F_2(Q^2,x)\right]$$

Let's now write θ in terms of x and y and we get

$$\sin^2\frac{\theta}{2} = \frac{(E-E')M_N}{2EE'}x = \frac{M_N}{2E'}xy$$

so we finally get

$$\frac{d^2\sigma}{dx\,dy} = \frac{8\pi M_N y E E' \alpha_{em}^2}{Q^4}\left[\frac{xy}{E'} F_1(Q^2,x) + \frac{1-\frac{M_N}{2E'}xy}{y_E} F_2(Q^2,x)\right]$$

$$\Rightarrow \boxed{\frac{d^2\sigma}{dx\,dy} = \frac{8\pi M_N E \alpha_{em}^2}{Q^4}\left[xy^2 F_1(Q^2,x) + \left(1-y-\frac{M_N xy}{2E}\right) F_2(Q^2,x)\right]}$$

Bjorken scaling

Experiments have shown that the structure function are ~~....~~ independent on Q^2 for $(Q^2 \geq 1\,GeV^2)$. This have suggested that the virtual photon γ^* is interacting with point like objects. When the charge distribution is a delta function and consequently the structure function are flat, it indicates that the interaction is with point-like objects. Experiments have also shown that

$F_2 \simeq 2x F_1$.

Heuristic interpretation (roughly speaking)

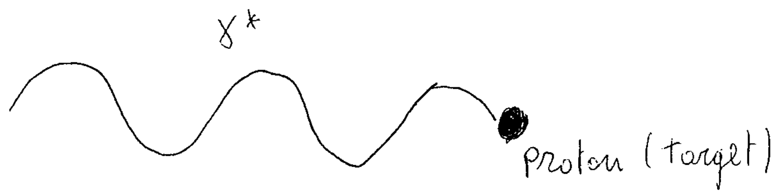
The photon wave-lenght is $\lambda \simeq \frac{1}{\sqrt{Q^2}}$

$\Rightarrow$ If I increase the momentum transfer Q, the resolution of the ~~....~~ virtual photon increases.

Let's distinguish the following cases ($\lambda \sim \frac{1}{\sqrt{Q^2}}$)

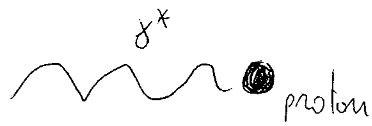
1) electron-proton scattering : Elastic case

a) λ is bigger then the size of the proton



In this case the virtual photon cannot resolve the inner structure of the proton $\Rightarrow$ the vertex γ^*-proton is the usual QED vertex (see pag 2)

b) If we increase Q^2 we get the wave lenght of the size comparable to the size of the proton



$\Rightarrow$ the virtual photon start to "sees" some charge distribution inside the proton $\Rightarrow$ form factors W_1 W_2 defined in page 5.

2) electron-proton scattering: inelastic (deep) case.

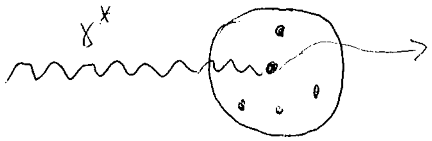
If we increase even more the resolution of the virtual photon, we can resolve the inner structure of the target (proton).

We introduced structure functions (in pag 7.) $W_1(Q^2, x)$, $W_2(Q^2, x)$

Increasing even more the Q^2 we reach a point where the scattering γ^*-proton is again point like $\Rightarrow$ the target is made of costituents that interact with γ^* as they were point like objects.

$\Rightarrow$ The structure functions are flat because they ar related to the charge distribution, which is a delta function for point-like objects,

So now we have the following picture



→ one of the constituents of the proton (target)

If we keep increasing Q^2 we still observe flat structure function ⇒

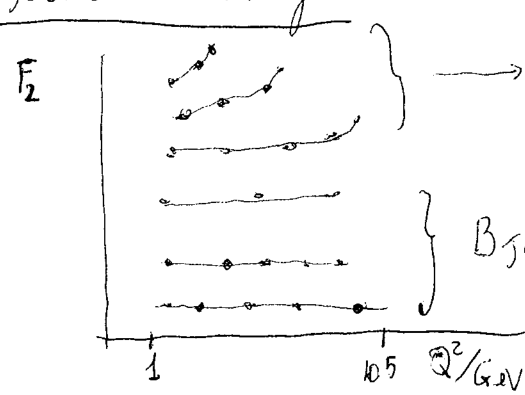
⇒ This behaviour is known as <u>Bjorken scaling</u> : the structure functions depend only on x_B and not on Q^2 ⇒

$$W_1(Q^2, x_B)\ ,\ W_2(Q^2, x_B) \longrightarrow W_1(x_B)\ ,\ W_2(x_B)$$

$$F_1(Q^2, x_B)\ ,\ F_2(Q^2, x_B) \longrightarrow F_1(x_B)\ ,\ F_2(x_B)\ .$$

- In the Bjorken scaling region the interaction among the constituents is neglected.
- If we increase the energy of the system γ^* – proton even more we reach a region where the interactions between the constituents cannot be neglected anymore ⇒ perturbative QCD corrections ⇒ ⇒ the structure functions start to deviate from straight line

⇒ Violation of <u>Bjorken scaling</u>



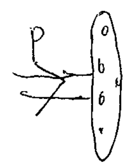
→ Violation of Bjorken scaling

dots : experimental data.
line : fit from theoretical prediction.

Bjorken scaling : $\underbrace{0.05 \lesssim x_B \lesssim 0.65}_{\text{roughly}}$

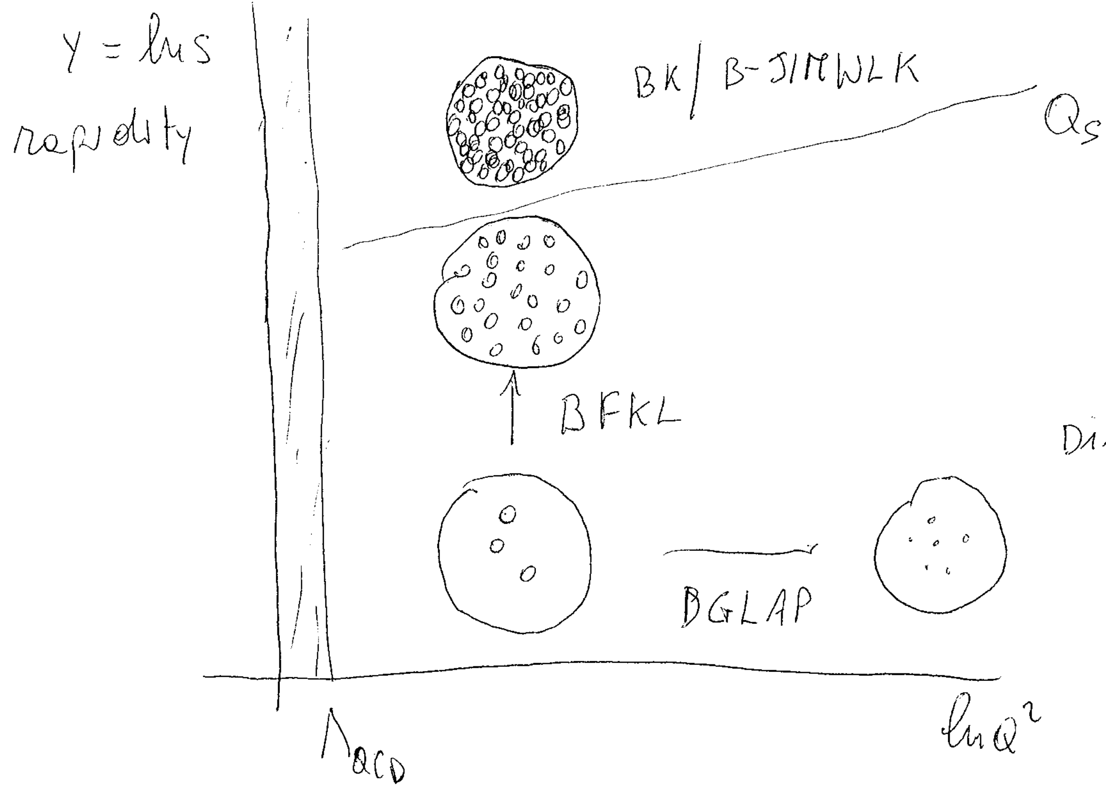
We said that the virtual photon acts as it were a microscope that can resolve the inner structure of the proton $\lambda \simeq \frac{1}{\sqrt{Q^2}}$

Let's suppose that the momentum of the γ^* is $q^\mu = (0, q^1, q^2, 0)$
it has only transverse components $\Rightarrow$ The virtual photon resolves inside the proton constituents of transverse size $x_\perp \sim \frac{1}{q_\perp}$.
Remember that we are in the infinite momentum frame where the proton is Lorentz contracted



We distinguish 2 cases $\begin{cases} \text{Bjorken limit: } Q^2 \gg \Lambda_{QCD} \quad x_B \simeq \frac{Q^2}{s} \text{ fixed} \\ \qquad\qquad\qquad\; s \gg \Lambda_{QCD} \\ \text{Regge limit: } s \gg Q^2 \gg \Lambda_{QCD} \quad x_B \simeq \frac{Q^2}{s} \to 0 \end{cases}$

1) if we increase energy s and momentum Q $\Rightarrow$ we resolve more and more constituents but of smaller size : Bjorken limit
2) if we fix Q^2 and increase energy $\Rightarrow$ we resolve more and more constituents but of the same size : Regge limit (high energy QCD)



Q_s is the ~~sca~~ scale that separate the dilute region from the saturation region (dense region where non linear effects are important)

DGLAP (DOKSHITZER-GRIBOV-LIPATOV-ALTARELLI-PARISI)

Evolution equation for structure functions in the Bjorken region
Evolution twards dilute region

BFKL (Balitsky, Fadin, Kuraev, Lipatov)

Evolution (linear) equation for structure function in the Regge limit

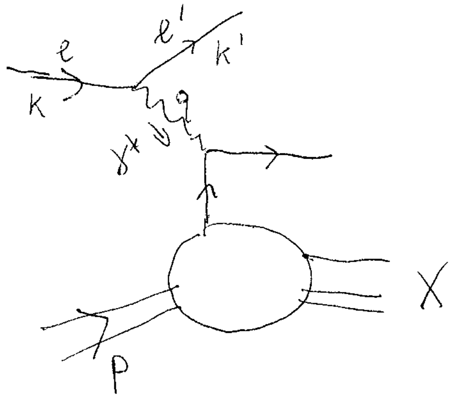
BK (Balitsky, Kovchegov)

Non Linear evolution equation for structure functions (high energy QCD)
(Regge limit)

Balitsky-JIMWLK (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner)
non linear evolution eq.

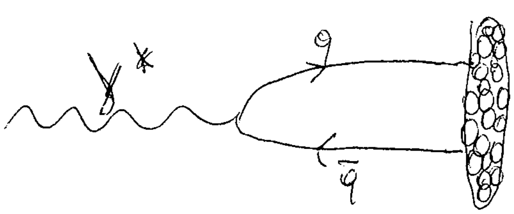
<u>Exercsize</u>: give an heuristic explanation of why Q_S (the saturation scale) grows as depicted in the picture.

<u>Bjorken limit</u>: <u>incoherent</u> interactions: the virtual photon interact with one of the constituents neglecting the interaction among the constituents



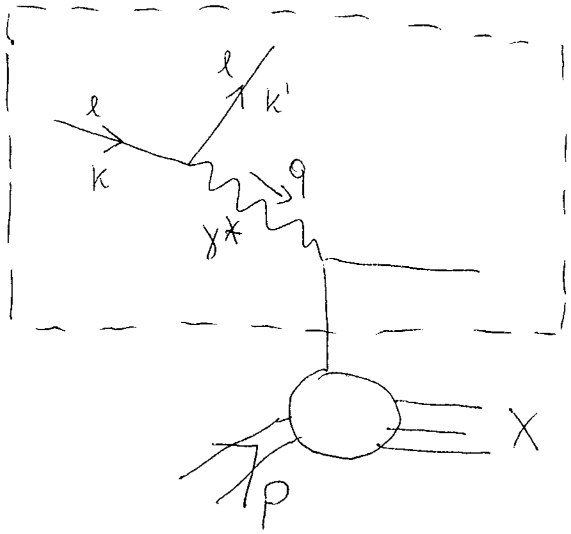
the proton "emit" a quark which participate to the interaction with the γ^*.

<u>Regge limit</u> (high-energy QCD): <u>coherent</u> interactions: the interactions among constituents cannot be neglected anymore due to the high-density of constituents in the target $\Rightarrow$ we chose a different frame: <u>Dipole</u> frame: the virtual photon split, befor the interaction, in a quark and anti-quark pair $\Rightarrow$ we may consider multiple interactions –



Project: Parton Model: Derive the relation (Bjorken Limit)

$$F_2 = 2 x_B F_1 \qquad \text{Callan-Gross relation}$$



$f_i(\xi_i)$: probability that the quark is "emitted" with momentum fraction ξ_i of the proton.

$$\frac{d\sigma}{dx_B dy} = \sum_i \int_0^1 d\xi_i \, f_i(\xi_i) \, \frac{d\sigma_i}{dx_B dy}$$

we derived $\frac{d\sigma}{dx_B dy}$ in pag 13: it was written in terms of F_1 and F_2

$\frac{d\sigma_i}{dx_B dy}$ is the cross section electron-quark

a) First derive the cross section electron-quark

$$\frac{d\sigma_i}{dQ^2} = \frac{4\pi \alpha_{em}^2 q_i^2}{Q^4} \left(\frac{s_i^2 + u_i^2}{2 s_i^2} \right)$$

q_i fraction of electron charge: charge of the "i" quark. s_i, u_i are the partonic Mandelstam variable: for each "i" quark.

b) rewrite $d\sigma_i$ in terms of s and u and dx and ~~d~~y ~~[illegible]~~

~~[illegible]~~ where s and u are the electron-proton Mandelstam invariants

$$\frac{d^2\sigma_i}{dx_B dy} = \left.\frac{dQ^2}{dy}\right|_{x_B} \frac{d^2\sigma_i}{dx_B dQ^2} = \frac{2\pi\alpha^2 q_i^2}{Q^4}\,\frac{s^2+u^2}{s^2}\,\delta(\xi_i - x_B)\,\frac{Q^2 s}{s+u}$$

ξ_i is the fraction of the proton momentum carried by the quark

At the end we have

$$\frac{d^2\sigma}{dx_B dy} = \sum_i \int_0^1 d\xi_i\, f_i(\xi_i)\,\frac{2\pi\alpha^2 q_i^2}{Q^4}\,\frac{s^2+u^2}{s^2}\,\delta(\xi_i - x_B)\,\frac{Q^2 s}{s+u}$$

$$= \frac{2\pi\alpha^2 s}{Q^4}\left[(y-1)^2+1\right]\sum_i f_i(x_B)\, q_i^2\, x_B$$

Note $\int_0^1 d\xi_i\, f_i(\xi_i) = 1$ normalization (conservation of probability)

- Comparing formula in pag 13 and the one we derived above we get

$$2\left[x_B y^2 F_1(Q^2,x) + (1-y)F_2(Q^2,x_B)\right] = \left[(y-1)^2+1\right]\sum_i f_i(x_B)\, q_i^2$$

Since Right Hand Side (RHS) is independent on Q^2 $\Rightarrow$

$\Rightarrow$ $F_1(Q^2, x_B) = F_1(x_B)$ and $F_2(Q^2, x_B) = F_2(x_B)$

and $\boxed{F_1(x) = \frac{1}{2}\sum_i f_i(x_B)\, q_i^2}$ $\boxed{F_2(x) = \sum_i f_i(x_B)\, q_i^2\, x_B}$

and $\boxed{F_2(x) = 2x\, F_1(x)}$ Callan-Gross relation

we obtained the Callan Gross relation in the parton model.
Observe that we obtained such relation assuming that the quark is a spin $\frac{1}{2}$ particle (when we get the electron-quark cross section).
In pag 13 we observed that experimentally $F_2 \simeq 2x F_1$
$\Rightarrow$ the ipotesis of quark being $\frac{1}{2}$ partical is verified experimentally!

- The structure functions mesure the density of constituents inside the nucleon (proton). At this order we also proved the Bjorken scaling since we obtained that the structure functions do not depend on Q^2.
- At higher order i.e. including gluons interactions we will get a violation of the Bjorken scaling $\Leftrightarrow$ the structure function will depend on Q^2 logarithmically $\Rightarrow$ get DGLAP evolution equation (evolution in $\log Q^2$). (look diagram in pag 17)