What you should know before you do Graph Theory Honours

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A note on sources:

The exercises in these notes, and many of the proofs, have largely been taken from existing sources, including Michael Henning's notes [Hen05] and books by Gary Chartrand [Cha85], Gary Chartrand and Linda Lesniak [CL96], and Frank Harary [Har69].

A note on conventions:

- Most graph theorists don't use the 'real' real numbers (like π, $\sqrt{3}$, and 121/55) very often. We do frequently work with integers, however. Throughout these notes, we therefore use the notation $[a, b]$ to indicate an *integer interval* rather than a closed interval, e.g., by $[3, 7]$ we mean the set $\{3, 4, 5, 6, 7\}$ rather than the set $\{x \in \mathbb{R} : 3 \leq x \leq 7\}$. Similarly, we shall use the notation $(3, 7)$ for the set $\{4, 5, 6\}$.
- To indicate the end of the proof of a *claim*, which is usually only part of the proof of a more substantial result, we use the symbol ©. For an example, see the proof of the Havel-Hakimi Theorem (Theorem 3).

Part I

Elementary stuff

Chapter 1

Introduction

1.1 Graphs

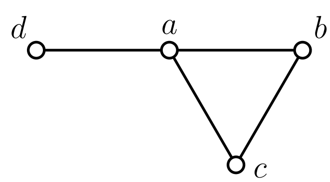
A *graph* G consists of:

1. A finite, nonempty set $V(G)$ of *vertices*[1], and,
2. A (possibly empty) set $E(G)$ of unordered pairs of vertices called *edges*.

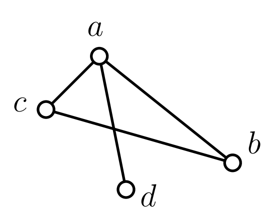
The set $V(G)$ is the *vertex set* of G. The set $E(G)$ is the *edge set* of G. Each edge is a 2-subset of the vertex set and so should be denoted with set notation. For convenience, however, we denote the edge joining the vertices a and b as ab rather than $\{a, b\}$. Since edges are unordered pairs of vertices, the edge ab is the same as the edge ba.

Graphs are frequently represented by pictures.

Example 1.1 Let G be the graph with $V(G) = \{a, b, c, d\}$ and $E(G) = \{ab, ac, ad, bc\}$. Here is one way to draw G:



Here is another picture of the same graph G:



Both of these pictures represent the same graph.

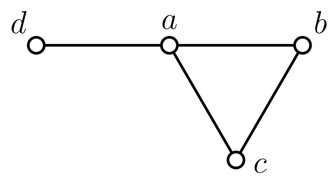
Some more notation: In the graph G, the edge ab *joins* the vertices a and b. Since a and b are joined by an edge, they are *adjacent* vertices, and a is said to be a *neighbour* of b. The *neighbourhood* of b is the set of all neighbours of b, i.e., the neighbourhood of b is the set $\{a, c\}$.

[1] The singular of vertices is *vertex*.

The vertices c and d are *nonadjacent*. The edge ab is *incident* with the vertices a and b, while the vertex a is *incident* with the edges ab, ac, and ad.

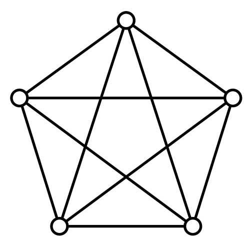
The *order* of a graph is the number of vertices, while the *size* of a graph is the number of edges. Usually, we use n for order and m for size. A graph of order n and size m is an (n, m) *graph*. If we want to make it clear that we mean the order and size of G (rather than some other graph), then we write $n(G)$ and $m(G)$.

Example 1.2 The graph G from Example 1.1 has order $n = 4$ and size $m = 4$:



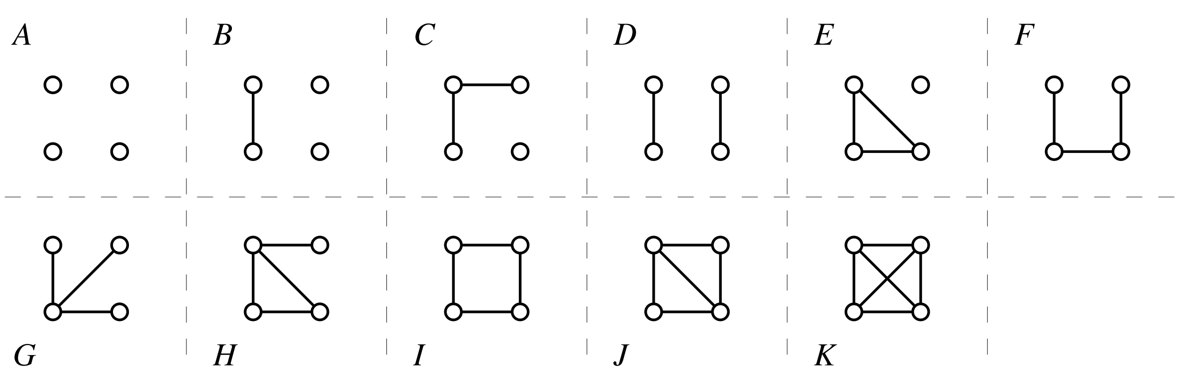
This graph, by the way, is sometimes called the *Martini Glass* or the *Alavi Graph*.

The graph below, which is an example of a *complete graph*, has order 5 and size 10. It is therefore a $(5, 10)$ graph.



Notice that we didn't give the vertices in this last graph names (like a, b, c). That's because the structure of the graph is clear from the picture, and the names that we give the vertices don't affect that structure. We'll explore this in more detail in Section 1.6.

Below are eleven graphs of order 4. The graphs $A - E$ are *disconnected graphs*, while the graphs $F - K$ are *connected*. A graph that has no edges is an *empty graph*, so graph A is an empty graph (of order 4). Graph K is a *complete graph* of order 4.



A graph of order 1 is called a *trivial graph*. A graph of order at least 2 is *nontrivial*.

1.2 Subgraphs

Let G and H be graphs. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a *subgraph* of G, denoted $H \subseteq G$.

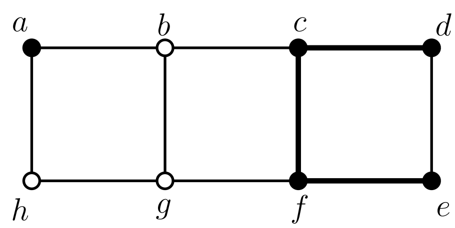
Example 1.3 Let G be the graph with

$$V(G) = \{a, b, c, d, e, f, g, h\}$$

and

$$E(G) = \{ab, bc, cd, de, ef, fg, gh, ah, bg, cf\}.$$

Let H be the graph with $V(H) = \{a, c, d, e, f\}$ and $E(H) = \{cd, cf, ef\}$. Then $H \subseteq G$. The figure below shows the graph G. The vertices of G that are also vertices of H are black, while the edges of G that are edges of H are thicker.



Let G be a graph and $S \subseteq V(G)$. The *subgraph induced by* S is the subgraph $G[S]$ having $V(G[S]) = S$ and where $uv \in E(G[S])$ if and only if $u, v \in S$ and $uv \in E(G)$.

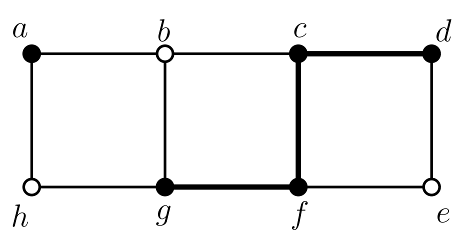
Example 1.4 Let G be the graph from Example 1.3 and let $S = \{a, c, d, f, g\}$. Then $G[S]$ is the graph with

$$V(G[S]) = \{a, c, d, f, g\}$$

and

$$E(G[S]) = \{cd, cf, fg\},$$

as indicated in the figure below:



If $H \subseteq G$ and there is a set $S \subseteq V(G)$ such that $H = G[S]$, then H is called an *induced subgraph* of G.

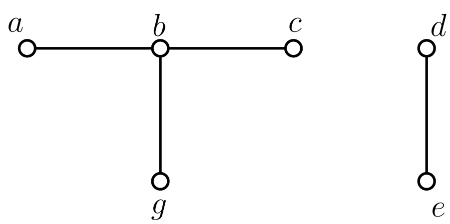
Example 1.5 The subgraph H from Example 1.3 is not an induced subgraph of G, because although d and e are in $V(H)$ and $de \in E(G)$, there is no edge between d and e in H. The subgraph

H' with $V(H') = \{a, c, d, f, g\}$ and $E(H') = \{cd, cf, fg\}$ is an induced subgraph of G, as was demonstrated in Example 1.4.

We can define an analogous concept for edges. Let G be a graph and $S \subseteq E(G)$. Then the *subgraph induced by* S is the subgraph $G[S]$ with $E(G[S]) = S$ and where $u \in V(G[S])$ if and only if u is incident with at least one edge in S. If H is a subgraph of G and there is a set $S \subseteq E(G)$ such that $H = G[S]$, then H is called an *edge-induced subgraph* of G.

Example 1.6 Let G be the graph from Example 1.3 and let $S = \{ab, bc, bg, de\}$. Then the subgraph induced by S has vertex set $\{a, b, c, d, e, g\}$ and edge set $\{ab, bc, bg, de\}$, as shown below:

$G[S]$:



Lastly, if $H \subseteq G$ and $V(H) = V(G)$, then H is a *spanning subgraph* of G.

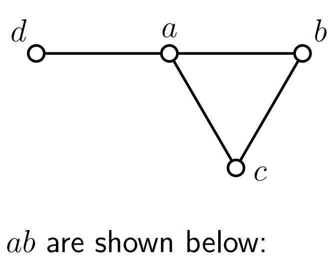
Example 1.7 Let G be the graph from Example 1.3. Then the graph H with $V(H) = V(G)$ and $E(H) = \{ab, bc, de, ef\}$ is a spanning subgraph of G.

If G is a nontrivial graph and $v \in V(G)$, then by $G - v$ we mean the subgraph of G obtained from G by deleting v and every edge incident with v. Similarly, if $e \in E(G)$, then $G - e$ is the subgraph obtained from G by deleting the edge e.

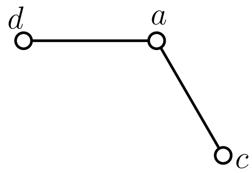
Example 1.8

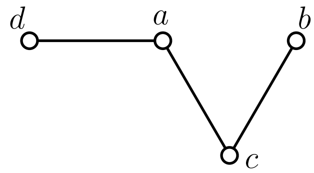
Consider once again the graph

G :



The graphs $G - b$ and $G - ab$ are shown below:

$G - b:$ 

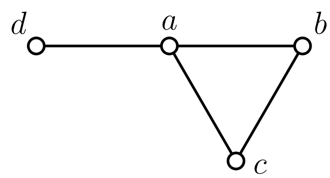
$G - ab:$ 

1.3 The degree of a vertex

1.3.1 Introduction

The *degree* of a vertex v is the number of edges incident with v. The degree of v is denoted $\deg v$ or, if we wish to be clear which graph we're referring to, $\deg_G v$. A vertex of degree 0 is called an *isolated vertex*. A vertex of degree 1 is called an *end-vertex*, while an edge that is incident with an end-vertex is called a *pendant edge*.

Example 1.9 Consider the graph shown below.



In this graph, we have

$$\begin{aligned}\deg a &= 3\\ \deg b &= 2\\ \deg c &= 2\\ \deg d &= 1\end{aligned}$$

We are now ready to give our first (very simple) result.

Theorem 1 ('First Theorem of Graph Theory')**.** *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When we add up the degrees of the vertices, each edge is counted exactly twice. $\square$

The 'First Theorem of Graph Theory' has the following Corollary. A vertex of even degree is called an *even vertex*. A vertex of odd degree is an *odd vertex*.

Corollary 2. *Every graph has an even number of odd vertices.*

Proof. Let G be a graph of size m. Let V_e be the set of even vertices of G and V_o the set of odd vertices. Then $V(G) = V_e \cup V_o$ and $V_e \cap V_o = \emptyset$, so by Theorem 1 we have

$$2m = \sum_{v \in V(G)} \deg v = \sum_{v \in V_e} \deg v + \sum_{v \in V_o} \deg v,$$

so

$$\sum_{v \in V_o} \deg v = 2m - \sum_{v \in V_e} \deg v. \tag{1.1}$$

For each $v \in V_e$, we know that $\deg v$ is an even number. The right hand side of Equation (1.1) is therefore an even number. Hence $\sum_{v \in V_o} \deg v$ is an even number and therefore, by an exercise, $|V_o|$ is an even number. $\square$

Let G be a graph. The *minimum degree of* G is

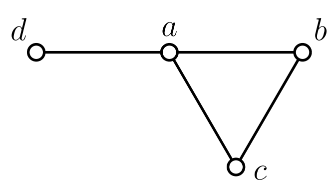
$$\delta(G) = \min\{\deg v : v \in V(G)\}$$

and the *maximum degree of* G is

$$\Delta(G) = \max\{\deg v : v \in V(G)\}.$$

Example 1.10 The graph G shown below

G :

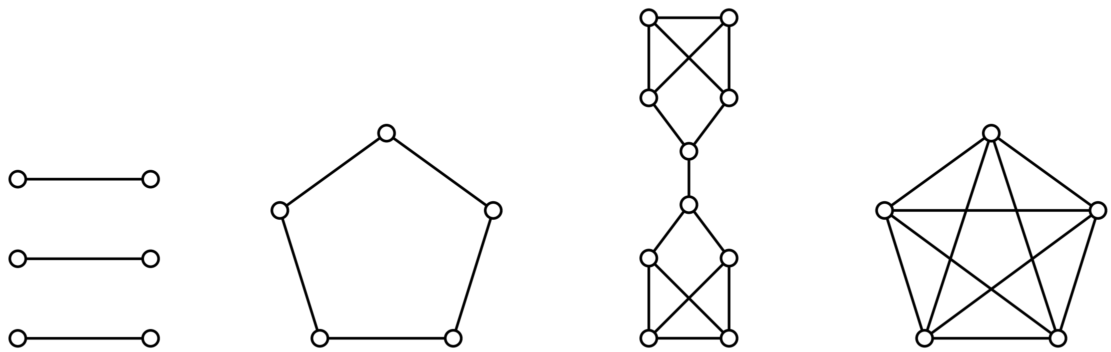


has

$$\delta(G) = 1$$
$$\Delta(G) = 3.$$

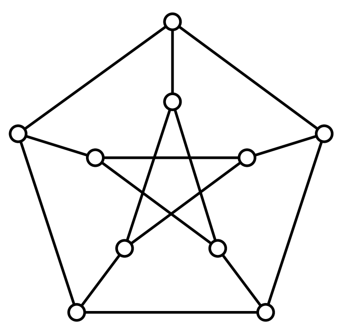
A graph is *regular* if every vertex has the same degree. If every vertex has degree r, then the graph is *r-regular* or *regular of degree r*.

Example 1.11 Here are some examples of regular graphs of different degrees:



1-regular 2-regular 3-regular 4-regular

Example 1.12 A graph that is 3-regular is also called *cubic*. Probably the most famous cubic graph is the *Petersen Graph*:

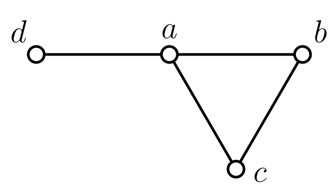


1.3.2 The degree sequence of a graph

If G is a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $\deg v_1 \geq \deg v_2 \geq \cdots \geq \deg v_n$, then $\deg v_1, \deg v_2, \ldots, \deg v_n$ is the *degree sequence* of G.

Example 1.13 Consider once again the graph

G :



Then $3, 2, 2, 1$ is the degree sequence of G.

Clearly, every graph has a degree sequence. But is every finite sequence of nonnegative integers the degree sequence of a graph?

Example 1.14 *Which of the following is the degree sequence of a graph?*

1. $s_1 : 4, 4, 3, 3, 3, 2, 1, 1.$
2. $s_2 : 6, 6, 3, 3, 3, 3.$
3. $s_3 : 3, 3, 3, 1.$

Solution:

1. The sequence s_1 has five odd entries. Therefore, by Corollary 2, it is not the degree sequence of a graph.
2. The sequence s_2 has six entries, so if it's the degree sequence of some graph G, then G has order 6. However, the maximum possible degree in a graph of order 6 is 5, and s_2 has two 6's. The sequence s_2 is therefore not the degree sequence of a graph.
3. Suppose that s_3 is the degree sequence of a graph G. Then G has order 4. Each of the three vertices of degree 3 is necessarily adjacent to the vertex of degree 1. However, the vertex of degree 1 is only adjacent to one other vertex. This is a contradiction and hence no such graph G exists. So s_3 is not the degree sequence of a graph.

1.3.3 The Havel-Hakimi Theorem

The preceding example shows that not every finite sequence of nonnegative integers is the degree sequence of a graph. The question of how to determine whether a given sequence of nonnegative integers is the degree sequence of a graph was first solved by Václav Havel [Hav55] in 1955. A few years later, S. Louis Hakimi [Hak62] independently discovered the same result, which now bears the names of both of these authors.

Theorem 3 (Havel-Hakimi Theorem)**.** *A sequence $s : d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ ($n \geq 2$ and $d_1 \geq 1$) is the degree sequence of a graph if and only if the sequence $s_1 : d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$ is the degree sequence of a graph.*

Proof. ($\Longleftarrow$) Assume first that s_1 is the degree sequence of a graph G. We must prove that s is also the degree sequence of a graph. Suppose that the vertices of G are $v_2, v_3, \ldots, v_n$ and that for $2 \leq i \leq d_1 + 1$ we have $\deg v_i = d_i - 1$, while for $d_1 + 2 \leq i \leq n$ we have $\deg v_i = d_i$. Let G' be the graph formed from G by adding a new vertex v_1 and joining it to the vertices $v_2, v_3, \ldots, v_{d_1+1}$. Then the new graph G' has degree sequence s and hence s is the degree sequence of a graph.

($\Longrightarrow$) Suppose now that there is at least one graph with degree sequence s. Amongst all such graphs, let G be one such that (i) $V(G) = \{v_1, v_2, \ldots, v_n\}$, (ii) for all $i \in \{1, 2, \ldots, n\}$, we have $\deg v_i = d_i$, and, (iii) the sum of the degrees of the vertices adjacent to v_1 is as large as possible. We claim that the neighbours of v_1 have degrees $d_2, d_3, \ldots, d_{d_1+1}$.

Proof of claim: Suppose, by way of contradiction, that the claim is not true. Then there are vertices v_i and v_j, where $i < j$, such that (i) $d_i > d_j$, and, (ii) v_1 is adjacent to v_j but not adjacent to v_i. Since $d_i > d_j$, there is a vertex $v_{i'} \neq v_j$ that is adjacent to v_i but not to v_j. Let G' be the graph obtained from G by deleting the edges v_1v_j and $v_iv_{i'}$ and adding the edges v_1v_i and $v_jv_{i'}$. The graph G' has the same degree sequence, s, as the graph G, but in G' the sum of the degrees of the neighbours of v_1 is higher than in G. This contradicts our choice of G, so it must be the case that the neighbours of v_1 have degrees $d_2, d_3, \ldots, d_{d_1+1}$, which proves the claim. ©

It follows that the graph $G - v_1$ has degree sequence s_1, which proves that s_1 is the degree sequence of a graph. □

By repeated use of the Havel-Hakimi Theorem, we may now determine whether a finite sequence of nonnegative integers is the degree sequence of a graph.

Example 1.15 *Is $s : 5, 3, 3, 3, 2, 2$ the degree sequence of a graph?*
Solution: Repeatedly using the Havel-Hakimi Theorem, we have

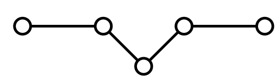
$$\begin{aligned} & s : 5,3,3,3,2,2 \text{ is the degree sequence of a graph} \\ \text{if and only if } & s_1 : 2,2,2,1,1 \text{ is the degree sequence of a graph} \\ \text{if and only if } & s_2 : 1,1,1,1 \text{ is the degree sequence of a graph.} \end{aligned}$$

The sequence $s_2 : 1, 1, 1, 1$ is clearly the degree sequence of the graph G_2 shown here

G_2 :

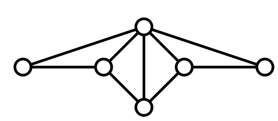
Hence s_2 is the degree sequence of a graph, and therefore s is the degree sequence of a graph. To construct a graph with degree sequence s, we work backwards from the graph with degree sequence s_2, using the ideas in the proof of the Havel-Hakimi Theorem. First, we construct a graph G_1 with degree sequence s_1 by adding a new vertex to G_2 and joining it to two vertices of degree 1 in G_2:

G_1 :



Now we construct G from G_1 by adding another new vertex and joining it to all five vertices of G_1:

G :



The graph G has degree sequence $s : 5, 3, 3, 3, 2, 2$, as required.

1.3.4 The Erdős-Gallai Theorem

Another useful characterization of degree sequences of graphs was obtained by Paul Erdős and Tibor Gallai [EG60] in 1960. We give a proof only of the necessity.

Theorem 4 (Erdős-Gallai Theorem). *A sequence $d_1, d_2, \ldots, d_n$ of nonnegative integers with $d_1 \geq d_2 \geq \cdots \geq d_n$ $(n \geq 2)$ is the degree sequence of a graph if and only if $\sum_{i=1}^{n} d_i$ is even and for each $k \in [1, n-1]$ we have*

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}.$$

Proof. We prove only the necessity, i.e., we prove that the degree sequence of a graph must satisfy the two conditions listed above. Let G be a nontrivial graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and degree sequence $d_1, d_2, \ldots, d_n$, where $d_1 \geq d_2 \geq \cdots \geq d_n$, and, for all $i \in [1, n]$, we have $\deg v_i = d_i$. We have already proved (Theorem 1) that $\sum_{i=1}^{n} d_i$ is even. Let $k \in [1, n-1]$ and let $V_k = \{v_1, v_2, \ldots, v_k\}$. Let E_1 be the set of edges that are incident with exactly one vertex in V_k and let E_2 be the set of edges that are incident with exactly two vertices in V_k. Then $\sum_{i=1}^{k} d_i = |E_1| + 2|E_2|$. Certainly, $|E_2| \leq k(k-1)/2$. Furthermore, every edge in E_1 is incident with exactly one vertex in $V(G) - V_k$, and no vertex $v_i \in V(G) - V_k$ is incident with more than $\min\{k, d_i\}$ edges of E_1. Hence $|E_1| \leq \sum_{i=k+1}^{n} \min\{k, d_i\}$, which proves the result. $\square$

1.4 Some classes of graphs

The *path* of order n $(n \geq 1)$ is the graph P_n with $V(P_n) = \{v_1, v_2 \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Some paths are shown in Figure 1.1.

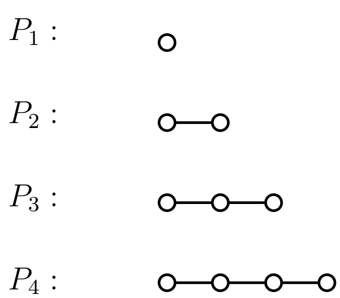


Figure 1.1: Some paths

The *cycle* of order n $(n \geq 3)$ is the graph C_n with $V(C_n) = \{v_1, v_2 \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_1, v_n\}$. Some cycles are shown in Figure 1.2.

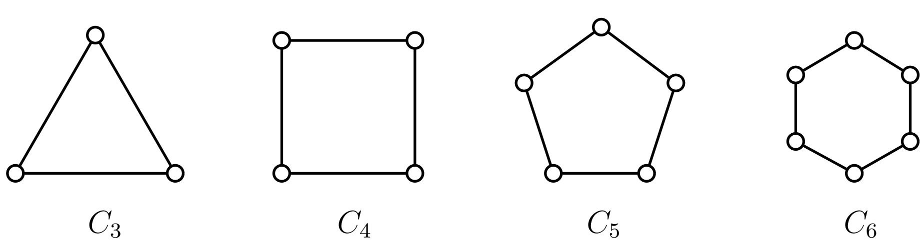


Figure 1.2: Some cycles

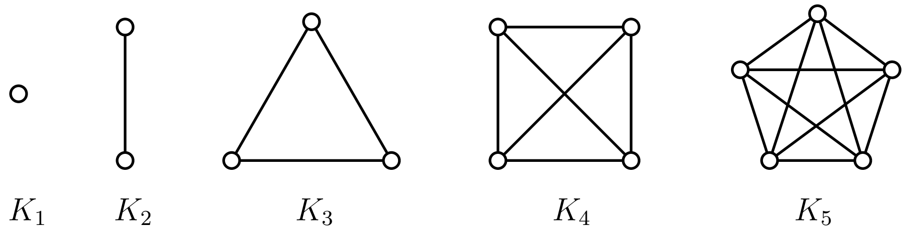


Figure 1.3: Some complete graphs

A graph is *complete* if every vertex is adjacent to every other vertex. The *complete graph of order* n ($n \geq 1$) is denoted K_n. Some complete graphs are shown in Figure 1.3.

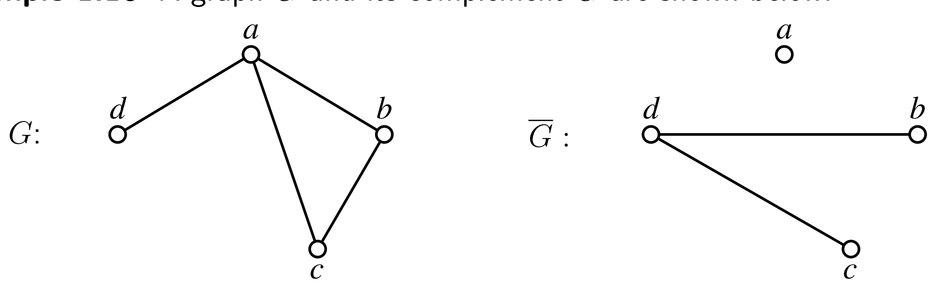
A graph in which every vertex is adjacent to every other vertex is complete. A graph in which at least one pair of vertices is not adjacent is *incomplete*.

1.5 Combining graphs

There are many different ways of generating new graphs from old. In this section, we explore some of these.

The *complement* of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$ and where two vertices are adjacent in $\overline{G}$ if and only if they are *not* adjacent in G.

Example 1.16 A graph G and its complement $\overline{G}$ are shown below:

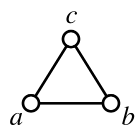


The *empty graph* of order n is $\overline{K}_n$, the complement of the complete graph of order n.

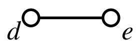
If G and H are graphs, then the *union* of G and H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. If $V(G) \cap V(H) = \emptyset$, then G and H are *disjoint*.

Example 1.17 Two disjoint graphs G and H are shown below, together with their union $G \cup H$.

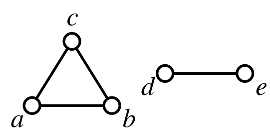
G:



H:



$G \cup H$:

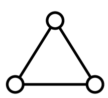


If G is a graph and k a positive integer, then by kG we mean the graph consisting of k disjoint 'copies' of G. To define this precisely, recall that the *cartesian product of two sets* A and B is the set $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. We may then define kG as the graph with $V(kG) = V(G) \times [1, k]$ and $E(kG) = \{(u, i)(v, i) : uv \in E(G) \text{ and } i \in [1, k]\}$.

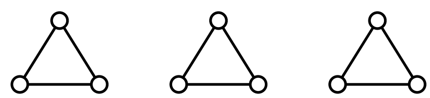
Example 1.18

The graphs C_3 and $3C_3$ are shown below:

C_3:



$3C_3$:

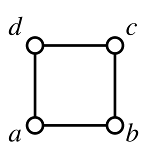


If G and H are disjoint graphs, then their *join* is the graph $G + H$ having $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{gh : g \in V(G) \text{ and } h \in V(H)\}$.

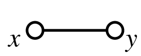
Example 1.19

The join $C_4 + P_2$ is shown below:

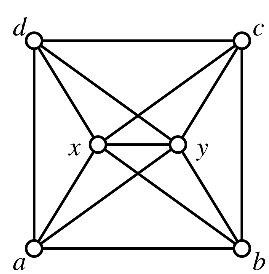
C_4:



P_2:



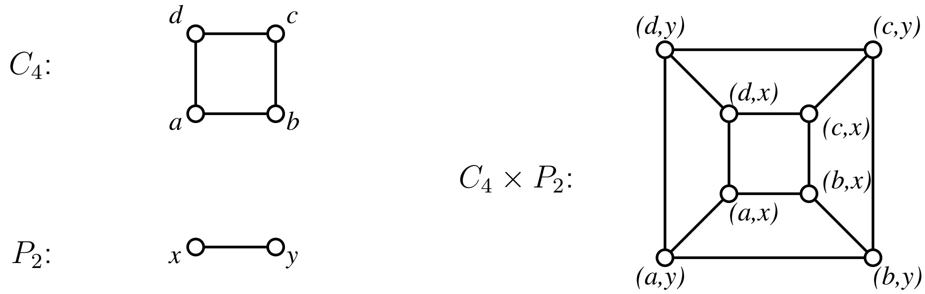
$C_4 + P_2$:



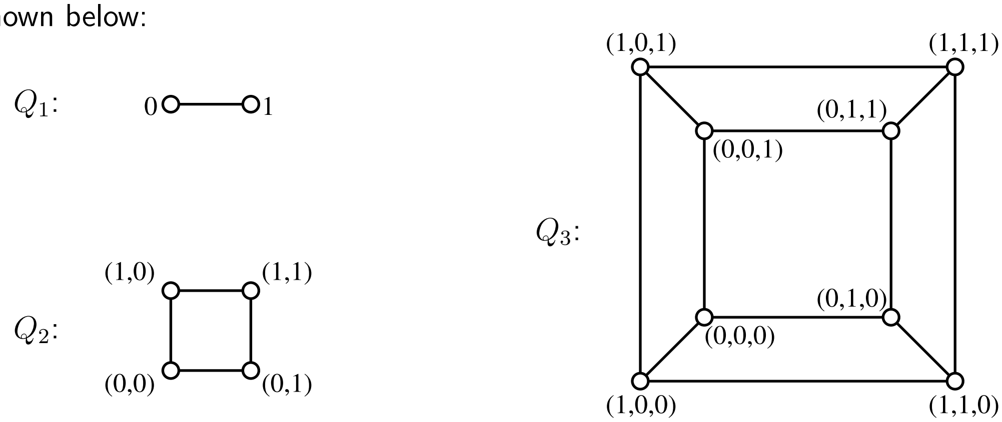
If G and H are graphs, then the *cartesian product* of G and H is the graph $G \times H$ in which

$$V(G \times H) = V(G) \times V(H), \text{ and,}$$
$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) : \text{(i) } g_1 g_2 \in E(G) \text{ and } h_1 = h_2, \text{ or, (ii) } g_1 = g_2 \text{ and } h_1 h_2 \in E(H)\}.$$

Example 1.20 The cartesian product $C_4 \times P_2$ is shown below:



Example 1.21 In this example, we introduce an important class of graphs. We define the *t-cube* recursively as the graph Q_t with $Q_1 = K_2$ and, for $t \geq 2$, $Q_t = Q_{t-1} \times K_2$. The first few cubes are shown below:



The t-cube is also known as a *hypercube*.

Another way to define the cubes is as follows: For a positive integer t, the vertex set of the graph Q_t consists of all ordered binary t-tuples, and two vertices in Q_t are adjacent if the ordered binary t-tuples they represent differ in exactly one position. This is illustrated in the preceding figure.

1.6 Isomorphic graphs

Two graphs G and H are *equal* if $V(G) = V(H)$ and $E(G) = E(H)$. Consider the three graphs G_1, G_2, G_3 shown in Figure 1.4.

Since $V(G_1) \neq V(G_2)$, clearly G_1 and G_2 are not equal. And while $V(G_1) = V(G_3)$, notice that $ac \in E(G_1)$ but $ac \notin E(G_3)$, so $E(G_1) \neq E(G_3)$, which means that $G_1 \neq G_3$.

It should be obvious, though, that these three graphs are, at a structural level, the same. The way we formalize this idea is as follows: If G and H are graphs, then an *isomorphism* from

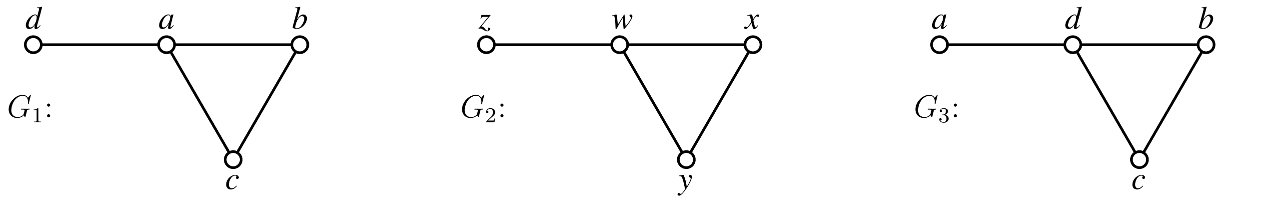
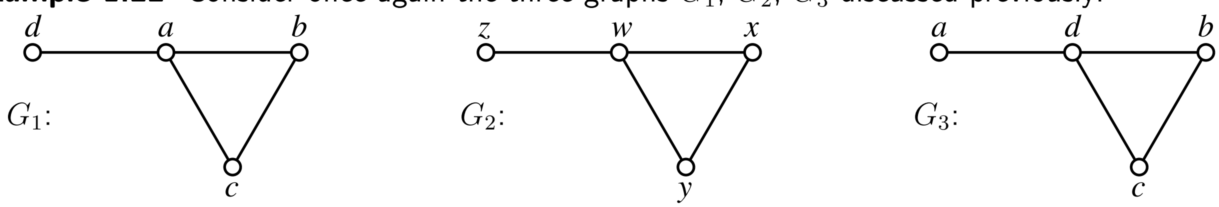


Figure 1.4: Three pairwise unequal but isomorphic graphs.

G to H is a bijection $f : V(G) \to V(H)$ such that for all $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If there is an isomorphism between G and H, then we say that G and H are *isomorphic* graphs, and we write $G \cong H$.

Example 1.22 Consider once again the three graphs G_1, G_2, G_3 discussed previously:



Let $f_{12} : V(G_1) \to V(G_2)$ be the function defined by

$$\begin{aligned} a &\mapsto w \\ b &\mapsto x \\ c &\mapsto y \\ d &\mapsto z \end{aligned}$$

Clearly, f_{12} is a bijection. We can also check that for all $u, v \in V(G_1)$ we have $uv \in E(G_1)$ if and only if $f_{12}(u)f_{12}(v) \in E(G_2)$ (e.g., $ad \in E(G_1)$, and $f_{12}(a)f_{12}(d) = wz \in E(G_2)$). So f_{12} is an isomorphism between G_1 and G_2, which means that $G_1 \cong G_2$, i.e., G_1 and G_2 are really the same graph with different labels on the vertices.

Notice that f_{12} is not the only isomorphism between G_1 and G_2: The function $f'_{12} : V(G_1) \to V(G_2)$ with rule

$$\begin{aligned} a &\mapsto w \\ b &\mapsto y \\ c &\mapsto x \\ d &\mapsto z \end{aligned}$$

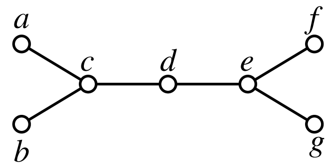
is also an isomorphism.

Similarly, one may find isomorphisms between G_1 and G_3 and between G_2 and G_3, showing that $G_1 \cong G_2 \cong G_3$.

Let G be a graph. An isomorphism from G to itself is called an *automorphism* of G. An automorphism of G is a permutation of $V(G)$ that preserves adjacency. We shall therefore use *cycle notation* to describe an automorphism.

Example 1.23 Consider the graph G shown below:

G :



One automorphism of G is the permutation α_1 with rule

$$\begin{aligned} a &\mapsto b \\ b &\mapsto a \\ c &\mapsto c \\ d &\mapsto d \\ e &\mapsto e \\ f &\mapsto f \\ g &\mapsto g \end{aligned}$$

This may be written more succinctly using cycle notation as $\alpha_1 = (ab)(c)(d)(e)(f)(g)$, or, if we agree that any vertex that does not appear is fixed (e.g., $\alpha_1(c) = c$), we may simply write $\alpha_1 = (ab)$. With this convention, we find that there are eight automorphisms of G :

$$\begin{aligned} \alpha_e &= (a) \qquad \text{(the identity permutation)} \\ \alpha_1 &= (ab) \\ \alpha_2 &= (fg) \\ \alpha_3 &= (ab)(fg) \\ \alpha_4 &= (af)(ce)(bg) \\ \alpha_5 &= (afbg)(ce) \\ \alpha_6 &= (ag)(ce)(bf) \\ \alpha_7 &= (agbf)(ce) \end{aligned}$$

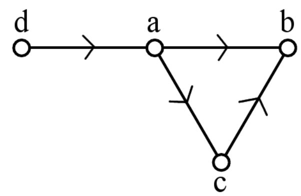
Each automorphism represents a *symmetry* of the graph: A way in which the bits of the graph may 'rearrange' themselves without changing the graph's structure. We shall have more to say about this later.

1.7 Digraphs

A *directed graph* or *digraph* D consists of a finite set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of distinct vertices called *arcs*. The number of vertices in a digraph is its *order*. The number of arcs is its *size*. When we draw a picture of a digraph, we put arrows on the arcs to indicate order, e.g., the arc (a, b) is drawn with the arrow pointing from a to b. We shall denote the arc (a, b) as $\overrightarrow{ab}$.

Example 1.24 Let D be the digraph with vertex set $V(D) = \{a, b, c, d\}$ and arc set $A(D) = \{\overrightarrow{ab}, \overrightarrow{ac}, \overrightarrow{cb}, \overrightarrow{da}\}$. A diagram of D is shown below:

D:



The digraph D has order 4 and size 4. Since $\overrightarrow{ab}$ is an arc of D, we say that a is *adjacent to* b, while b is *adjacent from* a.

Let D be a digraph and $v \in V(D)$. The *indegree* of v is the number of vertices from which v is adjacent, while the *outdegree* of v is the number of vertices to which v is adjacent. The indegree of v is denoted id v, while the outdegree of v is od v.

Example 1.25 In the digraph D in Example 1.24, we have id $a = 1$ and od $a = 2$. A vertex with outdegree 0, like the vertex b, is called a *sink*. A vertex of indegree 0, like the vertex d, is called a *source*.

If D is a digraph, then the *underlying graph* of D is the graph G obtained from D by replacing each (directed) arc with an (undirected) edge and then deleting any multiple edges which result. In other words, the underlying graph G has $V(G) = V(D)$ and $E(G) = \{uv : \overrightarrow{uv} \in A(D) \text{ or } \overrightarrow{vu} \in A(D)\}$.

Example 1.26 The underlying graph of the digraph in Example 1.24 is the frequently discussed Martini Glass (Example 1.1).

Exercises

1.1 Draw the graph G with vertex set $V(G) = \{a, b, c, d, e, f\}$ and edge set $E(G) = \{ab, bc, be, cf, de, af\}$.

1.2 Determine the order and size of each of the following:

a. The path P_n.

b. The cycle C_n.

c. The complete graph K_n.

d. The cube Q_t.

1.3 Let $c_1, c_2, \ldots, c_k$ be odd integers. Prove that $\sum_{i=1}^{k} c_i$ is odd if and only if k is odd.

1.4 Find a cubic graph of order 7 or prove that no such graph exists.

1.5 A graph G is *irregular* if no two vertices of G have the same degree. Prove that if G is an irregular graph, then G is trivial.

1.6 Suppose that you and your wife attend a party with three other married couples. Several handshakes take place. No-one shakes hands with himself (or herself) or with his (or her) spouse, and no-one shakes hands with the same person more than once. After all the handshaking is complete, you ask each person, including your wife, how many hands he or she shook. Each person gives a different answer.

a. How many hands did you shake?

b. How many hands did your wife shake?

1.7 Prove or disprove: For every graph G and every integer $r \geq \Delta(G)$, there is an r-regular graph H containing G as an induced subgraph.

1.8 Prove or disprove:

a. If two graphs are isomorphic, then they have the same degree sequence.

b. If two graphs have the same degree sequence, then they are isomorphic.

1.9 Determine whether each of the following sequences is the degree sequence of a graph. If it isn't, state why it isn't. If it is, construct a graph with that degree sequence. It is suggested, though not compulsory, that you apply both the Havel-Hakimi and Erdős -Gallai Theorems to each sequence.

a. $5, 5, 5, 3, 3, 2, 2, 2, 2, 2$.

b. $4, 4, 3, 2, 1, 0$.

c. $3, 3, 2, 2, 2, 2, 1, 1$.

d. $7, 4, 3, 3, 2, 2, 2, 1, 1, 1$.

1.10 Recall that an *equivalence relation* is a relation that is reflexive, symmetric, and transitive. Prove that $\cong$ is an equivalence relation on the set of all graphs. Of what significance are the equivalence classes in this relation?

1.11 We say that a word W_1 can be *transformed* into a word W_2 if W_2 can be obtained from W_1 by performing exactly one of the following two steps:

- Interchanging two letters of W_1;
- Replacing a letter in W_1 by another letter.

a. Let Ω be a set of words and R the relation on Ω given by: $W_1 R W_2$ if W_1 can be transformed into W_2. Which properties does this relation have? Is it an equivalence relation?

b. This situation can be modeled by a graph G_Ω in which V is the set of words and we join two vertices u and v with an edge if the word represented by u can be transformed into the word represented by v. Draw the graph G_Ω for the case when $\Omega = \{$aim, arm, arc, art, car, oar, act, cat, rat, tar, oat$\}$.

1.12 How many automorphisms does each of the following graphs have?

a. The path P_n.

b. The cycle C_n.

c. The complete graph K_n.

d. The empty graph $\overline{K}_n$.

1.13 Recall that the 'First Theorem of Graph Theory' (Theorem 1) states that if G is a graph of size m, then

$$\sum_{v \in V(G)} \deg v = 2m.$$

State and prove a 'First Theorem of Digraph Theory'.

1.14 Construct the digraph with vertex set $\{-3, 3, 6, 12\}$ in which $\overrightarrow{ij}$ is an arc iff $i \neq j$ and $i|j$.

1.15 For the following pairs G, H of graphs, draw $G + H$ and $G \times H$:

a. $G = K_5$ and $H = K_2$.

b. $G = \overline{K}_5$ and $H = \overline{K}_3$.

c. $G = C_5$ and $H = K_1$.

1.16 Find spanning subgraphs G_0, G_1, G_2, G_3 of the Petersen graph, where G_r is r-regular for $0 \leq r \leq 3$.

1.17 Prove that if G and $\overline{G}$ are both r-regular for some nonnegative integer r, then G has odd order.

1.18 A graph is *self-complementary* if it is isomorphic to its complement.

a. Determine all self-complementary graphs of order 5 or less.

b. If G is a self-complementary graph of order n, prove that $n \equiv 0 \pmod 4$ or $n \equiv 1 \pmod 4$.

1.19 Let n, k, r be positive integers with $r \leq k \leq n$ and let S be a set of cardinality n. Then $J(n, k, r)$ is the graph whose vertices are the k-subsets of S, and where two vertices are adjacent if the intersection of the two subsets to which they correspond has cardinality r.

a. Draw $J(5, 2, 0)$.

b. Determine the order of $J(n, k, r)$.

c. Prove that $J(n, k, r)$ is regular and determine the degree of the vertices. Hence find the size of $J(n, k, r)$.

d. Prove that $J(5, 2, 0)$ is (isomorphic to) the Petersen Graph.

Chapter 2

Distance

2.1 Walks, trails, paths, circuits, and cycles

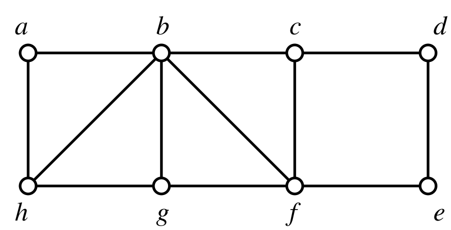
Let u and v be vertices in a graph G. A *u-v walk* is a finite, alternating sequence

$$u = v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k = v$$

of vertices $v_0, v_1, \ldots, v_k$ and edges $e_1, e_2, \ldots, e_k$ such that, for all $i \in [1, k]$, the edge e_i joins the vertices v_{i-1} and v_i. The number of edges in a walk is its *length*, and a walk of length zero is *trivial*.

Example 2.1 Consider the graph G shown below:

G:



The sequence $W_1 : a, ab, b, bc, c, bc, b, bg, g, fg, f$ is an $a - f$ walk of length 5.

For brevity, we shall usually write walks by listing the vertices but omitting the edges. For example, the walk W_1 in Example 2.1 could be rewritten as $W_1 : a, b, c, b, g, f$.

A *trail* is a walk in which no edge is repeated. A *path* is a walk in which no vertex is repeated. A walk that begins and ends at the same vertex is *closed*. A walk that is not closed is *open*. A nontrivial closed trail is called a *circuit*. A circuit with no repeated vertex except the first/last is a *cycle*.

Example 2.2

Consider again the graph G from Example 2.1. The walk W_1 from that example is neither a trail nor a path, since it contains repeated vertices and edges. The walk $W_2 : c, f, g, b, c, d$ is a trail, since no edge is repeated, but not a path, since the vertex c is repeated. The walk $W_3 : h, g, b, c, f$

is a path (and a trail). The walks W_1, W_2, and W_3 are all open, while the walk $W_4 : a, b, a, b, a, b, a$ is closed. The walk $W_5 : b, a, h, b, f, c, b$ is a circuit and the walk $W_6 : e, d, c, b, f, e$ is a cycle.

Every $u - v$ path is a $u - v$ trail, and every $u - v$ trail is (of course) a $u - v$ walk. It is not true, however, that every $u - v$ walk is a $u - v$ path. A walk W is said to *contain* a walk W' if W' is a subsequence of W.

Theorem 5. *Every $u - v$ walk in a graph contains a $u - v$ path.*

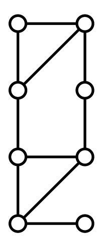
Proof. Let W be a $u - v$ walk in a graph G. If W is closed, then $W' : u$ is a (trivial) $u - v$ path and we are done. Suppose then that $u \neq v$. If no vertex of W is repeated, then W is a $u - v$ path, so we may suppose that at least one vertex x of G occurs at least twice on W. Suppose that $W : v_0, v_1, \ldots, v_k$ and let i and j, with $0 \leq i < j \leq k$, be the smallest and largest integers respectively such that $x = v_i = v_j$. Then $W' : v_0, v_1, \ldots, v_i, v_{j+1}, v_{j+2}, \ldots, v_k$ is a $u - v$ walk in which the vertex x is not repeated. If W' contains no repeated vertex, then W' is a $u - v$ path contained in W. If W' contains a repeated vertex, then, since W is finite, we may repeat the above procedure as many times as necessary until we obtain a $u - v$ path that is contained in W. □

2.2 Distance in graphs

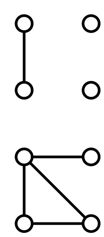
Let u and v be vertices in a graph G. We say that u is *connected to* v if there is a $u - v$ path in G. A graph G is *connected* if every pair of vertices of G is connected. A graph that is not connected is *disconnected*. A maximal connected subgraph[1] of G is called a *component* of G.

Example 2.3

G_1 :



G_2 :

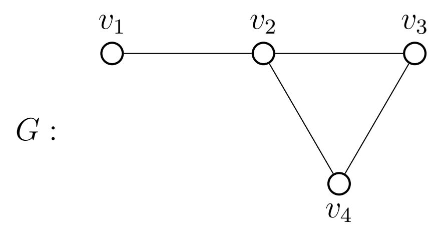


The graph G_1 is connected. The graph G_2, which has four components, is disconnected.

Let G be a connected graph and $u, v \in V(G)$. The *distance* from u to v, denoted $d(u, v)$, is the minimum length of a $u - v$ path. A $u - v$ path of minimum length is called a *shortest* $u - v$ path or a $u - v$ *geodesic*.

[1] A subgraph is *maximal* with respect to some property if it is not properly contained in any other subgraph with the same property. Hence, a maximal *connected* subgraph is a connected subgraph that is not properly contained in any other *connected* subgraph.

Example 2.4



In the graph G above, we have $d(v_1, v_1) = 0$, while $d(v_1, v_2) = 1$ and $d(v_1, v_3) = d(v_1, v_4) = 2$. The path v_1, v_2, v_3 is a $v_1 - v_3$ geodesic. The path v_1, v_2, v_4, v_3 is a $v_1 - v_3$ path but not a $v_1 - v_3$ geodesic.

Recall that a *metric space* is an ordered pair (S, f) consisting of a set S and a function $f : S \times S \to \mathbb{R}$ satisfying for all $x, y, z \in S$:

1. $f(x, y) \geq 0$,
2. $f(x, y) = 0$ if and only if $x = y$,
3. $f(x, y) = f(y, x)$, and,
4. $f(x, y) \leq f(x, z) + f(z, y)$ (the triangle inequality).

Theorem 6. *If G is a connected graph and $d : V(G) \times V(G) \to \mathbb{Z}$ is the standard distance function, then $(V(G), d)$ is a metric space.*

The proof of this is left as an exercise.

Let G be a connected graph and u a vertex of G. The *eccentricity* of the vertex u is

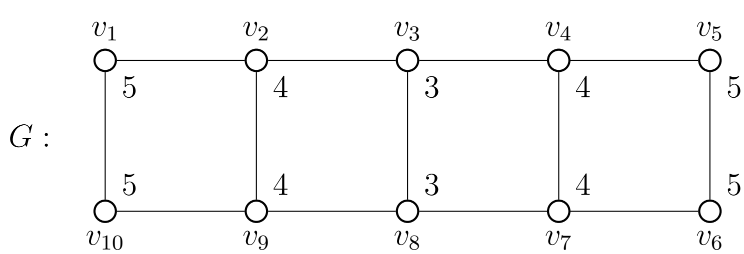
$$e(u) = \max\{d(u, v) : v \in V(G)\}.$$

The *radius* and *diameter* of the graph G are then defined as

$$\begin{aligned} \text{rad } G &= \min\{e(u) : u \in V(G)\} \\ \text{diam } G &= \max\{e(u) : u \in V(G)\}. \end{aligned}$$

If $e(u) = \text{rad } G$, then u is a *central vertex* of G, while if $e(u) = \text{diam } G$, then u is a *peripheral vertex* of G. The subgraph of G induced by the central vertices of G is the *center* of G, while the subgraph induced by the peripheral vertices is the *periphery*.

Example 2.5



In the graph G shown above, each vertex is labeled with its eccentricity. We have rad $G = 3$ and diam $G = 5$. The central vertices of G are v_3 and v_8, and the center of G, the subgraph induced by $\{v_3, v_8\}$, is isomorphic to K_2. The peripheral vertices of G are v_1, v_5, v_6, and v_{10}, and the periphery of G, the subgraph induced by $\{v_1, v_5, v_6, v_{10}\}$, is isomorphic to $2K_2$.

Theorem 7. *For every connected graph G,*

$$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G.$$

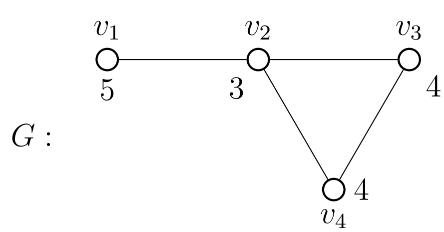
Proof. The first part of the chain of inequalities, that rad $G \leq$ diam G, follows directly from their definition. To prove that diam $G \leq 2$ rad G, let p_1 and p_2 be vertices of G for which $d(p_1, p_2) =$ diam G, and let c be a central vertex of G. Using the triangle inequality, we have diam $G = d(p_1, p_2) \leq d(p_1, c) + d(c, p_2) \leq e(c) + e(c) = 2$ rad G. □

For a vertex u in a connected graph G, the *total distance* of u is

$$d(u) = \sum_{v \in V(G)} d(u, v).$$

If $d(u) = \min\{d(v) : v \in V(G)\}$, then v is a *median vertex* of G. The subgraph of G induced by the median vertices is the *median* of G.

Example 2.6



Each vertex of the graph G shown above is labeled with its total distance. The only median vertex is v_2. The median of G consists of the graph induced by $\{v_2\}$, which is isomorphic to K_1.

2.3 Walks, connectivity, and distance in digraphs

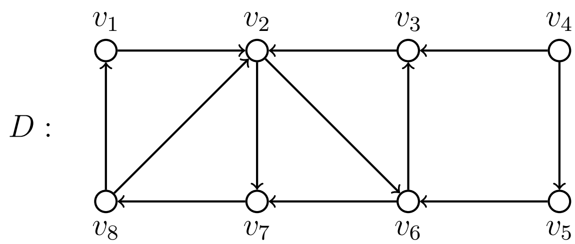
When discussing digraphs, we must be careful to distinguish between those situations in which we wish to take into account the directions of the edges, and those in which we do not.

Let D be a digraph and u, v vertices of D.

- A $u - v$ *directed walk* is a finite alternating sequence $u = v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k = v$ of vertices and arcs of D in which, for all $i \in [1, k]$, we have $a_i = \overrightarrow{v_{i-1}v_i}$.
- A $u - v$ *semi-walk* is a finite alternating sequence $u = v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k = v$ of vertices and arcs of D in which, for all $i \in [1, k]$, we have $a_i = \overrightarrow{v_{i-1}v_i}$ or $a_i = \overrightarrow{v_i v_{i-1}}$.

To summarize: In a directed walk, we must respect the directions of the arcs, while in a semi-walk, we need not.

Example 2.7



In the digraph D shown above, v_1, v_2, v_6, v_3, v_2 is both a $v_1 - v_2$ directed walk and a $v_1 - v_2$ semi-walk. On the other hand, v_1, v_8, v_2 is a $v_1 - v_2$ semi-walk, but not a $v_1 - v_2$ directed walk.

Analogously with undirected graphs, a $u - v$ *directed path* is a $u - v$ directed walk in which no vertex is repeated, while a $u-v$ *semi-path* is a $u-v$ semi-walk in which no vertex is repeated.

A digraph D is called *connected* or *weakly connected* if for every pair u, v of vertices of D there is a $u - v$ semi-path, i.e., if the underlying graph of D is connected. On the other hand, D is *strongly connected* if for every pair u, v of vertices there is a $u - v$ directed path.

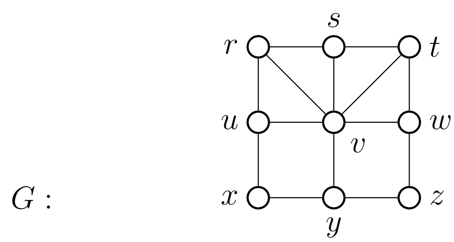
Example 2.8 The digraph in Example 2.7 is weakly connected (since the underlying graph is connected). It is not, however, strongly connected (there is no $v_1 - v_4$ directed path in D).

If D is a connected digraph and $u, v \in V(D)$, then $d(u, v)$, the *distance between u and v*, is the smallest number of arcs on a $u - v$ directed path. If D is not strongly connected, then there will be pairs u, v of vertices for which $d(u, v)$ is not well-defined; for convenience, we frequently define the distance between such pairs to be ∞.

Example 2.9 Returning to the digraph in Example 2.7, we have $d(v_1, v_8) = 3$ while $d(v_1, v_4) = \infty$.

Exercises

2.1 Consider the following graph:



Give an example of each of the following or explain why no such example exists:

a. An $x - y$ walk of length 6.

b. A $v - w$ trail that is not a $v - w$ path.

c. An $r - z$ path of length 2.

d. An $x - z$ path of length 3.

e. An $x - t$ path of length $d(x, t)$.

f. A circuit of length 10.

g. A cycle of length 8.

h. A geodesic whose length is diam G.

2.2 Prove that a closed walk of odd length contains an odd cycle.

2.3 Recall that an *equivalence relation* is a relation that is reflexive, symmetric, and transitive. If G is a graph, prove that *is connected to* is an equivalence relation on $V(G)$. Of what significance are the equivalence classes in this relation?

2.4 Prove Theorem 6.

2.5 If G is a graph of order n with $\delta(G) \geq \frac{n-1}{2}$, prove that G is connected.

2.6 Prove or disprove: If G is a connected graph and $v \in V(G)$, then v is a central vertex if and only if v is a median vertex.

2.7 Prove that if G is a disconnected graph, then diam $\overline{G} \leq 2$.

2.8 Prove or disprove: Every graph is the center of some connected graph.

2.9 For a connected graph G, the *average distance* $\mu(G)$ of the graph is the average distance between all pairs of vertices, i.e., $\mu(G) = \dfrac{2}{n(n-1)} \displaystyle\sum_{u,v \in V(G)} d(u, v)$. Prove or disprove: If G is a connected graph, then rad $G \leq \mu(G) \leq$ diam G.

2.10 Draw all connected graphs of order 5 in which the distance between every two distinct vertices is odd. Explain why you know that you have drawn all such graphs.

2.11 Let $P : u = v_0, v_1, \ldots, v_k = v$ be a $u - v$ geodesic in a connected graph G. Prove that $d(u, v_i) = i$ for each integer i with $1 \leq i \leq k$.

2.12 Let P and Q be two longest paths in a connected graph G. Prove that P and Q have at least one vertex in common.

2.13 Prove or disprove: Let G be a connected graph of diameter k. If P and Q are two geodesics of length k in G, then P and Q have at least one vertex in common.

2.14 Show that if G is a disconnected graph containing exactly two odd vertices, then these odd vertices must be in the same component of G.

2.15 Show that if G is a connected graph that is not regular, then G contains adjacent vertices u and v such that $\deg u \neq \deg v$.

2.16 A graph is *self-complementary* if it is isomorphic to its complement. Prove that every nontrivial self-complementary graph has diameter 2 or 3.

2.17 If u and v are adjacent vertices in a connected graph G, prove that $|e(u) - e(v)| \leq 1$.

2.18 Let D be a digraph. For a vertex v in D, the quantity $e^+(v) = \max\{d(v, x) : x \in V(D)\}$ is called the *out-eccentricity* of v.

a. Prove or disprove: Every vertex in D has finite out-eccentricity if and only if D is strongly connected.

b. Let u and v be vertices in D. Prove or disprove: If D is strongly connected and $\overrightarrow{uv} \in A(D)$, then $|e^+(u) - e^+(v)| \leq 1$.

Chapter 3

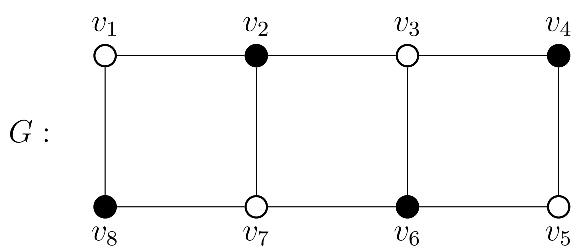
The structure of graphs

3.1 Bipartite graphs

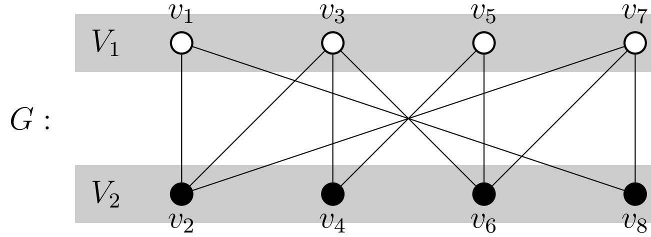
Recall that a collection $\{S_1, S_2, \ldots, S_k\}$ of sets is called *pairwise disjoint* if for all $i, j \in [1, k]$ with $i \neq j$, we have $S_i \cap S_j = \emptyset$. A *partition* of a set S is a finite, pairwise disjoint collection $\{S_1, S_2, \ldots, S_k\}$ of nonempty subsets of S such that $\bigcup_{i=1}^{k} S_i = S$.

A graph G is *bipartite* if there is a partition $\{V_1, V_2\}$ of $V(G)$ such that every edge of G joins a vertex of V_1 to a vertex of V_2. In other words, G is bipartite if we can partition $V(G)$ into two sets V_1 and V_2 such that both $G[V_1]$ and $G[V_2]$ are empty. The sets V_1 and V_2 are called *partite sets* and $\{V_1, V_2\}$ is called a *bipartition* of G.

Example 3.1

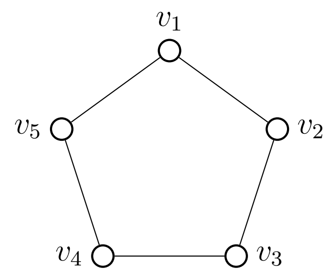


In the graph G shown above, choose $V_1 = \{v_1, v_3, v_5, v_7\}$ (the white vertices) and $V_2 = \{v_2, v_4, v_6, v_8\}$ (the black vertices). Then $\{V_1, V_2\}$ is a partition of $V(G)$. Furthermore, there is no pair of adjacent white vertices, and there is no pair of adjacent black vertices. This shows that G is bipartite. To make things clearer, we can redraw G:



Not all graphs are bipartite. Consider the graph C_5, shown below:

C_5 :



Suppose that C_5 is bipartite and let $\{V_1, V_2\}$ be a bipartition of C_5. Without loss of generality, we may assume that $v_1 \in V_1$. Since $v_1v_2 \in E(C_5)$, we must then have $v_2 \in V_2$. Continuing in this fashion, we argue that $v_3 \in V_1$ and $v_4 \in V_2$. What do we do with v_5? We can't have $v_5 \in V_1$, because $v_1 \in V_1$ and $v_1v_5 \in E(C_5)$. Similarly, we can't have $v_5 \in V_2$ since $v_4 \in V_2$ and $v_4v_5 \in E(C_5)$. Hence C_5 is not bipartite.

With a little thought, you should convince yourself that the problem we encountered when trying to construct a bipartition of C_5 will arise with any odd cycle. The following result shows that this is an important observation. If $P : v_1, v_2, \ldots, v_k$ is a path, then by $P[v_i, v_j]$ (where $i, j \in [1, k]$ and $i \leq j$) we mean the subpath of P induced by $\{v_i, v_{i+1}, \ldots, v_j\}$.

Theorem 8. *A graph is bipartite if and only if it does not contain an odd cycle.*

Proof. ($\Longrightarrow$) Suppose, to the contrary, that G is a bipartite graph that contains an odd cycle $v_1, v_2, \ldots, v_k, v_1$. Let $\{V_1, V_2\}$ be a bipartition of G and suppose, without loss of generality, that $v_1 \in V_1$. Arguing as in Example 3.1, we find that $v_i \in V_1$ when i is odd, while $v_i \in V_2$ when i is even. But then both the vertices v_1 and v_k are in V_1. Since $v_1v_k \in E(G)$, this is impossible.

($\Longleftarrow$) Let G be a graph that contains no odd cycle. Notice that G is bipartite if and only if each of its components is bipartite, so we assume that G is connected. Let $u \in V(G)$ and define

$$V_1 = \{v \in V(G) : d(u, v) \text{ is odd}\},$$
$$V_2 = \{v \in V(G) : d(u, v) \text{ is even}\}.$$

We claim that $\{V_1, V_2\}$ is a bipartition of G. Certainly, $\{V_1, V_2\}$ is a partition of $V(G)$. It remains to prove that every edge of G joins a vertex of V_1 to a vertex of V_2. Suppose, by way of contradiction, that there are two vertices $x, y \in V_1$, say, such that $xy \in E(G)$. Let P_x be a $u - x$ geodesic and let P_y be a $u - y$ geodesic. Notice that $u \in V(P_x) \cap V(P_y)$ and, since $d(u, x)$ and $d(u, y)$ have the same parity[1], we cannot have $x \in V(P_y)$ or $y \in V(P_x)$. Let z be that vertex in $V(P_x) \cap V(P_y)$ for which $d(z, x)$ is a minimum. If $z' \in V(P_x) \cap V(P_y)$ and $d(z', y) < d(z, y)$, then because P_x and P_y are geodesics, we must have $d(z', x) < d(z, x)$. Hence, there is no vertex in $V(P_x) \cap V(P_y)$ that is closer to y than z. Thus, z is the only vertex of $P_x[z, x]$ that belongs to P_y, and z is the only vertex of $P_y[z, y]$ that belongs to P_x. Now, $d(u, x) = d(u, z) + d(z, x)$ and $d(u, y) = d(u, z) + d(z, y)$, so we have $d(u, x) - d(u, y) = d(z, x) - d(z, y)$. Since $d(u, x)$ and $d(u, y)$ have the same parity, this implies that $d(z, x)$ and $d(z, y)$ have the same parity, and hence that $d(z, x) + d(z, y)$ is even. By following P_x in reverse from x to z, then P_y from z to y, and then using the edge xy, we obtain a cycle of length $d(z, x) + d(z, y) + 1$, a contradiction. $\square$

[1]Two integers have the same *parity* if they are both even or both odd.

For positive integers r and s, we define the *complete bipartite graph* $K_{r,s}$ to be the graph $\overline{K_r}+\overline{K_s}$. In other words, $K_{r,s}$ is a bipartite graph in which there is a partite set V_1 of cardinality r and a partite set V_2 of cardinality s, and in which every vertex in V_1 is adjacent to every vertex in V_2. The complete bipartite graph $K_{2,4}$ is shown in Figure 3.1. A complete bipartite

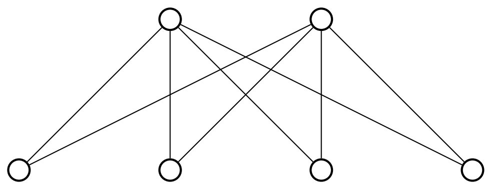


Figure 3.1: The complete bipartite graph $K_{2,4}$.

graph of the form $K_{1,r}$ is called a *star*. The star $K_{1,3}$ is called a *claw*.

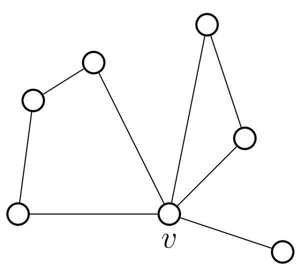
We can extend the idea of a bipartite graph. If there is a partition $\{V_1, V_2, \ldots, V_k\}$ of a graph G such that for all $i \in [1, k]$, the subgraph $G[V_i]$ is empty, then G is called *k-partite*.

3.2 Cut-vertices, bridges, and blocks

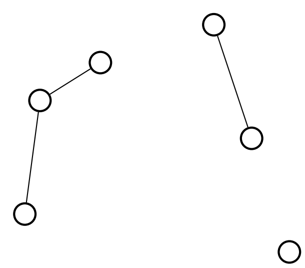
Let G be a connected graph. If v is a vertex of G for which $G - v$ is disconnected, then v is called a *cut-vertex*. If e is an edge of G for which $G - e$ is disconnected, then e is called a *bridge*.

Example 3.2

G :

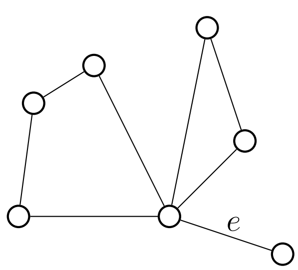


$G - v$:

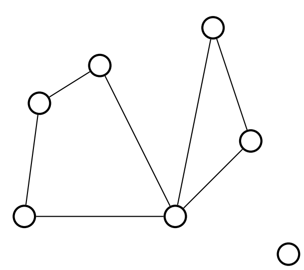


The graph G shown above is connected while the graph $G - v$ is disconnected, so v is a cut-vertex. In fact, v is the only cut-vertex of G.

G :



$G - e$:



Since G is connected and $G - e$ is disconnected, e is a bridge. The edge e is the only bridge of G.

Theorem 9. *A vertex v in a connected graph G is a cut-vertex if and only if there are vertices u and w $(u, w \neq v)$ such that v lies on every $u - w$ path.*

Proof. ($\Longrightarrow$) Suppose first that v is a cut-vertex of G. Then $G - v$ is disconnected, which means that there are vertices u and w in $G - v$ such that there is no $u - w$ path in $G - v$. Hence there is no $u - w$ path in G that does not contain v.

($\Longleftarrow$) Suppose now that there are vertices u and w with $u, w \neq v$ such that v lies on every $u - w$ path in G. Then there is no $u - w$ path in $G - v$, which means that $G - v$ is disconnected. Hence v is a cut-vertex. □

An analogous result holds for bridges.

Theorem 10. *An edge e in a connected graph G is a bridge if and only if there are vertices u and w such that e lies on every $u - w$ path.*

Another interesting result for bridges is the following.

Theorem 11. *An edge e in a connected graph G is a bridge if and only if e does not lie on a cycle of G.*

Proof. ($\Longrightarrow$) Suppose first that $e = uv$ is a bridge of G. Then $G - e$ is disconnected, the vertices u and v lie in different components of $G - e$, and there is no $u - v$ path in $G - e$. Suppose now, to the contrary, that e lies on a cycle $C : u, v, v_1, v_2, \ldots, v_k, u$ in G. But then $v, v_1, v_2, \ldots, v_k, u$ is a $v - u$ path in $G - e$, a contradiction. Hence e does not lie on a cycle in G.

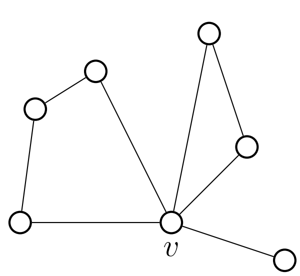
($\Longleftarrow$) Suppose now that e does not lie on a cycle in G. Suppose, by way of contradiction, that e is not a bridge. Then $G - e$ is connected. Hence there is a $u - v$ path P in $G - e$. But then P together with the edge e is a cycle in G that contains the edge e, a contradiction. □

Inspired by Theorem 11, we make the following definition: An edge e in a graph G is said to be a *cycle edge* if e lies on a cycle of G. Theorem 11 states that every edge of a connected graph is either a bridge or a cycle edge.

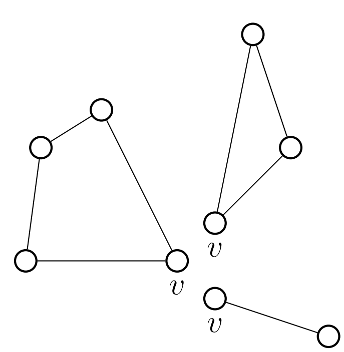
A connected graph that contains at least one cut-vertex is *separable*. A connected graph that contains no cut-vertices is called *non-separable*. A maximal non-separable subgraph of a connected graph is called a *block*, i.e., a block is a connected subgraph that has no cut-vertices and that is not properly contained in a subgraph with no cut-vertices. If G is non-separable, then G has no cut-vertices and hence G contains exactly one block. For this reason, a non-separable graph is sometimes called a *block*.

Example 3.3

G :



The three blocks of G :



The graph G from Example 3.2 has three blocks. The only cut-vertex of G, the vertex v, occurs in all three blocks. It's important to distinguish between the role the vertex v plays as a vertex of G and the role it plays as a vertex in each of the three blocks. In the graph G, the vertex v is a cut-vertex since $G - v$ is disconnected. On the other hand, let G' be any of the three blocks of G. Then $G' - v$ is *not* disconnected, so v is *not* a cut-vertex of G'. Notice, however, that in any connected subgraph of G that properly contains G', the vertex v is a cut-vertex. Hence G' is a maximal connected subgraph that is non-separable, i.e., a block.

Two distinct blocks have at most one vertex in common. If two distinct blocks have a vertex in common, then that vertex is a cut-vertex. We now present some interesting characterizations of blocks. The proof of the first lemma is left as an exercise.

Lemma 12. *If G is a graph of order at least 3 that contains a bridge, then G contains a cut-vertex.*

Theorem 13. *A graph G of order at least 3 is a block if and only if for every pair u, v of vertices of G, there is a cycle containing both u and v.*

Proof. ($\Longleftarrow$) Suppose that every pair of vertices of G lie on a common cycle and, to the contrary, that G contains a cut-vertex x. Then by Theorem 9, there are vertices u and v $(u, v \neq x)$ such that x lies on every $u - v$ path. By assumption, the vertices u and v lie on a cycle C. However, since $C - x$ is connected, there is a $u - v$ path in G that does not contain the vertex x, a contradiction. Hence G contains no cut-vertex.

($\Longrightarrow$) Suppose now that G has no cut-vertices and, to the contrary, that there is at least one pair of vertices that do not lie on a common cycle. Amongst all such pairs of vertices, choose u and v such that $d(u, v)$ is a minimum. By Lemma 12, no edge incident with u is a bridge. Hence, by Theorem 11, every edge incident with u lies on a cycle, i.e., every neighbour of u lies on a cycle containing u. Hence $d(u, v) \geq 2$. Let P be a $u - v$ geodesic and v' the vertex of P that is adjacent to v (since $d(u, v) \geq 2$, we are certain that $u \neq v'$). Since $d(u, v') < d(u, v)$, the vertex v' lies on a cycle C containing u. Furthermore, since v' is not a cut-vertex, there is a $v - u$ path P' that does not contain v'. Let t be the first vertex of P' that is on C. Let C' be the $t - v'$ path in C that contains u. By following P' from v to t, then C' from t through u to v', then using the edge between v' and v, we obtain a cycle containing v. This is a contradiction. Hence no such pair u, v of vertices exists and the result is proved. □

Let u and v be vertices in a connected graph G. Two $u - v$ paths P_1 and P_2 are called *internally disjoint* if $V(P_1) \cap V(P_2) = \{u, v\}$. Theorem 13 implies the following.

Corollary 14. *A graph G of order at least 3 is a block if and only if for every pair u, v of distinct vertices of G, there are (at least) two internally disjoint $u - v$ paths in G.*

The following result was proved by Frank Harary and Robert Norman in 1953 [HN53].

Theorem 15. *The center of a connected graph lies in a single block.*

Proof. Suppose, to the contrary, that there is a connected graph G and central vertices c_1, c_2 of G such that c_1 and c_2 do not lie in the same block. Let P be a $c_1 - c_2$ path. Then P is not contained in a single block. Consequently, P contains a vertex x which belongs to two different blocks, and hence x is a cut-vertex. Let G_1 and G_2 be the components of $G - x$ that contain, respectively, c_1 and c_2. Let x' be an eccentric vertex[2] of x. Since x is a cut-vertex, there is at least one component, say G_1, of $G - x$ that contains no vertices from any $x - x'$ path. This implies that $d(c_1, x') = d(c_1, x) + d(x, x') > d(x, x')$. This contradicts the fact that c_1 is a central vertex, and hence no such graph G exists. □

[2]If $d(x, x') = e(x)$, then x' is called an *eccentric vertex* of x.

Exercises

3.1 Prove that if G is a regular bipartite graph with partite sets V_1 and V_2, then $|V_1| = |V_2|$.

3.2 Prove or disprove: For every positive integer t, the hypercube Q_t is bipartite.

3.3 Let G be a graph of order at least 5. Prove that at most one of G and $\overline{G}$ is bipartite.

3.4 Prove that a graph G is connected if and only if for every partition $\{V_1, V_2\}$ of $V(G)$, there is an edge of G joining a vertex of V_1 to a vertex of V_2.

3.5 Prove that every nontrivial connected graph contains at least two vertices that are not cut-vertices.

3.6 Prove that if v is a cut-vertex of a connected graph G, then v is not a cut-vertex of $\overline{G}$.

3.7 Let G be a connected graph. Prove that a vertex v of G is a cut-vertex if and only if there are neighbours u and w of v such that v is on every $u - w$ path.

3.8 Let G be a connected graph in which every vertex has even degree. Prove that G does not contain a bridge.

3.9 Prove Lemma 12.

3.10 An *end-block* in a graph G is a block that contains exactly one cut-vertex of G. Prove that if G is separable, then G has at least two end-blocks.

3.11 Let G be a graph of order $n \geq 3$ in which, for every pair u, v of nonadjacent vertices, we have $\deg u + \deg v \geq n$. Prove that G is a block.

3.12
a. Prove or disprove: A graph G of order 3 or more is connected if and only if G contains two distinct vertices u and v such that $G - u$ and $G - v$ are connected.
b. Prove or disprove: Every connected graph of order 4 or more contains three distinct vertices u, v, and w such that $G - u$, $G - v$, and $G - w$ are connected.

3.13 What is the maximum number of cutvertices in a graph of order n?

3.14 Let G be a cubic graph. Prove that G has a cutvertex if and only if G has a bridge.

3.15 Prove that every cubic graph with a bridge has order at least 10, and prove that this result is *sharp*, i.e., find a cubic graph of order 10 containing a bridge.

3.16 Prove that if G is a graph of order $n \geq 3$ with $\delta(G) \geq n/2$, then G is a block.

3.17 Prove or disprove: If G is a connected graph with cutvertices and u and v are vertices of G such that $d(u, v) = \operatorname{diam} G$, then no block of G contains both u and v.

3.18 Let v be a vertex of a connected graph G. Prove that v is a cut-vertex of G if and only if there exists a partition $\{V_1, V_2\}$ of $V(G) - \{v\}$ such that for every pair v_1, v_2 of vertices with $v_1 \in V_1$ and $v_2 \in V_2$, every $v_1 - v_2$ path contains v.

3.19 Let G be a connected graph of order at least 3. Prove that G is a block if and only if for every vertex $v \in V(G)$ and edge $e \in E(G)$, there is a cycle containing both v and e.

Chapter 4

Trees

A graph that contains no cycles is *acyclic*. Graphs that are acyclic are also called *forests*. A *tree* is a connected acyclic graph. Hence, a forest is a graph in which each component is a tree. Figure 4 shows all of the trees of order at most 5.

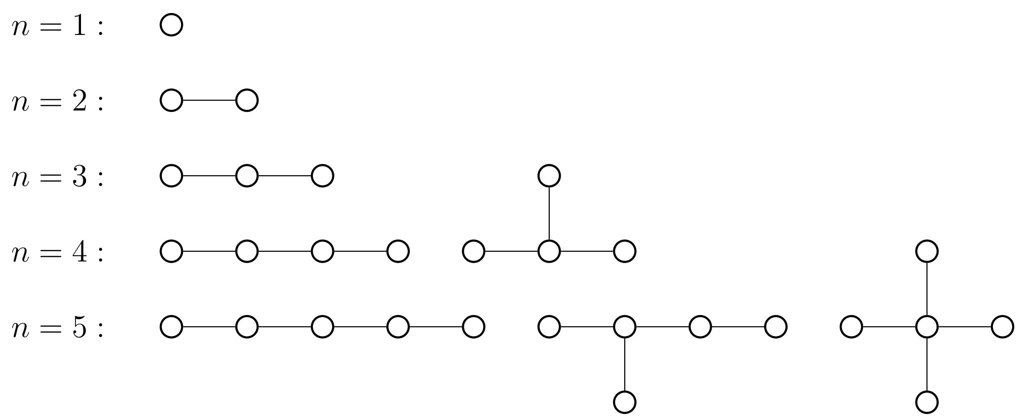


Figure 4.1: The trees of order $n \leq 5$.

Recall that a vertex of degree 1 is called an end-vertex.

Theorem 16. *Every nontrivial tree has at least two endvertices.*

Proof. Let T be a nontrivial tree and $P : v_1, v_2, \ldots, v_k$ a maximal path in T (i.e., a path that is not properly contained in another path). If $\deg v_1 \geq 2$, then there is a vertex x which is adjacent to v_1 but, since T is acyclic, not on P. By starting at x, using the edge xv_1 to move to v_1, and then the path P, we obtain a path that properly contains P, which is a contradiction. Hence, $\deg v_1 = 1$. Similarly, $\deg v_k = 1$. $\square$

Every block of order at least 3 contains a cycle. Since a tree is acyclic, every block in a nontrivial tree is isomorphic to K_2. The next result therefore follows in a straightforward way from Theorem 15.

Theorem 17. *The center of a tree is isomorphic to K_1 or K_2.*

A tree whose center is (isomorphic to) K_1 is sometimes called a *central tree*, while a tree whose center is (isomorphic to) K_2 is sometimes called a *bicentral tree*.

There are a number of well-known characterizations of trees.

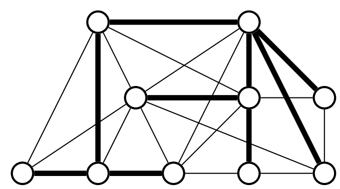
Theorem 18. *Every tree of order n has size $n-1$.*

Proof. Our proof will be by induction on n. There is only one tree of order 1, and that tree (K_1) has size 0, so when $n = 1$ the statement is true. Suppose now that $n > 1$ and that the statement is true for every tree of order less than n. Let T be a tree of order n and size m. By Theorem 16, T has an end-vertex v. No end-vertex is a cut-vertex, so $T - v$ is a tree of order $n(T - v) = n - 1$ and size $m(T - v) = m - 1$. By the Inductive Hypothesis, $m(T - v) = n(T - v) - 1$. Hence $m - 1 = n - 1 - 1$, which implies that $m = n - 1$. □

Corollary 19. *A forest of order n with k components has size $n - k$.*

If G is a connected graph, then a *spanning tree* of G is a spanning subgraph of G that is a tree.

Example 4.1



In the connected graph shown above, the edges of a spanning tree are indicated as darker lines.

Theorem 20. *Every connected graph has a spanning tree.*

Theorem 21. *Let G be a graph of order n and size m. If any two of the following three statements are true, then G is a tree:*

(i). *G is connected.*

(ii). *G is acyclic.*

(iii). $m = n - 1$.

Proof. ((i) and (ii)) If G is connected and acyclic, then by definition G is a tree.

((i) and (iii)) Suppose that G is connected and that $m = n - 1$. By Theorem 20, the graph G has a spanning tree T. By Theorem 18, the tree T has size $n - 1$, the same as G. Since T is a subgraph of G, we must therefore have $G = T$, i.e., G is a tree.

((ii) and (iii)) Since G is acyclic, G is a forest. Let k be the number of components of G. Then, by Corollary 19, $k = 1$, i.e., G is connected. Hence, G is a tree. □

Theorem 22. *If T is a tree of order k and G is a graph with $\delta(G) \geq k-1$, then T is a subgraph of G.*

Proof. Our proof will be by induction on k. Certainly the result is true when $k = 1$. Suppose now that $k > 1$ and that the result is true for trees of order less than k, i.e., if T is a tree with $n(T) < k$ and G is a graph with $\delta(G) \geq n(T) - 1$, then T is a subgraph of G. Let T be a tree of order k and G a graph with $\delta(G) \geq k - 1$. Let v be an end-vertex of T and v' the neighbour of v. Then $T - v$ is a tree of order less than k. Since $\delta(G) \geq k - 1 > n(T - v) - 1$, by the Inductive Hypothesis, the tree $T - v$ is a subgraph of G. In the graph G, the vertex v' has degree at least $\delta(G) \geq k - 1$. However, in the tree $T - v$, the vertex v' has degree at most $k - 2$. Hence there is a vertex x of G to which v' is adjacent in G but not adjacent in $T - v$. Then by adding the vertex x and the edge xv' to $T - v$ we obtain a tree that is isomorphic to T and contained in G. $\square$

Exercises

4.1 Draw all trees of order 6.

4.2 Prove that if every vertex of a graph G has degree at least 2, then G contains a cycle.

4.3 Prove or disprove: If G is a graph of order n and size m with three cycles, then $m \geq n+2$.

4.4 Find a graph of order n and size $n-1$ that is not a tree.

4.5 Prove that if G is a connected graph of order n and size m, then $m \geq n-1$.

4.6 Suppose that T is a tree of order $n \geq 4$ that contains only vertices of degree 1 and 3. Prove that T contains $(n-2)/2$ vertices of degree 3.

4.7 Prove Corollary 19.

4.8 Prove Theorem 20.

4.9 Prove that the following statements are equivalent for an (n, m) graph G:

a. G is a tree.
b. Every two vertices of G are joined by a unique path.
c. G is connected and $m = n-1$.
d. G is acyclic and $m = n-1$.
e. G is acyclic and for every pair u, v of distinct nonadjacent vertices of G, the graph $G + uv$ has exactly one cycle.

4.10 A graph that is connected and has exactly one cycle is called *unicyclic*. Prove that the following four statements are equivalent:

a. G is unicyclic.
b. G is connected and $m = n$.
c. For some edge e of G, the graph $G - e$ is a tree.
d. G is connected and the set of edges that are not bridges form a cycle.

4.11 Prove that a tree of order at least 3 has diameter 2 if and only if it is a star.

4.12 Prove or disprove:

a. If G has diameter 2, then it has a spanning star.
b. If G has a spanning star, then it has diameter 2.

4.13 Prove that G is a forest if and only if every induced subgraph of G contains a vertex of degree at most 1.

4.14 Prove Theorem 17.

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