Worked Examples Set 4

Q.1. Dispersion [Griffiths, Problem 9.22]

Shallow water is nondispersive; the waves travel at a speed that is proportional to the square root of the depth. In deep water, however, the waves can't "feel" all the way down to the bottom – they behave as though the depth were proportional to λ. [i.e. $v \propto \sqrt{d}$ and $d \propto \lambda$, so write $v = \alpha\sqrt{\lambda}$] Show that the wave velocity of deep water waves is *twice* the group velocity.

(b) In quantum mechanics, a free particle of mass m traveling in the x-direction is described by the wave function

$$\Psi(x,t) = Ae^{i(px-Et)/\hbar}$$

where p is the momentum, and $E = p^2/2m$ is the kinetic energy. Calculate the group velocity and the wave velocity. Which one corresponds to the classical speed of the particle? Note that the wave velocity is *half* the group velocity.

Q.1. *Solution* (a)

We have $v = \alpha\sqrt{\lambda}$, α constant. But $\lambda = 2\pi/k$ and $v = \omega/k$

so $\omega = \alpha k\sqrt{2\pi/k} = \alpha\sqrt{2\pi k} = \alpha\sqrt{2\pi}\, k^{1/2}$ *dispersion relation*

Group vel. $v_g = \frac{d\omega}{dk} = \alpha\sqrt{2\pi}\ \frac{1}{2}k^{-1/2} = \frac{1}{2}\alpha\sqrt{\frac{2\pi}{k}} = \frac{1}{2}\alpha\sqrt{\lambda} = \frac{1}{2}v$

i.e. Wave vel. $v = 2v_g$

(b) The wave function is given as $Ae^{i(px-Et)/\hbar}$ but it should be of the form $Ae^{i(kx-\omega t)}$ so we have $\left(\frac{p}{\hbar}x - \frac{E}{\hbar}t\right) = (kx - \omega t)$ where the x and t factors must equate separately. Then

$k = \frac{p}{\hbar}$ and $\omega = \frac{E}{\hbar} = \frac{p^2}{2m\hbar}$ since $E = \frac{p^2}{2m}$ and substituting for p,

$\omega = \frac{k^2\hbar}{2m}$ *dispersion relation* . Then $v = \frac{\omega}{k} = \frac{E}{p} = \frac{p}{2m} = \frac{\hbar k}{2m}$ and

$v_g = \frac{d\omega}{dk} = \frac{2k\hbar}{2m} = \frac{\hbar k}{m} = \frac{p}{m}$. We see $v = \frac{v_g}{2}$. Now $p = mv_c$ where v_c is the classical velocity, so this must be v_g

Q.2. Polarization [Hecht 8.1]

Describe the state of polarization of

(a) $\vec{E} = \hat{x}\, E_0 \cos(kz - \omega t) - \hat{y}\, E_0 \cos(kz - \omega t)$

(b) $\vec{E} = \hat{x}\, E_0 \sin(kz - \omega t) + \hat{y}\, E_0 \cos(kz - \omega t - \pi/4)$

Solution: (a) $-\cos\theta = \cos(\theta + \pi)$ so rewrite as

$\vec{E} = \hat{x}\, E_0 \cos(kz - \omega t) + \hat{y}\, E_0 \cos(kz - \omega t + \pi)$

Equal amplitudes $(E_x = E_y)$ and E_y lags E_x by π .

This corresponds to linear polarization with negative slope, i.e. $\mathcal{P}$-state at $135°$ or $-45°$ since amplitudes are equal.

(b) $\vec{E} = \hat{x}\, E_0 \sin(kz - \omega t) + \hat{y}\, E_0 \cos(kz - \omega t - \pi/4)$

Again equal amplitudes and E_y leads E_x by $\pi/4$.

This is equivalent to E_x leading E_y by $7\pi/4$.

This is an ellipse tilted at $+45°$ (since amplitudes are equal) and is right handed since $\delta = 7\pi/4 > \pi$.

Q.3. Polarization [Hecht 8.4]

Write an expression for a $\mathcal{P}$-state EM wave with angular frequency ω and amplitude E_0 propagating along the x-axis with its plane of vibration at $25°$ to the xy-plane. The disturbance is zero at $t = 0$ and $x = 0$.

Solution: The y and z amplitudes are $E_{0y} = E_0 \cos 25°$ and $E_{0z} = E_0 \sin 25°$ ($\vec{E}$ is at $25°$ to the y-axis).
From the last sentence we require $\cos(kx - \omega t + \delta) = 0$ when $x = 0$ and $t = 0$. This means $\delta = \pi/2$.
So the wave function for this linearly polarized wave is
$\vec{E}(x, t) = (0.91\hat{y} + 0.42\,\hat{z})\, E_0 \cos(kx - \omega t + \pi/2)$

Q.4. Quartz is a suitable material for making retarders, as it is birefringent with $n_{\text{fast}} = 1.5440$ and $n_{\text{slow}} = 1.5530$. In order to make a half-wave plate from quartz for use with hydrogen Hα (wavelength 656 nm in vacuum), what must the minimum thickness be?

Solution: For a half-wave plate we require a phase difference of π corresponding to $\lambda/2$. For thickness d,

$$\delta = d(k_{\text{slow}} - k_{\text{fast}}) = d\left(\frac{2\pi}{\lambda_{\text{slow}}} - \frac{2\pi}{\lambda_{\text{fast}}}\right)$$

and wavelength in material $\lambda = \lambda_0/n$ so

$$\delta = d\left(\frac{2\pi n_{\text{slow}}}{\lambda_0} - \frac{2\pi n_{\text{fast}}}{\lambda_0}\right) = \frac{2\pi d}{\lambda_0}(n_{\text{slow}} - n_{\text{fast}})$$

$$d = \frac{\lambda_0 \delta}{2\pi(n_{\text{slow}} - n_{\text{fast}})} = \frac{656\times10^{-9}\times\pi}{2\pi(1.553-1.544)} = 3.6 \times 10^{-5}\ \text{m}$$