Worked Examples Set 3

Q.1. EM Waves in Non-Conductors

Determine the (real) electric and magnetic fields for a plane EM wave of amplitude $5.0\ \mathrm{V/m}$, frequency $5.0\ \mathrm{MHz}$, (and phase angle zero) that is travelling in the $+z$-direction and polarized in the $+y$-direction, propagating in water ($n = 1.333$).

Solution: As before, the fields are of the form

$$\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)} \quad ; \quad \vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$$

given $\vec{E}_0 = (5.0\ \mathrm{V/m})\hat{y}$ [polarization is direction of $\vec{E}$]

$f = 5.0\ \mathrm{MHz}$, so $\omega = 2\pi f = 10\pi \times 10^6\ \mathrm{s}^{-1}$

$n = 1.33$ so $\omega/k = v = c/n = \frac{3.0\times10^8}{1.333} = 2.5 \times 10^8\ \mathrm{m/s}$

and then $k = \omega/v = \frac{10\pi\times10^6}{2.5\times10^8} = 0.126\ \mathrm{m}^{-1}$ ($\lambda = 50\ \mathrm{m}$)

Q.1. *Solution* [continued]

Now $\vec{B} = (\hat{k} \times \vec{E})/v$ with $\vec{E}_0 = (5.0\ \mathrm{V/m})\hat{y}$ and $\hat{k} = \hat{z}$

So $\vec{B}_0 = \left(\frac{E_0}{v}\right)(\hat{z} \times \hat{y}) = \left(\frac{5.0}{2.5\times10^8}\right)(-\hat{x}) = (20\ \mathrm{nT})(-\hat{x})$

The fields can then be written

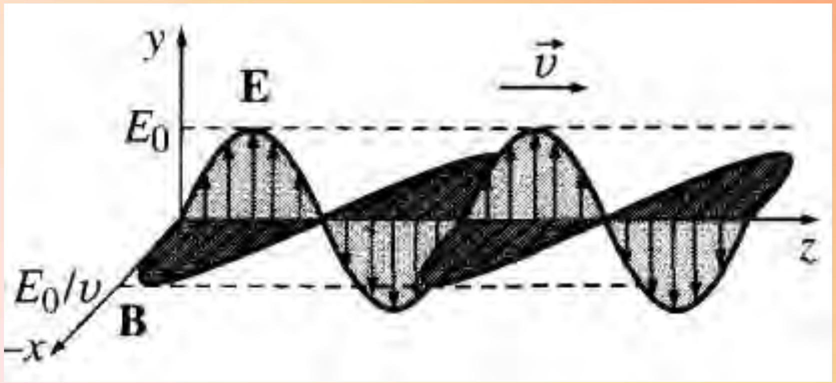
$$\vec{E}(z,t) = (5.0\ \mathrm{V/m})\hat{y}\exp[(0.126\ \mathrm{m}^{-1})z - (10\pi\ (\mu\mathrm{s})^{-1})t]$$

$$\vec{B}(z,t) = (20\ \mathrm{nT})(-\hat{x})\exp[(0.126\ \mathrm{m}^{-1})z - (10\pi\ (\mu\mathrm{s})^{-1})t]$$

with the real fields being

$$\vec{E}(z,t) = (5.0\ \mathrm{V/m})\hat{y}\cos[(0.126\ \mathrm{m}^{-1})z - (10\pi\ (\mu\mathrm{s})^{-1})t]$$

$$\vec{B}(z,t) = (20\ \mathrm{nT})(-\hat{x})\cos[(0.126\ \mathrm{m}^{-1})z - (10\pi\ (\mu\mathrm{s})^{-1})t]$$



Q.2. Reflection and Transmission Coefficients [Hecht 4.21]

A light wave that is linearly polarized in the plane of incidence is incident at $30°$ on a crown glass ($n = 1.52$) plate in air. Compute (a) the amplitude reflection and transmission coefficients; (b) the (intensity) refelection and transmission coefficients R and T. Compare your results with the graphs on ED-08 slides 17 and 18 respectively. [Assume $\mu_{\text{glass}} = \mu_0$]

Solution: (a) We first need θ_T from Snell's law:

$$\theta_T = \sin^{-1}[(1.0/1.52)\sin 30°] = 19.2°$$

Now if $\mu_2 = \mu_1 = \mu_0$, then $\beta = n_2/n_1 = 1.52$ here.

And $\alpha = \cos\theta_T/\cos\theta_I = \cos 19.2°/\cos 30° = 1.09$

Then $r_\parallel = \frac{E_{0R}}{E_{0I}} = \frac{\alpha-\beta}{\alpha+\beta} = \frac{1.09-1.52}{1.09+1.52} = \frac{-0.43}{2.61} = -0.165$

and $t_\parallel = \frac{E_{0T}}{E_{0I}} = \frac{2}{\alpha+\beta} = \frac{2}{1.09+1.52} = \frac{2}{2.61} = 0.766$

The values look close to those on the graph (for $n = 1.50$).

Q.2. *Solution* [continued]

(b) $$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = (-0.165)^2 = 0.027$$

$$T = \frac{I_T}{I_I} = \left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{v_2\varepsilon_2}{v_1\varepsilon_1}\frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

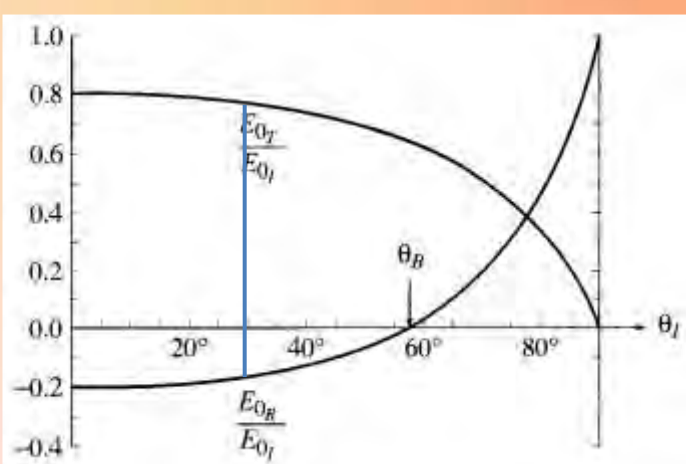
$$= 1.09 \times 1.52 \times (0.766)^2 = 0.972$$

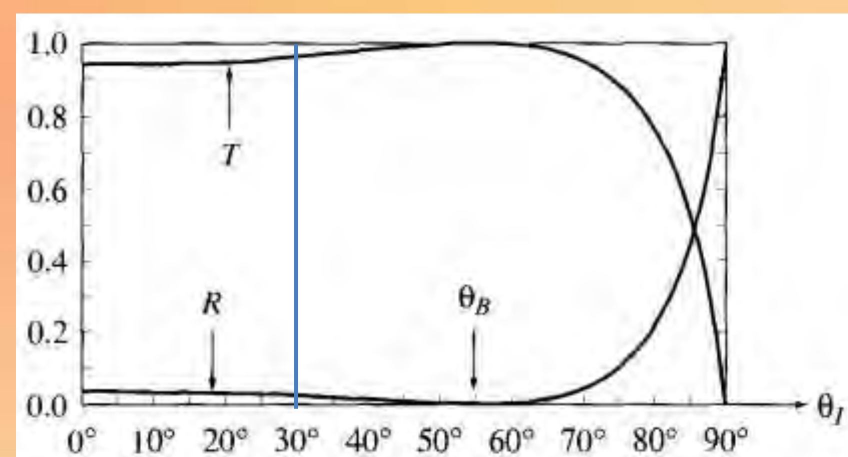
[check: $R + T = 0.999$: close enough to 1.]

It's hard to tell, but they seem to agree with the graph.

(c) What is Brewster's angle for this glass?

Solution: $\theta_B = \tan^{-1}(n_2/n_1) = \tan^{-1}(1.52) = 56.7°$



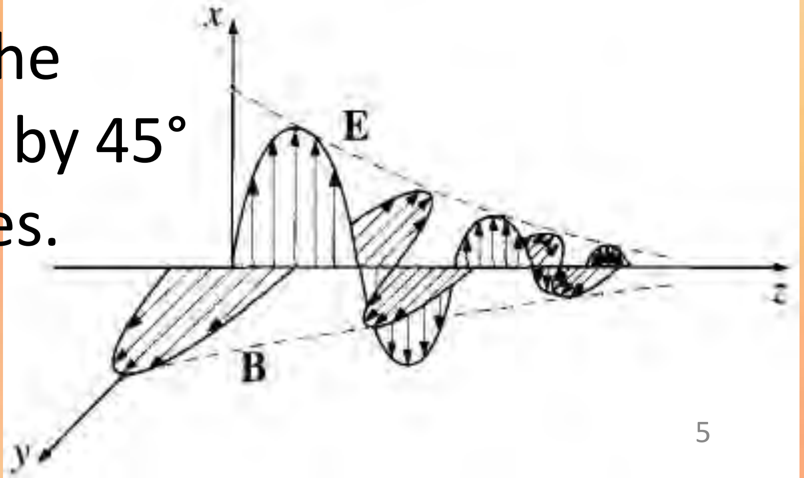


Q.3. EM Waves in Conductors [Griffiths Problem 9.19]

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\varepsilon$) is $(2/\sigma)\sqrt{\varepsilon/\mu}$ (independent of frequency). Find the skin depth (in metres) for (pure) water. [$\varepsilon_r = 80.1$; $\mu_r = 1 - 9.0 \times 10^{-6}$ (i.e. take $\mu_r = 1$) ; $\sigma = 4.0 \times 10^{-6}\ (\Omega.\mathrm{m})^{-1}$]

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\varepsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nm) for a typical metal ($\sigma \sim 10^7\ (\Omega.\mathrm{m})^{-1}$) in the visible range ($\omega \approx 10^{15}$ Hz), assuming $\varepsilon \approx \varepsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

(c) Show that in a good conductor the magnetic field lags the electric field by 45° and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).



Q.3. *Solution* (a) Use the binomial expansion for the square root in the κ eqn. $\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$: $[\sigma \ll \omega\varepsilon]$

$$\kappa \cong \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2-1\right]^{1/2} = \omega\sqrt{\frac{\varepsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\varepsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

Then $d = \frac{1}{\kappa} = \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$ Q.E.D.

For water, inserting the given values,

$$d = \frac{2}{4.0\times 10^{-6}}\sqrt{\frac{80.1\times 8.85\times 10^{-12}}{4\pi\times 10^{-7}}} = 1.19\times 10^{4}\ \mathrm{m}$$

(b) Now $(\sigma/\varepsilon\omega)^2$ dominates [is $\gg 1$], so we can see from the eqns. that $k=\kappa$ and $\lambda = \frac{2\pi}{k} = \frac{2\pi}{\kappa} = 2\pi d$ or $d = \frac{\lambda}{2\pi}$ Q.E.D.

Q.3. *Solution* (b) [continued]

For the 'typical metal', inserting the given values,

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma\omega}{2}} = \sqrt{\frac{(4\pi\times10^{-7})(10^{15})(10^{7})}{2}}$$

$= 8 \times 10^7\ \mathrm{m}^{-1}$ so $d = 1/\kappa = 1.3 \times 10^{-8}\ \mathrm{m} = 13\ \mathrm{nm}$

Metals are opaque because the fields do not penetrate far.

(c) In (b) we had $k = \kappa$, so $\phi = \tan^{-1}(\kappa/k) = \tan^{-1} 1 = 45°$

Ratio $\dfrac{B_0}{E_0} = \dfrac{K}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\dfrac{\sigma}{\varepsilon\omega}\right)^2}} \cong \sqrt{\varepsilon\mu\dfrac{\sigma}{\varepsilon\omega}} = \sqrt{\dfrac{\mu\sigma}{\omega}}$

For 'typical metal' $\dfrac{B_0}{E_0} = \sqrt{\dfrac{(4\pi\times10^{-7})(10^{7})}{10^{15}}} = 1.1\times10^{-7}\ \mathrm{s/m}$

In vacuum, $\dfrac{B_0}{E_0} = \dfrac{1}{c} = 3.3\times10^{-9}\ \mathrm{s/m}$ so $300\times$ larger in metal