# Worked Examples Set 2

**Q.1**. Application of Maxwell's eqns. [Griffiths Problem 7.42]

In a perfect conductor the conductivity $\sigma$ is infinite, so from 'Ohm's law' $\vec{J} = \sigma\vec{E}$ , $\vec{E} = 0$ . Any net charge must be on the surface (as in electrostatics for an imperfect conductor).

(a) Show that the magnetic field is constant inside a perfect conductor.

*Solution*: Faraday's law: $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ so if $\vec{E} = 0$ then $\partial\vec{B}/\partial t = 0$ , i.e. $\vec{B}$ is independent of $t$, i.e. constant.

(b) Show that the magnetic flux through a perfectly conducting loop is constant.

*Solution*: Faraday's law in integral form is $\oint_c \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}$

**Q.1**. *Solution (b)* [continued]

Inside the perfectly conducting loop $\vec{E} = 0$ , so $\frac{d\Phi_m}{dt} = 0$

i.e. Magnetic flux $\Phi_m$ is independent of $t$, i.e. constant.

(c) A superconductor is a perfect conductor with the additional property that the constant $\vec{B}$-field inside is actually zero.

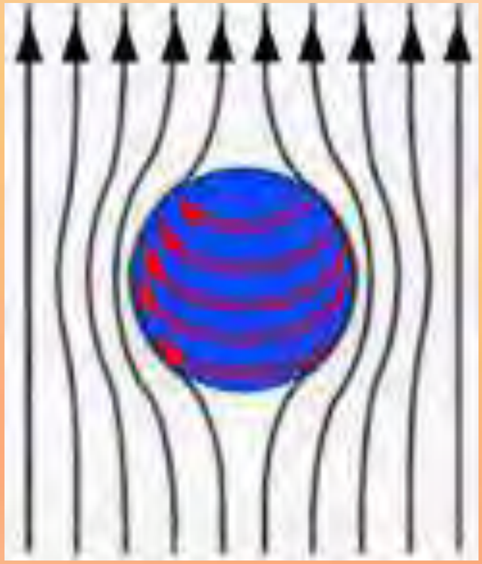
Show that the current in a superconductor is confined to the surface (there are no volume currents).

*Solution*: Maxwell-Ampere law:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0\, \partial \vec{E} / \partial t$$

so if $\vec{E} = 0$ and $\vec{B} = 0$ (given) then $\vec{J} = 0$

i.e. there are no volume currents and so any currents must be surface currents ($\vec{K}$).



**Q.2.** Potentials and Gauge Transformations

[Griffiths Problems 10.3 and 10.5]

(a) Find the fields, and the charge and current distributions, corresponding to potentials $V(\vec{r},t) = 0$ , $\vec{A}(\vec{r},t) = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r}$

(b) Use the gauge function $\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r}$ to transform the potentials in (a), and comment on the results.

*Solution*: (a) $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = 0 + \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$

$\vec{B} = \nabla \times \vec{A} = 0$ [$\vec{A} = A\hat{r}$ (radial component only) which does not depend on $\theta$ or $\phi$, i.e. $\partial A_r/\partial\theta = 0$ , $\partial A_r/\partial\phi = 0$]

$$\nabla\times\mathbf{A} = \left(\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] , \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right] , \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial A_\theta}\right]\right)$$

Fields of a stationary point charge, strange potentials.

**Q.2.** *Solution* [continued]

Charge distribution: we have a point charge $q$ , which can be written as a (3D) delta function : $\rho = q\delta^3(\vec{r})$ ;

current density $\vec{J} = 0$ (since $\vec{B} = 0$)

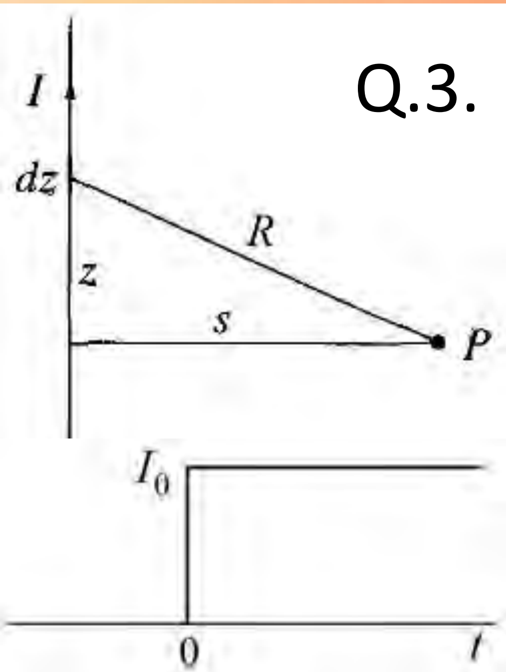
(b) Gauge transformation: $\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r}$

$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right) = \frac{q}{4\pi\varepsilon_0 r}$$

$$\vec{A}' = \vec{A} + \nabla\lambda = -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r} + \left(-\frac{1}{4\pi\varepsilon_0}qt\right)\left(-\frac{1}{r^2}\hat{r}\right)$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r} + \frac{1}{4\pi\varepsilon_0}\frac{qt}{r^2}\hat{r} = 0$$

So this transformation transforms the strange potentials of (a) into the more standard potentials of a static point charge.




# Q.3. Retarded Potentials

[Griffiths, Example 10.2]

A long straight wire has $I(t) = 0\,,\ \ t \leq 0$
and $I(t) = I_0\,,\ \ t > 0$ ('switch on' at $t = 0$).
Find the resulting $\vec{E}$ and $\vec{B}$ fields.

*Solution*: Assume wire is neutral, so $V = 0$.

Retarded vector potential at $P$ (see figure) is

$\vec{A}(s,t) = \frac{\mu_0}{4\pi}\hat{z}\int_{-\infty}^{+\infty}\frac{I(t_r)}{R}dz$ [in same direction ($\hat{z}$) as current $I$]

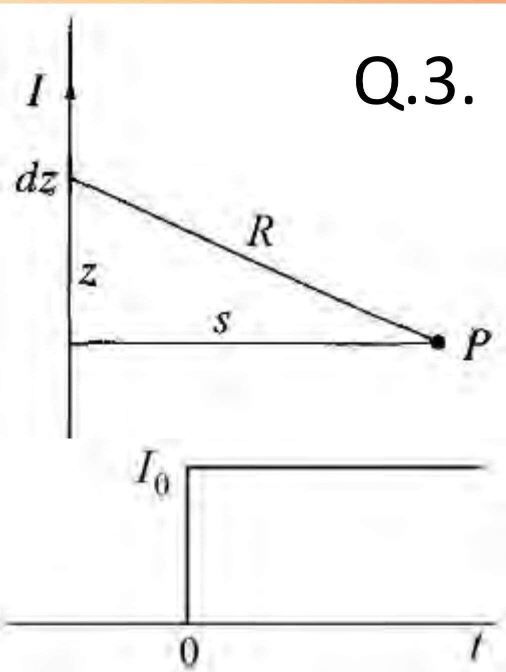
For the field to reach $P$ takes time $s/c$, so for $t < s/c\,,\ \vec{A} = 0$

For $t > s/c$ the 'signal' takes time $R/c$ to travel along $R$, so range of $z$ contributing to $\vec{A}(s,t)$ at $P$ is $|z| \leq \sqrt{(ct)^2 - s^2}$

Outside this range $t_r = t - R/c < 0$ so $I(t_r) = 0$

[Note the $|z| \Rightarrow$ contributions from above & below the $\perp$ point]

Q.3. *Solutions* [continued]




$$\Rightarrow \vec{A}(s,t) = \left(\frac{\mu_0 I_0}{4\pi}\hat{z}\right) 2\int_0^{\sqrt{(ct)^2-s^2}} \frac{dz}{\sqrt{s^2+z^2}}$$

$$= \left(\frac{\mu_0 I_0}{2\pi}\hat{z}\right)\left[\ln\left(\sqrt{s^2+z^2}+z\right)\right]_0^{\sqrt{(ct)^2-s^2}}$$

$$= \left(\frac{\mu_0 I_0}{2\pi}\hat{z}\right)\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)$$

The derivatives of $\vec{A}$ $(= A\hat{z})$ give the fields:

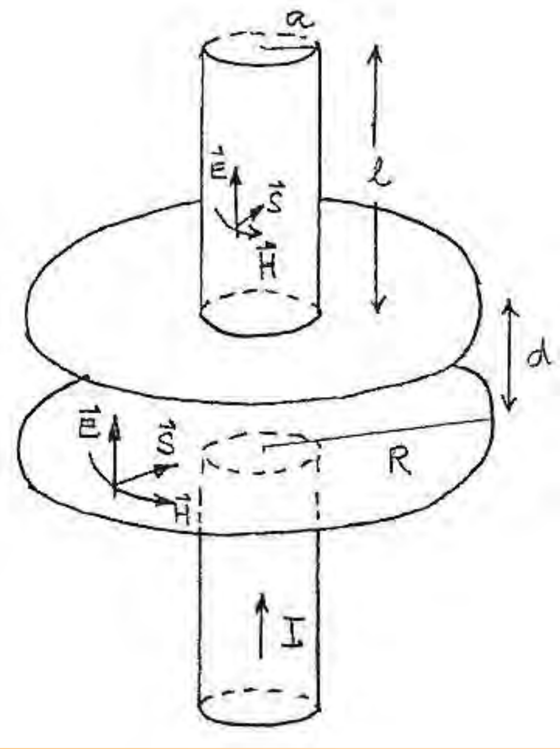
$$\vec{E}(s,t) = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\hat{z}$$

$$\vec{B}(s,t) = \nabla \times \vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2-s^2}}\hat{\phi}$$

Note that as $t \to \infty$ , $\vec{E} \to 0$ , $\vec{B} \to \dfrac{\mu_0 I_0}{2\pi s}\hat{\phi}$ (the static fields)

# Q.4. Poynting Vector

[*Feynman Lectures in Physics*, Vol. II, 27.5]
A capacitor with circular plates, radius $R$, separation $d$, is being charged. Find the rate at which its stored energy is increasing and relate this to the Poynting vector flux.




*Solution*: We assume the field between the plates is uniform. The energy stored is

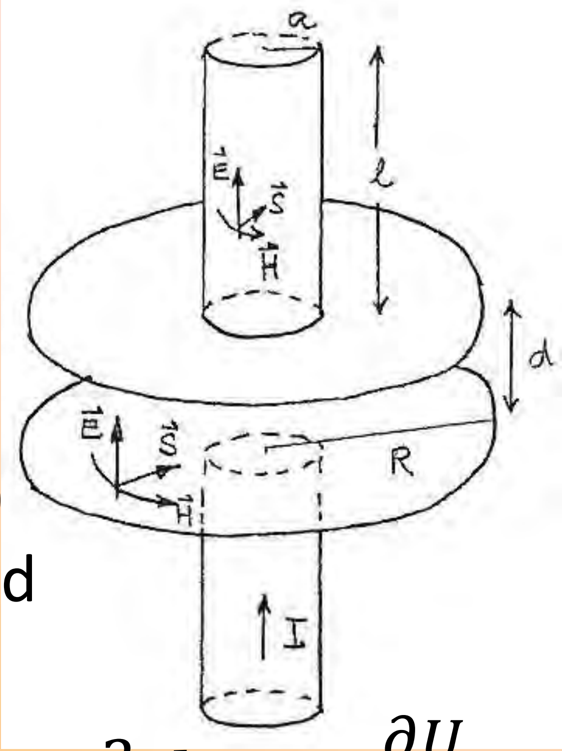
$U = \int_{\mathcal{V}} \frac{1}{2}\varepsilon_0 E^2\, d\mathcal{V} = \frac{1}{2}\varepsilon_0 E^2\, \pi R^2 d$ and this increases at the rate $\frac{\partial U}{\partial t} = \pi R^2 d \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2\right) = \pi R^2 d \varepsilon_0 E \frac{\partial E}{\partial t}$

Now in the capacitor space $\vec{J}_f = 0$ and Maxwell-Ampere $\rightarrow$

$\oint \vec{H}.d\vec{l} = \int \frac{\partial \vec{D}}{\partial t}.d\vec{a}$ i.e. $H.2\pi R = \varepsilon_0 \frac{\partial E}{\partial t}.\pi R^2$ so $H = \frac{R}{2}\varepsilon_0 \frac{\partial E}{\partial t}$

[$\vec{H}$ is azimuthal, field lines around the cylindrical air space.]

Q.4. *Solution* [continued]




We have $\dfrac{\partial U}{\partial t} = \pi R^2 d\varepsilon_0 E\dfrac{\partial E}{\partial t}$, $H = \dfrac{R}{2}\varepsilon_0\dfrac{\partial E}{\partial t}$

Now the Poynting vector is $\vec{S} = \vec{E} \times \vec{H}$

and we see that it points inwards; this is the direction of energy flow! (note $\vec{E} \perp \vec{H}$ )

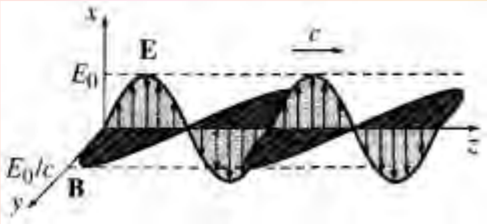
The Poynting vector flux, through the curved surface that defines the capacitor space , is

$$\oint \vec{S}.d\vec{a} = -EH.2\pi Rd = -\varepsilon_0 E\frac{\partial E}{\partial t}\pi R^2 d = -\frac{\partial U}{\partial t}$$

[we have to compute the integral over a closed surface, but the flux through the ends of the 'cylinder' (the plates) is zero , since $\vec{S}$ ∥ surface; the integral is negative because flux is **in**.]

This is just **Poynting's theorem** with $\vec{J}_f = 0$ : the rate at which energy flows in = the rate at which the (electric in this case) field energy is increasing. Energy flows in from outside!

## Q.5. EM Waves




[Griffiths, Problem 9.9 ]

Write down the (real) electric and magnetic fields for a monochromatic plane wave of amplitude $E_0$, frequency $\omega$, (and phase angle zero) that is: (a) travelling in the $-x$-direction and polarized in the $+z$-direction;

(b) travelling in the direction from the origin to the point (1,1,1), with polarization parallel to the $xz$-plane.

In each case, sketch the wave, and give the explicit Cartesian components of $\vec{k}$ and $\hat{n}$.

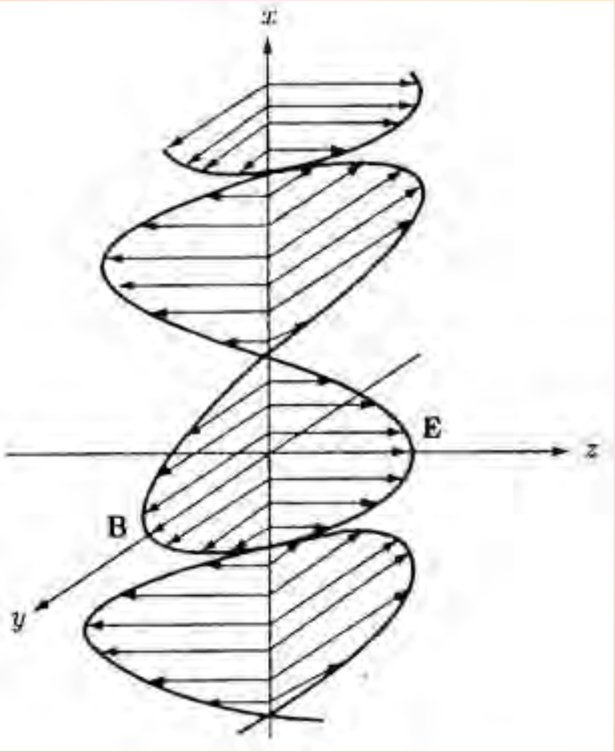
*Solution*: (a) The frequency is specified so we should give $k$ in terms of $\omega$ ; the wave direction is $-\hat{x}$ , so $\vec{k} = -(\omega/c)\,\hat{x}$

Then $\vec{k}.\vec{r} = -(\omega/c)\,\hat{x} \cdot (x\hat{x} + y\hat{y} + z\hat{z}) = -(\omega/c)\,x$

$\vec{E}$ is in the given direction of polarization, $\hat{z}$

$\vec{B}$ is in the direction $\hat{k} \times \hat{n} = (-\hat{x}) \times \hat{z} = \hat{y}$

Q.5. *Solution [continued]*




Thus the fields are

$\vec{E}(x,t) = E_0\hat{z}\, \cos\left[(\omega/c)x + \omega t\right]$

$\vec{B}(x,t) = (E_0/c)\hat{y}\, \cos\left[(\omega/c)x + \omega t\right]$

(b) The unit vector from the origin to the point $(1,1,1)$ is $(\hat{x}+\hat{y}+\hat{z})/\sqrt{3}$ , so

$\vec{k} = (\omega/c)\,(\hat{x}+\hat{y}+\hat{z})/\sqrt{3}$

Polarization direction $\hat{n}$ is ‖ $xz$-plane, so must have the form $\alpha\hat{x}+\beta\hat{z}$ ; $\hat{n}.\hat{k}=0$, so must have $\alpha=-\beta$;

$\hat{n}$ is a unit vector, so $\alpha = 1/\sqrt{2}$ and thus $\hat{n} = (\hat{x}-\hat{z})/\sqrt{2}$

This is the direction of $\vec{E}$ ; the direction of $\vec{B}$ is $\hat{k}\times\hat{n}$ :

$$\hat{k}\times\hat{n} = \frac{1}{\sqrt{6}}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 1 & 1 & 1\\ 1 & 0 & -1\end{vmatrix} = \frac{(-\hat{x}+2\hat{y}-\hat{z})}{\sqrt{6}}$$

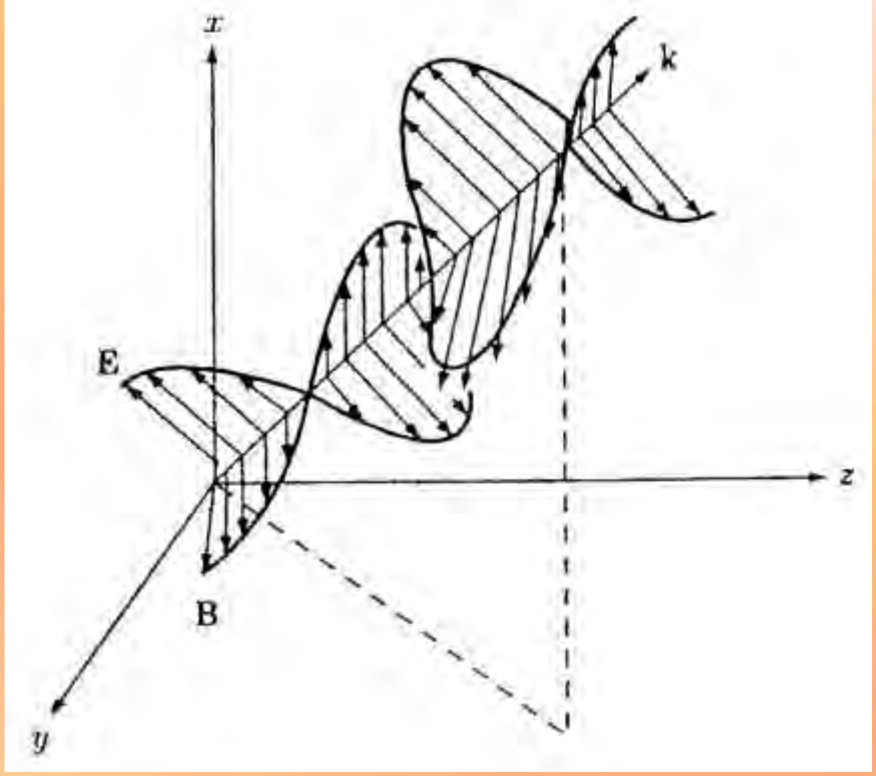
Q.5. *Solution [continued]*

Thus the fields are

$$\vec{E}(x,y,z,t) = E_0 \cos\left[\frac{\omega(x+y+z)}{\sqrt{3}c} - \omega t\right]\left(\frac{\hat{x}-\hat{z}}{\sqrt{2}}\right)$$

$$\vec{B}(x,t) = \frac{E_0}{c}\cos\left[\frac{\omega(x+y+z)}{\sqrt{3}c} - \omega t\right]\left(\frac{(-\hat{x}+2\hat{y}-\hat{z})}{\sqrt{6}}\right)$$

This is what this wave would look like; sometimes this might be quite difficult to draw!




[Part (b) is far more intricate than any question you would get in a test or exam.]

# Q.6. Energy in EM Waves

The "solar constant", the average intensity of radiation from the Sun at the top of the Earth's atmosphere, is $1.34\ \mathrm{kW/m^2}$. If this radiation was a linearly polarized monochromatic (single frequency) plane wave (propagating in free space, of course), find the amplitudes of $\vec{E}$ and $\vec{B}$ for this wave.

*Solution*: Intensity (in $\mathrm{W/m^2}$) is the (time averaged) Poynting vector: $I = \langle S \rangle = \frac{1}{2} c\ \varepsilon_0 {E_0}^2$ Inserting the values,

$$E_0 = \sqrt{\frac{2I}{c\varepsilon_0}} = \sqrt{\frac{2 \times 1.34 \times 10^3}{3.0 \times 10^8 \times 8.85 \times 10^{-12}}} = 1.0 \times 10^3\ \mathrm{V/m}$$

$E_0 = 1\ \mathrm{kV/m}$ ! The magnetic field is

$$B_0 = \frac{E_0}{c} = \frac{1.0 \times 10^3}{3.0 \times 10^8} = 3.3 \times 10^{-6}\ \mathrm{T} = 3.3\ \mu\mathrm{T}$$