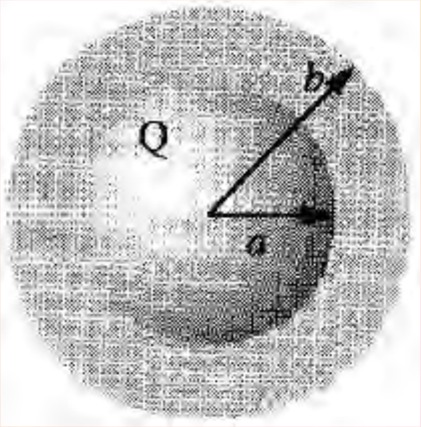
Worked Examples Set 1



Q.1. [Griffiths Example 4.5 and Problem 4.26]

A metal sphere of radius a carries a charge Q.

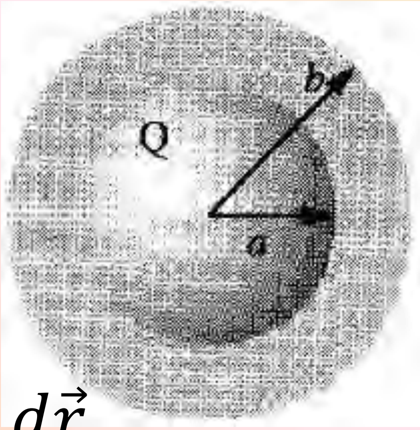
It is surrounded, out to radius b, by a linear dielectric with permittivity ε. (a) Find the potential at the centre. ($V = 0$ at ∞)

Solution: We know $\vec{E} = 0$ inside the sphere, and $\vec{E} = -\nabla V$ so V at the centre $=$ V at the surface of the sphere. This we can get by integrating $\vec{E}$ from ∞ (where $V = 0$) to $r = a$.

We should use Gauss's law, but for $\vec{E}$ we will need to include $\vec{P}$. We can use Gauss's law for free charges, since we put free charge Q on the sphere. This gives $\vec{D}$ (Gaussian sphere, rad. r):

$$\oint \vec{D} \cdot d\vec{a} = Q_{f\text{enc}} = Q \quad \Rightarrow \quad \vec{D} = \frac{Q}{4\pi r^2}\hat{r} \text{ for all } r > a$$

Q.1. *Solution* [continued]



Now $\vec{D} = \varepsilon\vec{E}$ in dielectric, $\vec{D} = \varepsilon_0\vec{E}$ in vacuum

$$\Rightarrow \; \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \;\; (r > b) \;, \; \frac{Q}{4\pi\varepsilon r^2}\hat{r} \;\; (a < r < b)$$

The potential at the centre is then $V = -\int_{\infty}^{0}\vec{E}\cdot d\vec{r}$

$$V = -\int_{\infty}^{b}\frac{Q}{4\pi\varepsilon_0 r^2}dr - \int_{b}^{a}\frac{Q}{4\pi\varepsilon r^2}dr - \int_{a}^{0} 0\,dr$$

$$= \frac{Q}{4\pi}\left\{\left[\frac{1}{\varepsilon_0 r}\right]_{\infty}^{b} + \left[\frac{1}{\varepsilon r}\right]_{b}^{a}\right\} = \frac{Q}{4\pi}\left\{\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right\}$$

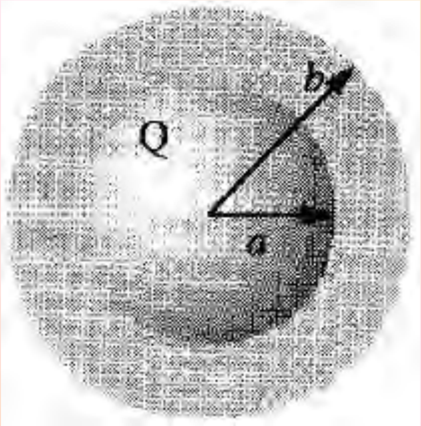
(b) Find the bound surface charge densities on the inside and outside surfaces of the dielectric.

Solution: The dielectric is linear, so $\vec{P} = \chi_e\varepsilon_0\vec{E} = \frac{\chi_e\varepsilon_0 Q}{4\pi\varepsilon r^2}\hat{r}$

This is constant so $\rho_b = -\nabla\cdot\vec{P} = 0$; there is only $\sigma_b = \vec{P}\cdot\hat{n}$

On the outer surface $r = b$ and $\hat{n} = \hat{r}$ so $\sigma_{b(\mathrm{out})} = \frac{\chi_e\varepsilon_0 Q}{4\pi\varepsilon b^2}$

Q.1. *Solution* [continued]



On the inner surface $r = a$ and $\hat{n} = -\hat{r}$ (out w.r.t. dielectric) so $\sigma_{b(\text{in})} = -\frac{\chi_e \varepsilon_0 Q}{4\pi\varepsilon a^2}$

[Can you easily check total bound charge = 0 ?]

(c) Find the energy stored in this system.

Solution: $U = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\mathcal{V} = \frac{1}{2}\int DEd\mathcal{V}$ with $\vec{D}$ and $\vec{E}$ from (a);

$D = \frac{Q}{4\pi r^2}$ (all r) , $E = \frac{Q}{4\pi\varepsilon r^2}$ $(a < r < b)$, $\frac{Q}{4\pi\varepsilon_0 r^2}$ $(r > b)$

Integrate over 'spherical shell' $4\pi r^2 dr$ from $a \to b \to \infty$:

$$U = \frac{1}{2}\left(\frac{Q}{4\pi}\right)^2 \left\{\int_a^b \frac{1}{r^2}\frac{1}{\varepsilon r^2} 4\pi r^2 dr + \int_b^\infty \frac{1}{r^2}\frac{1}{\varepsilon_0 r^2} 4\pi r^2 dr\right\}$$

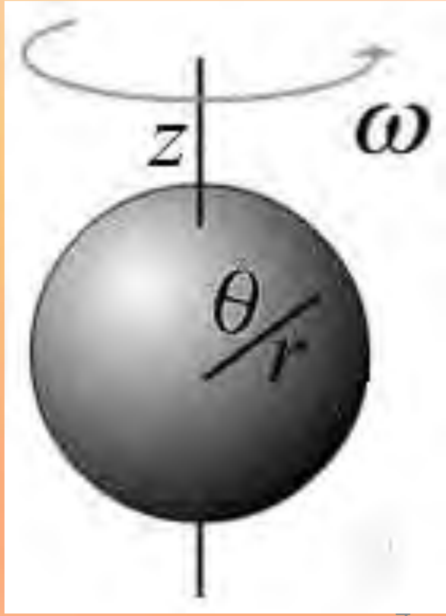
$$= \frac{Q^2}{8\pi}\left\{\frac{1}{\varepsilon}\left[\frac{-1}{r}\right]_a^b + \frac{1}{\varepsilon_0}\left[\frac{-1}{r}\right]_b^\infty\right\} = \frac{Q^2}{8\pi\varepsilon_0}\left\{\frac{1}{\varepsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{b}\right\}$$

$$= \frac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left\{\frac{1}{a} + \frac{(\varepsilon_r - 1)}{b}\right\}$$

Q.2. [Griffiths Problem 5.6]

(a) A phonograph record carries a uniform surface density of "static electricity" σ. If it rotates at constant angular velocity ω, what is the surface current density K (magnitude) at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\vec{J}$ at any point (r, θ, ϕ) within the sphere. [Hint: $\vec{J} = \rho\vec{v}$]



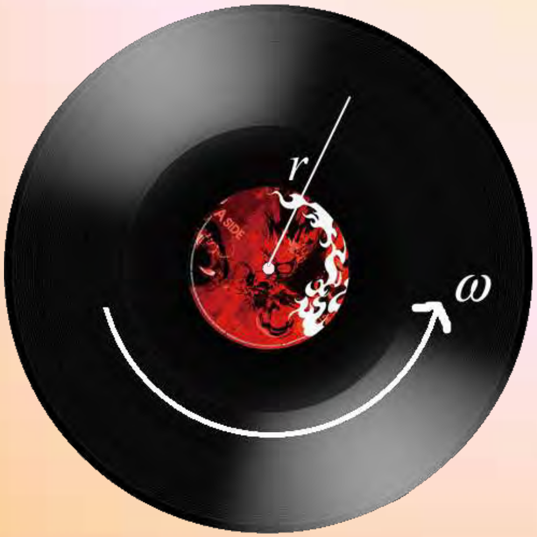
Q.2. *Solution*:

(a) (see ED-04-Magnetostatics, slide 2):

Surface current density $K = \sigma\, v$

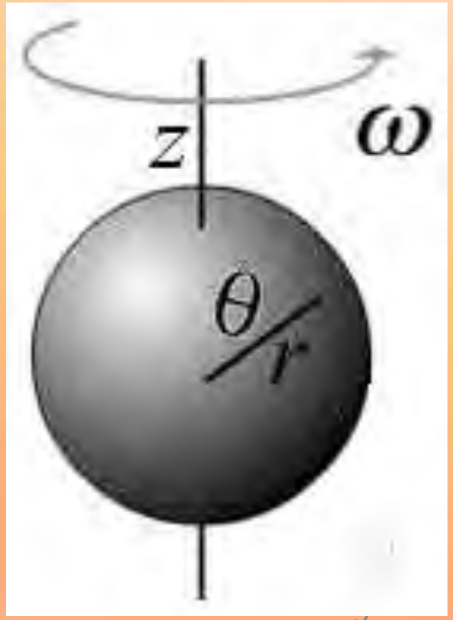
At radius r , $v = r\omega$ so very simply

$$K = \sigma r \omega$$



(b) Let's take the hint: $\vec{J} = \rho\vec{v}$

Volume charge density $\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3} = \frac{3Q}{4\pi R^3}$



Now $\vec{v}$ will be in the $\hat{\phi}$ direction (azimuthal) and a point distance r from the centre ($0 \le r \le R$) will rotate in a circle of radius $r \sin\theta$.

Thus $\vec{v} = \omega\, r \sin\theta\ \hat{\phi}$

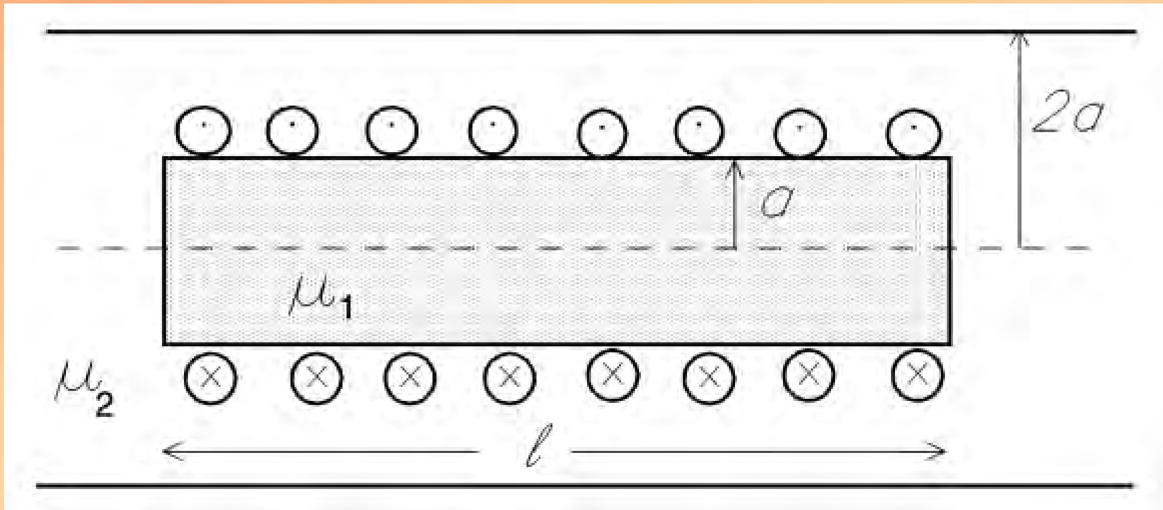
and so $\vec{J} = \rho\vec{v} = \frac{3Q}{4\pi R^3}\, \omega\, r \sin\theta\, \hat{\phi}$

Q.3. A solenoid is made by winding n turns of wire per unit length onto a paramagnetic rod of permeability μ_1 which has a radius a and length $l \gg a$. This solenoid is surrounded by a region of permeability μ_2 which extends to infinity parallel to the axis and radially out to radius $2a$. Outside this region the magnetic field is zero. (a) Use flux conservation ($\oint \vec{B} \cdot d\vec{S} = 0$) to show that, well away from the ends, the magnetic field inside the solenoid is three times that outside.

(b) Show that the field inside the solenoid has magnitude

$$B = n\, I \frac{3\mu_1\mu_2}{\mu_1+3\mu_2}$$

where I is the current in the solenoid.

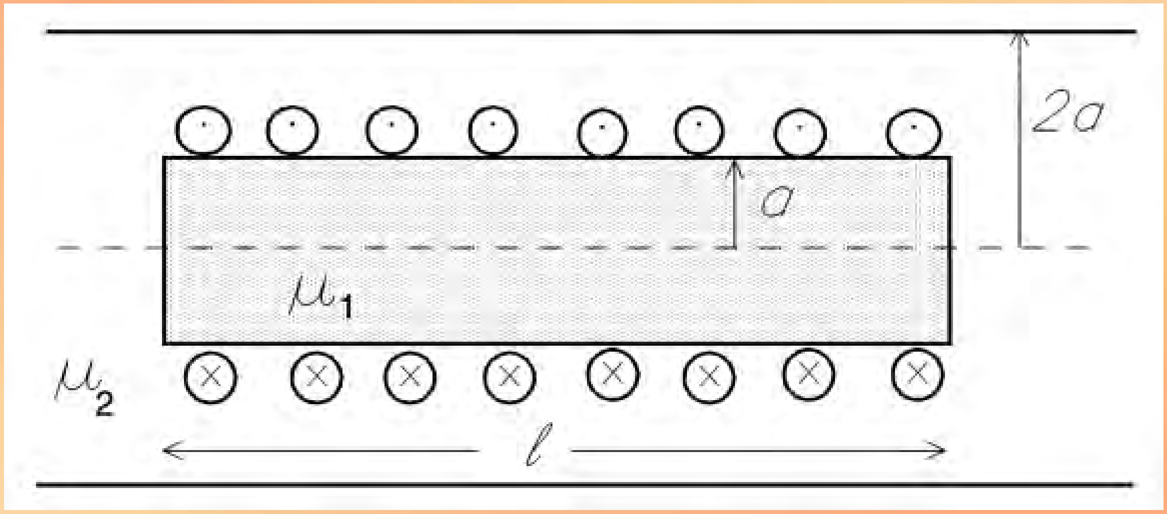


Q.3. *Solution*: (a) Since $\vec{B} = 0$ outside the region and $\nabla \cdot \vec{B} = 0$, all field lines emerging from the end of the solenoid must loop back through region 2 to conserve the flux. Thus

$$\Phi = \int_{S_1} \vec{B}_1 \cdot d\vec{S}_1 = \int_{S_2} \vec{B}_2 \cdot d\vec{S}_2$$

Assuming the magnetic field is uniform (away from the ends) this means (solving the integrals)

$$B_1 \pi a^2 = B_2(\pi(2a)^2 - \pi a^2) \quad \Longrightarrow \quad B_1 = 3B_2$$



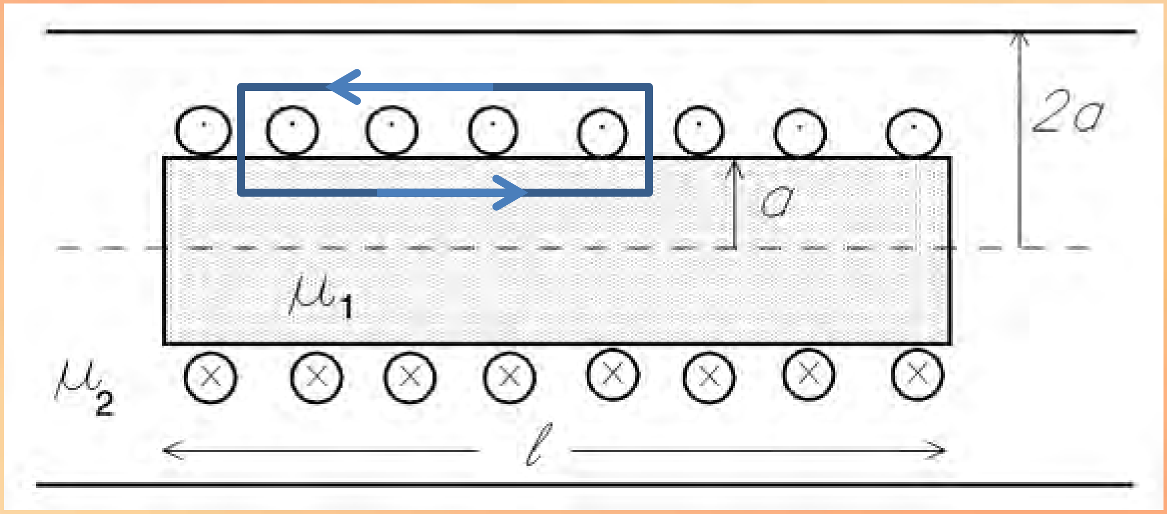
Q.3. *Solution*: (b) Apply Ampere's circuital law to the anticlockwise rectangular path shown: $\oint \vec{H}.d\vec{l} = n\,I\,dl$

(n turns per unit length, $n\,dl$ times current I in length dl)

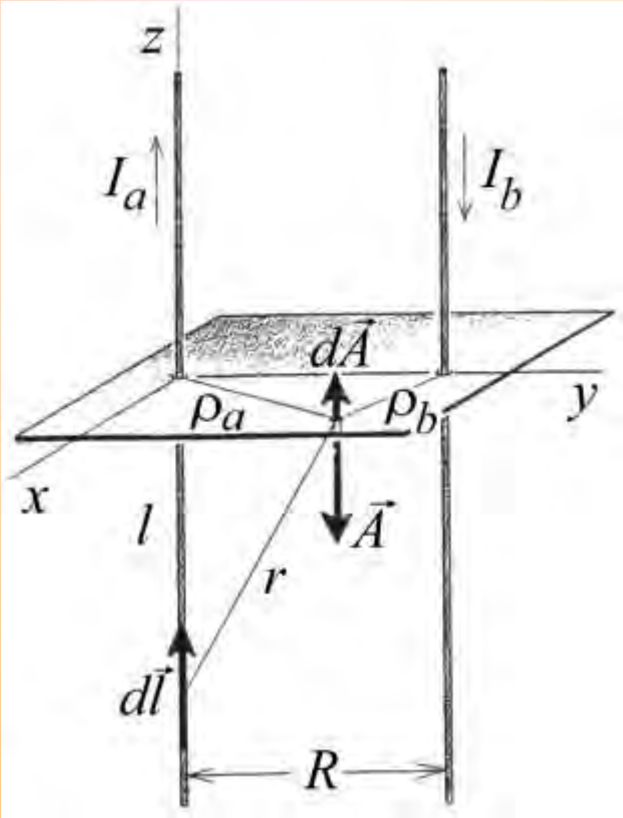
i.e. $H_1 dl + H_2 dl = n\,I\,dl$ ($\vec{H}$ parallel to $d\vec{l}$ in both media)

i.e. $\dfrac{B_1}{\mu_1} + \dfrac{B_2}{\mu_2} = n\,I$ But $B_1 = 3B_2$ from part (a) so

$$B_1\left(\frac{1}{\mu_1} + \frac{1}{3\mu_2}\right) = n\,I \quad \Longrightarrow \quad B = \left(\frac{3\mu_1\mu_2}{\mu_1 + 3\mu_2}\right) n\,I$$



Q.4. [A rather long and involved Example from Lorrain & Corson, Ch. 7] The figure shows two long parallel wires separated by distance R and carrying equal and opposite currents I_a and I_b.



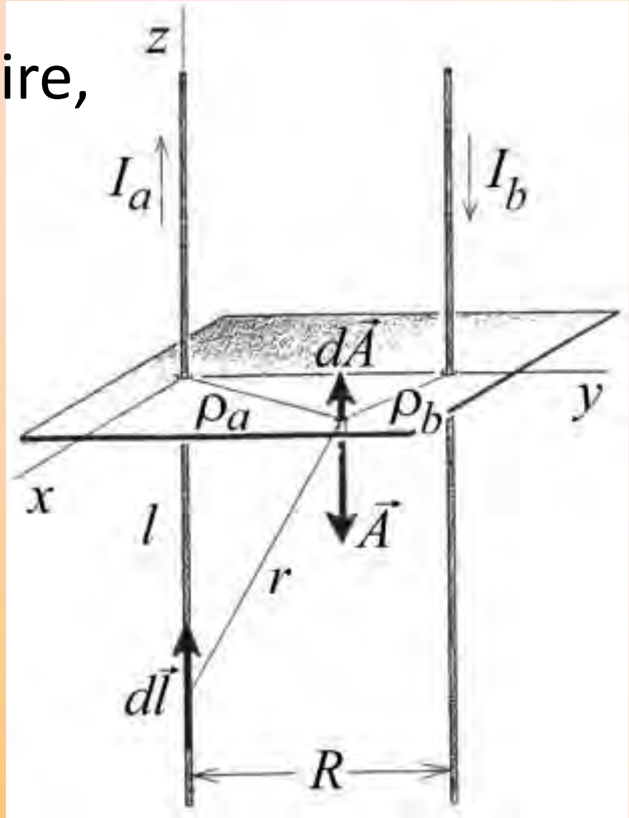
(a) Calculate the magnetic vector potential $\vec{A}$ as a function of position.

(b) Calculate the magnetic field $\vec{B}$ as a function of position.

(c) What is $\vec{B}$ at the midpoint between the two wires?

[Hint: Begin with one wire of finite length $2L$, and first find the equation for $\vec{A}$ for this length (assume distance from the wire $\rho \ll L$). Let the distances be ρ_a and ρ_b from the two wires, and add the two vector potentials. Then let $L \longrightarrow \infty$ for a long wire.]

Q.4. *Solution*: (a) For one long straight wire, $\vec{A} = \int \frac{\mu_0 I}{4\pi} \frac{d\vec{l}}{r}$ where $Id\vec{l}$ is always in the z-direction here. For length $2L$, integrate from 0 to L and multiply by 2 :



$$A_z = 2\frac{\mu_0 I}{4\pi}\int_0^L \frac{dl}{\sqrt{l^2+\rho^2}}$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln\left\{l+\sqrt{l^2+\rho^2}\right\}\right]_0^L$$

$$= \frac{\mu_0 I}{2\pi}\left[\ln L\left\{1+\sqrt{1+\rho^2/L^2}\right\} - \ln\rho\right]$$

$$\approx \frac{\mu_0 I}{2\pi}[\ln 2L - \ln\rho] = \frac{\mu_0 I}{2\pi}\ln\frac{2L}{\rho}$$

if $\rho^2 \ll L^2$

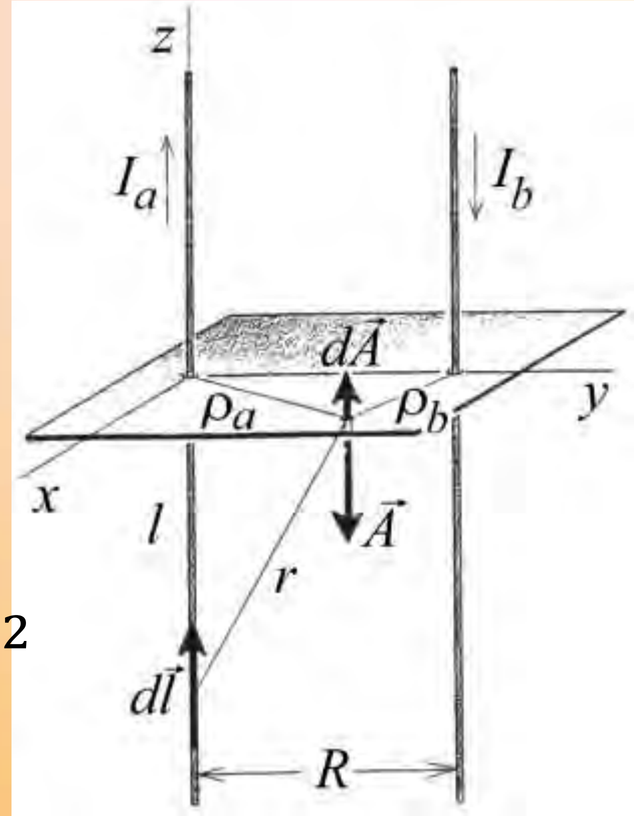
Q.4. *Solution* [continued]

For the two wires

$$A_z = \frac{\mu_0 I}{2\pi}\left(\ln\frac{2L}{\rho_a} - \ln\frac{2L}{\rho_b}\right)$$

$$= \frac{\mu_0 I}{2\pi}\ln\left(\frac{2L}{\rho_a}\cdot\frac{\rho_b}{2L}\right) = \frac{\mu_0 I}{2\pi}\ln\frac{\rho_b}{\rho_a}$$

$${\rho_a}^2 = x^2 + y^2 \;\; ; \;\; {\rho_b}^2 = x^2 + (R-y)^2$$

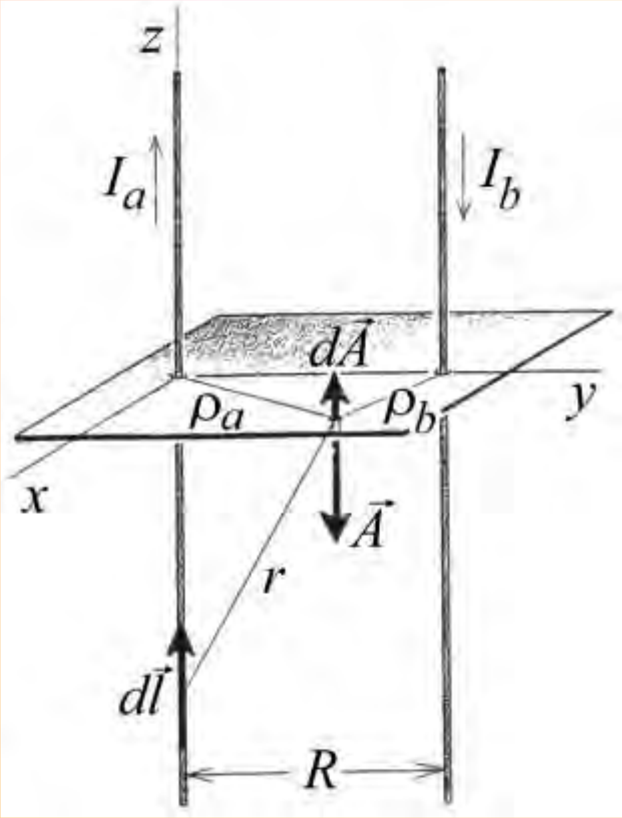


Recall: $\ln(x) = \frac{1}{2}\ln(x^2)$, so

$$A_z = \frac{\mu_0 I}{4\pi}\ln\left(\frac{\rho_b}{\rho_a}\right)^2$$; substituting for ρ_a , ρ_b

$$A_z = \frac{\mu_0 I}{4\pi}\ln\frac{x^2 + (R-y)^2}{x^2 + y^2}$$

Q.4. *Solution* [continued] (b)



$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y}\right)\hat{x} - \left(\frac{\partial A_z}{\partial x}\right)\hat{y}$$ where

$$A_z = \frac{\mu_0 I}{4\pi} \ln \frac{x^2+(R-y)^2}{x^2+y^2}$$ so

$$\frac{\partial A_z}{\partial y} = -\frac{\mu_0 I}{2\pi}\left[\frac{R-y}{x^2+(R-y)^2} + \frac{y}{x^2+y^2}\right] = B_x$$

$$\frac{\partial A_z}{\partial x} = -\frac{\mu_0 I}{2\pi}\left[\frac{x}{x^2+(R-y)^2} + \frac{x}{x^2+y^2}\right] = B_y$$

[x and y components of the curl respectively]

Q.4. *Solution* [continued] (c)

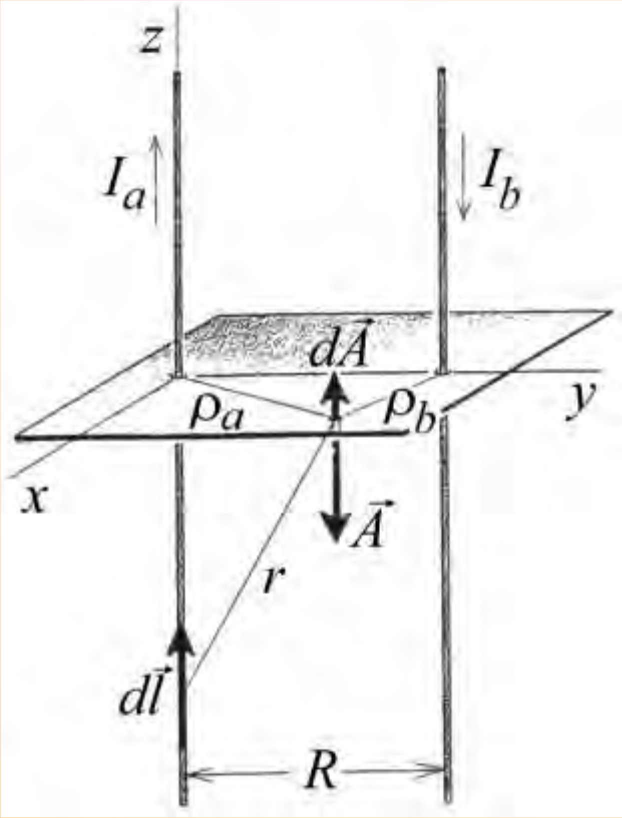
At the midpoint between the two wires

$x = 0,\ \ y = R/2$ (and $R - y = R/2$)

and so from (b)

$$B_x = -\frac{\mu_0 I}{2\pi}\left[\frac{1}{R/2} + \frac{1}{R/2}\right] = -\frac{2\mu_0 I}{\pi R}$$

$$B_y = 0$$



Note $\vec{B}$ is in the negative x-direction.

[Check that this is what you would expect by applying the right hand rule, using the given directions of the currents.]