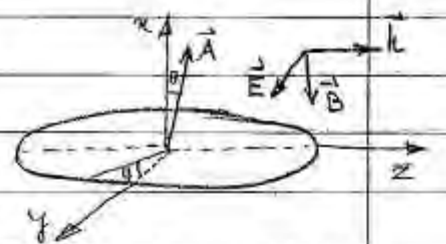
NASSP Honours Electrodynamics Part 1

Tutorial Problem Set 3: EM Waves **SOLUTIONS**

Q1 (a) Faraday's law: $\mathcal{E} = -d\Phi/dt$

or $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$

where $\vec{E}$ is induced ("non-electrostatic") field.



We need to find the rate of change of magnetic flux through the loop (RHS of the equation).

Given $\vec{E} = E_0 e^{i(kz-\omega t)}\,\hat{y}$, so the wave is travelling in the z-direction and $\vec{E}$ is in the y direction.

Now $\vec{B} = \frac{1}{\omega}\vec{k}\times\vec{E} = \frac{k}{\omega}\,\hat{z}\times E_0 e^{i(kz-\omega t)}\,\hat{y} = \frac{k}{\omega}E_0 e^{i(kz-\omega t)}(-\hat{x})$

[or $E_0/c \ldots$] since $\hat{z}\times\hat{y} = -\hat{x}$.

We want the real $\vec{B}$ field, so taking the real part,

$B_{real} = \mathcal{R}\left[|\vec{B}|\right] = \frac{k}{\omega}E_0\cos(kz-\omega t)$

Now we assume $a \ll \lambda$ so that $\vec{B}$ can be considered constant over the area of the loop, and we can put $\vec{B} = \vec{B}$ at $z=0$: $\vec{B} = \frac{k}{\omega}E_0\cos\omega t\,(-\hat{x})$

The angle between $\vec{B}$ and $\vec{A}$ (area vector $\perp$ loop) is $180°-\theta$ so the flux through the loop is

$\Phi_m = \int \vec{B}\cdot d\vec{a} = \vec{B}\cdot\vec{A}$ [assuming $\vec{B}$ uniform over A]

$= BA\cos(180°-\theta) = -BA\cos\theta$ [or say flux is negative]

$= -\pi a^2\cos\theta\; k/\omega\; E_0\cos\omega t$

Then $\mathcal{E} = -N\frac{d\Phi_m}{dt} = N\pi a^2\cos\theta\,\frac{k}{\omega}E_0(-\omega\sin\omega t)$

[for N turns]

$= -N\pi a^2\cos\theta\, kE_0\sin\omega t$

(b) $f = 6$ MHz so $\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^6} = 50$ m ; $k = \frac{2\pi}{\lambda} = 0.126\,m^{-1}$

$a = 0.5$ m, $N = 100$, $E_0 = 50\times10^{-3} = 0.05$ V/m

Max. $\mathcal{E}$ is when $\sin\omega t = 1$ (or -1)

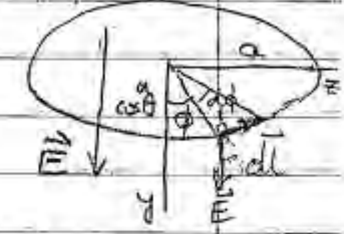
$\Rightarrow \mathcal{E}_{max} = N\pi a^2\cos\theta\, kE_0$

$= 100\,\pi\times0.5^2\cos 37°\times0.126\times0.05$

$= 0.40$ V

Comment on Q. 1 (a) :

Note that in Faraday's law $\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$ the electric field on the LHS is the <u>induced</u> field as a result of the changing $\vec{B}$.
So it is not valid to integrate the electric field of the wave around the loop.
In any case, if you do... [VIEW FROM ABOVE]



$\vec{E} = E_0 e^{i(kz-\omega t)}\hat{y}$, and if we assume the field is uniform over the loop, this applies to the direction as well as the magnitude.

$$\vec{E}\cdot d\vec{l} = E_0 e^{i(kz-\omega t)}\hat{y}\cdot(\hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz)$$
$$= E_0 e^{i(kz-\omega t)}\,dy \quad \text{of course}$$

where $dy = dl \sin\phi = (a\,d\phi)\sin\phi$ [arc length dl]

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = \int_0^{2\pi} E_0 e^{i(kz-\omega t)}\, a\sin\phi\, d\phi$$
$$= aE_0 e^{i(kz-\omega t)}\int_0^{2\pi}\sin\phi\, d\phi$$
$$= aE_0 e^{i(kz-\omega t)}\Big[-\cos\phi\Big]_0^{2\pi} = 0 \;!$$

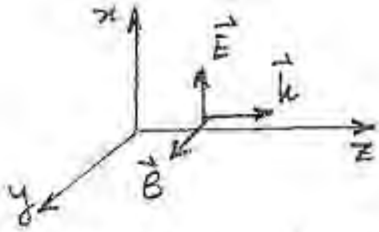
Q3 (a) $\lambda = \dfrac{c}{f} = \dfrac{3\times10^8}{5\times10^6} = 60\text{ m}$

(b) Intensity $I = \langle S\rangle = \frac{1}{2}c\epsilon_0 E_0^2 = 20\text{ W/m}^2$

$$\Rightarrow E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2\times20}{3\times10^8\times8.85\times10^{-12}}} = 123\text{ V/m}$$

and $B_0 = \dfrac{E_0}{c} = \dfrac{123}{3\times10^8} = 4.09\times10^{-7}\text{ T} = 409\text{ nT}$

Q.2.



Force on moving charge given by Lorentz force

$$\vec{F} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$$

Let us choose $\vec{E}$ to be in the x direction;
then $\vec{B}$ is in the y direction (given wave $\vec{k}$ in z dirn.)

i.e. $\vec{E} = E_0 e^{i(kz-\omega t)}\hat{x}$ and $\vec{B} = \frac{E_0}{c} e^{i(kz-\omega t)}\hat{y}$

Particle velocity $\vec{u}$ in arbitrary direction:

$$\vec{u} = u_x \hat{x} + u_y \hat{y} + u_z \hat{z}$$

Then

$$\vec{u} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_x & u_y & u_z \\ 0 & \frac{E_0}{c} & 0 \end{vmatrix} = -u_z \frac{E_0}{c}\hat{x} + u_x \frac{E_0}{c}\hat{z}$$

and so

$$\vec{F} = q\left(E_0 \hat{x} - u_z \frac{E_0}{c}\hat{x} + u_x \frac{E_0}{c}\hat{z}\right)$$

$$= q\left(E_0\left(1 - \frac{u_z}{c}\right)\hat{x} + E_0 \frac{u_x}{c}\hat{z}\right)$$

If particle travels in z direction, $u_x = 0$
and $\vec{F} = qE_0\left(1 - \frac{u_z}{c}\right)\hat{x}$, i.e. $/\!/ \ \vec{E}$

$\vec{F}$ would go to zero if $u_z = c$.

Q.4.

$d = 1/K$, $K = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left[\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 \right]^{1/2}$ (imaginary part of complex wave no.)

$\sigma = 4.0\,S/m$ $[S = \Omega^{-1}]$, $\varepsilon = \varepsilon_r \varepsilon_o = 81 \times 8.85 \times 10^{-12}\,F/m$,

$\mu = \mu_r \mu_o = 1 \times 4\pi \times 10^{-7}\,H/m$, $\omega = 2\pi f$, $f = 2 \times 10^4$ (a), 2×10^8 (b)

(a) $\frac{\sigma}{\varepsilon\omega} = \frac{4.0}{81 \times 8.85 \times 10^{-12} \times 2\pi \times 2 \times 10^4} = 4.44 \times 10^4 \gg 1$, so

$[\;\;]^{1/2} = \sqrt{4.44 \times 10^4} = 211$

$\sqrt{\varepsilon\mu/2} = \sqrt{81 \times 8.85 \times 10^{-12} \times 2\pi \times 10^{-7}} = 2.12 \times 10^{-8}$

$\Rightarrow K = 2\pi \times 2 \times 10^4 \times 2.12 \times 10^{-8} \times 211 = 0.562\ m^{-1}$

$\Rightarrow d(20\,kHz) = 1/K = 1.78\ m$

(b) ω is now 10^4 larger, so $\sigma/\varepsilon\omega$ will be 10^4 smaller $= 4.44$

Now $[\;\;]^{1/2} = \left[\sqrt{1 + 4.44^2} - 1\right] = 3.55$; $\sqrt{\varepsilon\mu/2}$ is the same

$\Rightarrow K = 2\pi \times 2 \times 10^8 \times 2.12 \times 10^{-8} \times 3.55 = 94.55$

$\Rightarrow d(200\,MHz) = 1/K = 1.06 \times 10^{-2}\ m = 1.06\ cm$

(c) $d(200\,MHz) \ll d(20\,kHz)$ $\Rightarrow$ use VLF to communicate with submarines!

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Tutorial Problem Set 3: EM Waves SOLUTION: Q.5

Q. 5.

We are assuming $\mu_1 = \mu_2 = \mu_0$. Then $\beta = \mu_1 n_2/\mu_2 n_1 = n_2/n_1 = 2.42$ here.
We want to plot a graph of $E_{0r}/E_{0i} = (\alpha - \beta)/(\alpha + \beta)$ and $E_{0t}/E_{0i} = 2/(\alpha + \beta)$
So we need α as a function of θ_i :

$$\alpha = \frac{\cos\theta_t}{\cos\theta_i} = \frac{\sqrt{1-(\sin\theta_t)^2}}{\cos\theta_i} = \frac{\sqrt{1-(\sin\theta_i)^2/\beta^2}}{\cos\theta_i}$$ by Snell's law ($n_1 \sin\theta_i = n_2 \sin\theta_t$)

This equation can easily be used to calculate α for a given θ_i , e.g. every 10°, in a spreadsheet.

Then we can calculate $E_{0r}/E_{0i} = (\alpha - \beta)/(\alpha + \beta)$ and $E_{0t}/E_{0i} = 2/(\alpha + \beta)$
[see spreadsheet]

(a) Amplitudes at normal incidence : when $\theta_i = 0$, then $\alpha = 1$ from above eqn. and

$$E_{0r}/E_{0i} = (\alpha - \beta)/(\alpha + \beta) = (1 - 2.42)/(1 + 2.42) = -1.42/3.42 = -0.415$$

$$E_{0t}/E_{0i} = 2/(\alpha + \beta) = 2/(3.42) = 0.585$$

(b) At Brewster's angle, $E_{0r}/E_{0i} = (\alpha - \beta)/(\alpha + \beta) = 0$ when $\alpha = \beta$.

i.e. $\frac{\sqrt{1-(\sin\theta_i)^2/\beta^2}}{\cos\theta_i} = \beta$. Put $\cos\theta_i = \sqrt{1 - (\sin\theta_i)^2}$, square and solve for

$(\sin\theta_i)^2 = (1 - \beta^2)/(1/\beta^2 - \beta^2)$ Hence $\theta_B = \sin^{-1}\sqrt{-4.86/-5.69} = 67.5°$

(c) At the crossover angle, $E_{0r}/E_{0i} = E_{0t}/E_{0i}$, i.e. $(\alpha - \beta)/(\alpha + \beta) = 2/(\alpha + \beta)$

so $\alpha - \beta = 2$ i.e. $\alpha = \beta + 2 = 4.42$ where by def. $\alpha = \frac{\cos\theta_2}{\cos\theta_1}$

so $\alpha^2 = \frac{1-(\sin\theta_2)^2}{1-(\sin\theta_1)^2} = \frac{1-(\sin\theta_1)^2/\beta^2}{1-(\sin\theta_1)^2}$ (using Snell's law)

or $\alpha^2(1 - (\sin\theta_1)^2) = 1 - \frac{(\sin\theta_1)^2}{\beta^2}$

where $\alpha = 4.42$ and $\beta = \alpha - 2 = 2.42$, so we have

$$4.42^2 - 4.42^2(\sin\theta_1)^2 = 1 - 2.42^{-2}(\sin\theta_1)^2$$

or $18.54 = 19.37(sin\,\theta_1)^2$ so $\sin\theta_1 = \sqrt{18.54/19.37}$ and $\theta_1 = 78°$

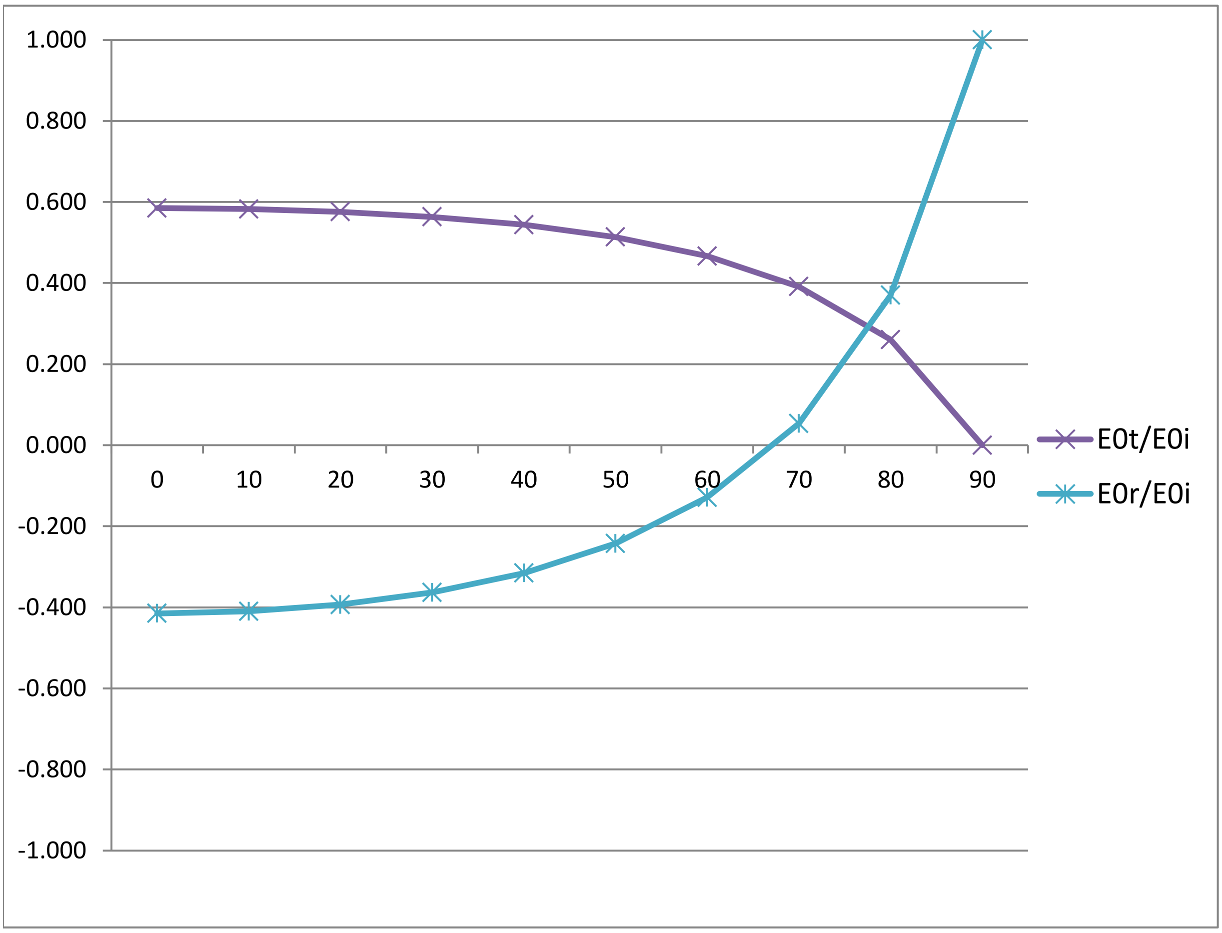
The plot shows the cross-over angle to be just less than 80°.

Plot of Fresnel Equations for $\beta = n_2/n_1 = 2.42$ (air/diamond)

E **parallel** to plane of incidence: $E_{0r}/E_{0i} = (\alpha - \beta) / (\alpha + \beta)$

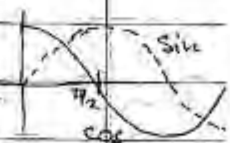
$E_{0t}/E_{0i} = 2 / (\alpha + \beta)$

θ deg	θ rad	α	β	E_{0r}/E_{0i}	E_{0t}/E_{0i}
0	0	1.000	2.420	-0.415	0.585
10	0.175	1.013	2.420	-0.410	0.583
20	0.349	1.053	2.420	-0.393	0.576
30	0.524	1.130	2.420	-0.363	0.563
40	0.698	1.259	2.420	-0.316	0.544
50	0.873	1.476	2.420	-0.242	0.513
60	1.047	1.868	2.420	-0.129	0.466
70	1.222	2.694	2.420	0.054	0.391
80	1.396	5.260	2.420	0.370	0.260
90	1.571	3.60E+16	2.420	1.000	0.000

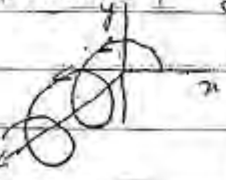


Q6. The x and y amplitudes are equal;
E_y lags E_x by $\pi/2$. This means it should be a circularly polarized wave.
But left or right?

$$\vec{E} = E_0\left[\hat{x}\cos(kz-\omega t) + \hat{y}\cos(kz-\omega t+\pi/2)\right]$$
$$= E_0\left[\hat{x}\cos(kz-\omega t) + \hat{y}\sin(kz-\omega t)\right]$$



This is a right circularly polarized wave
["what happens to E_x happens to E_y $\pi/2$ later"]



[A left circularly polarized wave is given by $\vec{E} = E_0\left[\hat{x}\cos(kz-\omega t) - \hat{y}\sin(kz-\omega t)\right]$]

Q.7.

For quarter-wave plate we require phase diff $\pi/2$ (corr. to $\lambda/4$).

i.e. $\delta = d(k_{slow} - k_{fast}) = d\left(\frac{2\pi n_{slow}}{\lambda_{vac}} - \frac{2\pi n_{fast}}{\lambda_{vac}}\right)$

i.e. $\frac{2\pi d}{\lambda_{vac}}(n_{slow} - n_{fast}) = \frac{\pi}{2}$

$$\Rightarrow d = \frac{\cancel{\pi} \times 589\times10^{-9}}{4\cancel{\pi}(1.5990-1.5940)} = 2.945\times10^{-5}\text{ m}$$

$\simeq$ 29.5 microns.