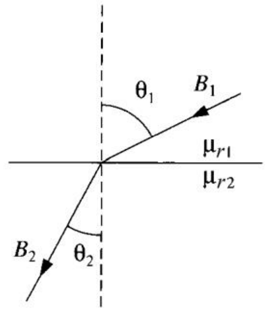
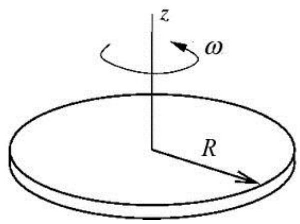
NASSP Honours Electrodynamics Part 1

Tutorial Problem Set 2: SOLUTIONS



Q.1. The boundary conditions (see ED-05, slides 8-9) are
$B_{n1} = B_{n2}$ (normal comp.) and $H_{t1} = H_{t2}$ (tangential comp.)
i.e. $B_1 \cos\theta_1 = B_2 \cos\theta_2$ and $H_1 \sin\theta_1 = H_2 \sin\theta_2$
Now $B_1 = \mu_1 H_1 = \mu_{r1}\mu_0 H_1$ and $B_2 = \mu_2 H_2 = \mu_{r2}\mu_0 H_2$
[Linear materials, so μ is a constant, $\vec{B} = \mu\vec{H}$, vectors in same direction and $B = \mu H$.]
Divide H eqn. by B eqn. : $\dfrac{H_1 \sin\theta_1}{B_1 \cos\theta_1} = \dfrac{H_2 \sin\theta_2}{B_2 \cos\theta_2}$ i.e . $\dfrac{H_1 \tan\theta_1}{\mu_{r1}\mu_0 H_1} = \dfrac{H_2 \tan\theta_2}{\mu_{r2}\mu_0 H_2}$
Cancel the H's and μ_0 : $\dfrac{\tan\theta_1}{\mu_{r1}} = \dfrac{\tan\theta_2}{\mu_{r2}}$ i.e. $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\mu_{r1}}{\mu_{r2}}$ QED
For the electric field, we had the equivalent relationship $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}$

Q.2. Disc $R = 10^7$ m , $\omega = 0.03$ rad.s^{-1} , and we want $B = 0.4$ T.



(a) Consider annulus, width dr at radius r : area $da = 2\pi r\, dr$;
charge density σ (= ne : we want n, the number of electrons per unit area)
so the charge on this annulus is $dq = \sigma\, da = \sigma\, 2\pi r\, dr$
The disc rotates at angular velocity ω , so the period of rotation is $2\pi/\omega$
and the current due to the annulus is $I = \dfrac{dq}{dt} = \dfrac{\sigma\, 2\pi r\, dr}{2\pi/\omega} = \sigma\omega r\, dr$
A circular current loop of radius r produces $B = \frac{\mu_0 I}{2r}$ at the centre (along the axis)
Thus $dB = \frac{\mu_0 I}{2r} = \frac{\mu_0 \sigma\omega r\, dr}{2r} = \frac{\mu_0\sigma\omega\, dr}{2}$ due to annulus dr and for the whole disc

$$B = \int_0^R \frac{\mu_0\sigma\omega}{2} dr = \frac{\mu_0\sigma\omega R}{2}$$ since σ and ω are constant.

We solve this for σ for $B = 0.4$ T : $\sigma = \dfrac{2B}{\mu_0\omega R} = \dfrac{2\times 0.4}{4\pi\times10^{-7}\times0.03\times10^{7}} = 2.1$ C/m^2
So the no. of electrons / m^2 is $n = \frac{\sigma}{e} = \frac{2.1}{1.6\times10^{-19}} = 1.3\times10^{19}$ m^{-2}

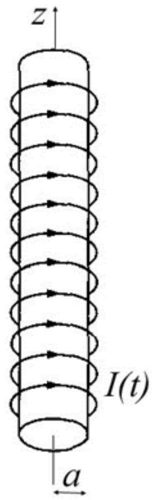
(b) Current due to the whole disc is $I = \frac{dQ}{dt}$ where total charge $Q = \sigma A = ne\,\pi R^2$
and $T = 2\pi/\omega$: $I = \frac{1.3\times10^{19}\times1.6\times10^{-19}\times\pi\times10^{14}}{2\pi/0.03} = 3.1\times10^{12}$ A

(c) These are *electrons* rotating in the $\hat{\phi}$ direction as indicated, which means conventional I is in the $-\hat{\phi}$ direction. By the right hand rule, then, $\vec{B}$ will be in the $-z$ direction in the figure.

Q.3. For a long solenoid, $\vec{B} = \mu_0 n I\hat{z}$ inside (n = no. of turns per unit length)
if I is in the $\hat{\phi}$ direction as shown, and is uniform. $\vec{B} = 0$ outside.
(i) Inside the solenoid, i.e. for $s < a$, we want $\vec{E}(s)$ and we assume cylindrical symmetry,
i.e. $\vec{E}$ is independent of ϕ . So we apply Faraday's law to a circular loop
of radius s : $\oint \vec{E}\cdot d\vec{l} = E\, 2\pi s = -\dfrac{d\Phi_m}{dt} = -\dfrac{d(BA)}{dt} = -\mu_0 n \dfrac{dI(t)}{dt}\pi s^2$

$\Rightarrow \qquad E(s) = -\mu_0 n\, s\dfrac{dI(t)}{dt}$ or as a vector $\vec{E}(s) = -\mu_0 n\, \dfrac{s}{2}\dfrac{dI(t)}{dt}\hat{\phi}$

i.e. if I is in the $\hat{\phi}$ direction, then $\vec{E}$ is in the $-\hat{\phi}$ direction. (Check Lenz's law!)



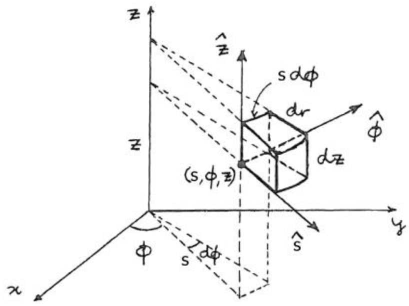
(ii) Outside the solenoid, i.e. for $s > a$, a similar loop will enclose all the flux over the entire cross-sectional area of the solenoid, i.e. πa^2 replaces πs^2 above [remember $\vec{B} = 0$ outside so we do not add any more flux.]

and $$\oint \vec{E}\cdot d\vec{l} = E\, 2\pi s = -\frac{d\Phi_m}{dt} = -\frac{d(BA)}{dt} = -\mu_0 n\frac{dI(t)}{dt}\pi a^2$$

$\Rightarrow \qquad E(s) = -\mu_0 n\, \dfrac{a^2}{2s}\dfrac{dI(t)}{dt}$ or as a vector $\vec{E}(s) = -\mu_0 n\, \dfrac{a^2}{2s}\dfrac{dI(t)}{dt}\hat{\phi}$

Q.4. (a) We have $\vec{M} = k\hat{z}$, (k constant); we require $\vec{J}_e = \nabla\times\vec{M}$ and

$\vec{K}_e = \vec{M}\times\hat{n}$ [Griffiths notation cylindrical co-ords. $(s, \phi, z) \rightarrow$]



Curl in cylindrical coordinates:

$$\nabla\times\vec{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial\{s v_\phi\}}{\partial s} - \frac{\partial v_s}{\partial\phi}\right)\hat{z}$$

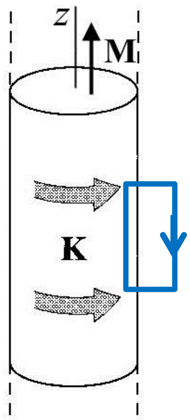
Since $\vec{M} = k\hat{z}$, we have $M_\phi = 0$ and $M_s = 0$ and

$$\nabla\times\vec{M} = \left(\frac{1}{s}\frac{\partial M_z}{\partial\phi}\right)\hat{s} - \left(\frac{\partial M_z}{\partial s}\right)\hat{\phi}$$

However $M_z = k$ so $\partial M_z/\partial\phi = 0$ and $\partial M_z/\partial s = 0$ so $\vec{J}_e = \nabla\times\vec{M} = 0$

At all points on the curved surface of the cylinder $\hat{n} = \hat{s}$ and so

$\vec{K}_e = \vec{M}\times\hat{n} = k\hat{z}\times\hat{s} = k\hat{\phi}$ i.e. the surface current is azimuthal.



(b) Ampere's Law: $\oint\vec{B}\cdot d\vec{l} = \mu_0\int\vec{J}\cdot d\vec{a}$ for volume currents and $\oint\vec{B}\cdot d\vec{l} = \mu_0 K\, L$ for surface currents [rectangular Amperian loop length l ∥ surface but ⊥ $\vec{K}$ (Griffiths Fig 5.33, p.226)] Clearly since $\vec{J}_e = 0$ there is no $\vec{B}$ due to volume currents. However the surface current $\vec{K}_e$ will produce a $\vec{B}$-field.

Apply Ampere's law to the rectangular loop length L inside and outside the surface.

The cylinder is "infinitely long", so we assume $\vec{B} = 0$ outside. Then Ampere's law gives

$\oint\vec{B}\cdot d\vec{l} = B\,L$ (inside) + 0 (top) + 0 (outside) + 0 (bottom) [top and bottom are ∥ $\vec{K}$]

$= BL = \mu_0 I_{\text{enc}} = \mu_0 KL \qquad \Rightarrow \qquad B = \mu_0 K$

$\vec{B}$ must be in the $\hat{z}$ direction, same as $\vec{M}$ i.e. $\vec{B} = \mu_0 K\,\hat{z}$

(c) By definition $\vec{H} = \vec{B}/\mu_0 - \vec{M}$ i.e. $\vec{B} = \mu_0(\vec{H} + \vec{M})$

Applied $\vec{H} = 9k\,\hat{z}$ so $\vec{B} = \mu_0(9k\,\hat{z} + k\,\hat{z}) = \mu_0(10k)\,\hat{z}$

By def. $\mu_r = 1 + \chi_m$ where $\chi_m = M/H = k/9k = 1/9 = 0.11$ so $\mu_r = 1.11$

or: $\mu_r = B/(\mu_0 H) = 10k/9k = 1.11$

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Q.5.* (a) The full Maxwell-Ampere Law in differential form is $\nabla \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \varepsilon_0\, \partial \vec{E}/\partial t$.

Rewrite this equation in integral form using Stokes' theorem. Show your steps.

Integrate over surface S:

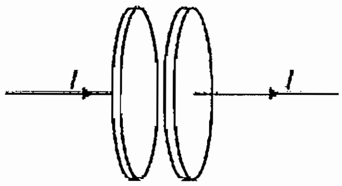
$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J}_f \cdot d\vec{a} + \mu_0 \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t} \cdot d\vec{a}$$

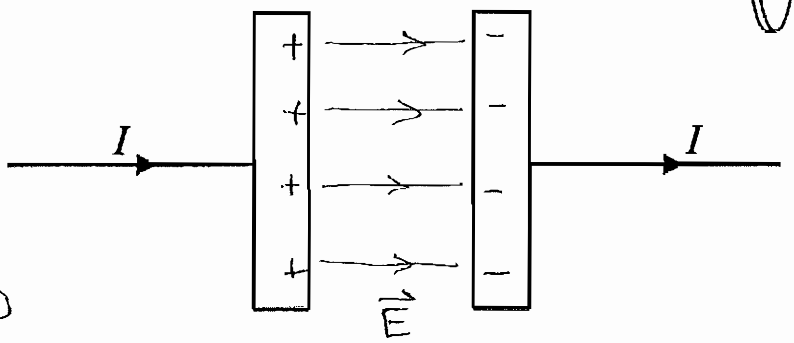
Apply Stokes' thm. to LHS: (C bounds surface S)

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_S \vec{E} \cdot d\vec{a}$$

(also putting $\int_S \vec{J}_f \cdot d\vec{a} = I$, total current).

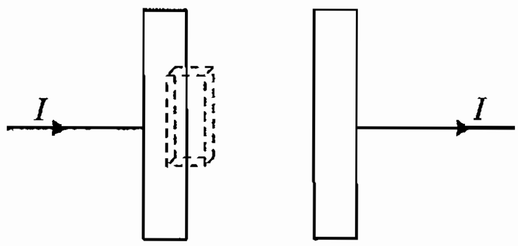
(b) Consider a capacitor in the process of charging up. The circular plates have radius R, area $A = \pi R^2$, and are so close together that fringe effects can be ignored. A current I is flowing in the long, straight wires (which are assumed to be very good conductors). Sketch the $\vec{E}$-field between the plates in the diagram below, which shows the plates edge-on. Is $\partial E/\partial t$ between the plates *positive*, *negative* or *zero*?





$$\frac{\partial E}{\partial t} > 0$$

(c) Consider the surface of an imaginary volume (dashed lines, at right) that partly encloses the left capacitor plate. For this closed surface, is the *total* flux of the current density $\oint \vec{J} \cdot d\vec{a}$ *positive*, *negative* or *zero*? Briefly explain your answer.



$\oint \vec{J} \cdot d\vec{a} < 0$ for this closed surface.

Flux JA in on left, zero out on right.

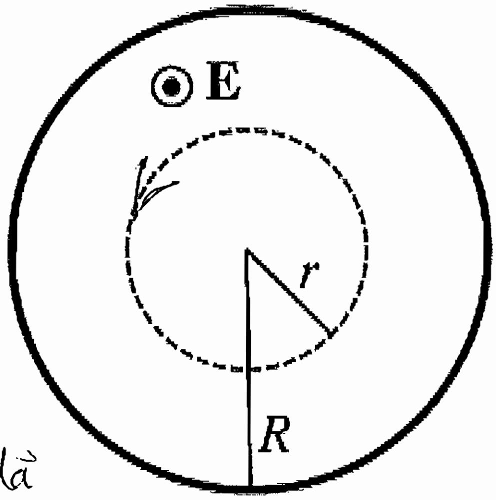
(g) Write down a relationship between the current I flowing in the wires and the rate of change of the charge per unit area on the plates $\partial\sigma/\partial t$.

$$I = \frac{dQ}{dt} = A\frac{\partial\sigma}{\partial t}$$

(h) Use this information to find a relationship between the current I flowing in the wires and the rate of change of the electric field between the plates $\partial E/\partial t$.

$$\Rightarrow \quad I = \varepsilon_0 A \frac{\partial E}{\partial t} \qquad (\text{since } \sigma = \varepsilon_0 E)$$

(i) Use the diagram at right to indicate the direction of the magnetic field everywhere along the dashed loop, at a distance r from the center of the plates, shown here face-on.



$\circlearrowleft$ by RH rule ($\vec{E}$ out)

(j) Use the Maxwell-Ampere Law to derive a formula for the magnetic field at a distance $r < R$ from the center of the plates. Express your final answer in terms of the current I flowing in the wires.

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

Here $I = 0$; S is bounded by C, so area $= \pi r^2$

$$\Rightarrow B \cdot 2\pi r = \mu_0 \varepsilon_0 \cdot \frac{\partial E}{\partial t} \cdot \pi r^2$$

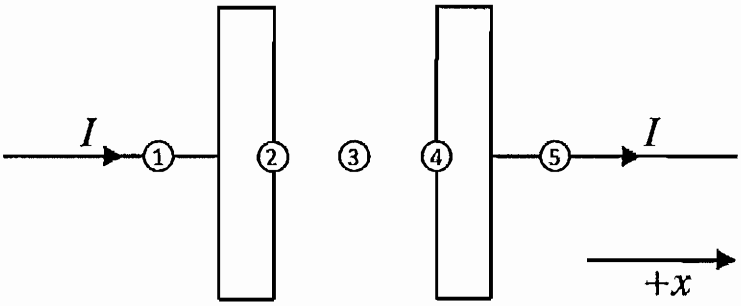
$$\Rightarrow \quad B = \frac{\mu_0\varepsilon_0 r}{2}\frac{\partial E}{\partial t}$$

and $\frac{\partial E}{\partial t} = \frac{I}{\varepsilon_0 A}$ from (h),

$A = \pi R^2$, total area

$$\Rightarrow \quad B = \frac{\mu_0 \varepsilon_0 r I}{2\varepsilon_0 A}$$

i.e. $$B = \frac{\mu_0 I r}{2\pi R^2}$$



(d) For each of the five points in the diagram above (labeled 1-5), fill out the table below to indicate whether the quantity in each row is *positive* (+), *negative* (−) or *zero* (0) at that point. Be sure your answers are consistent with the continuity equation $\nabla \cdot \vec{J} + \partial\rho/\partial t = 0$.

	1	2	3	4	5
$\partial\rho/\partial t$	0	+	0	−	0
$\nabla \cdot \vec{J}$	0	−	0	+	0

Explain briefly how your entries in each column are consistent with charge conservation.

(1) I the same all along the wire; no charge accumulation

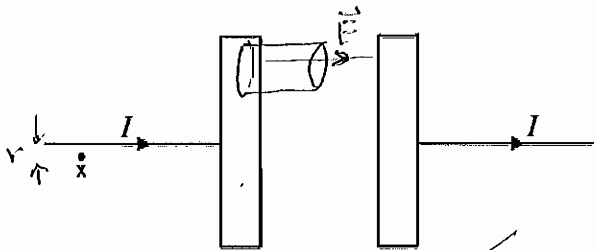
(2) Current flows in; +charge builds up on plates

(3) No current, no charge in capacitor space.

(4) − charge build up on plate, current flows out

(5) As for (1)

(e) Use the Maxwell-Ampere law to derive a formula for the magnetic field at the point "x" indicated in the diagram, a short distance r from the wire. [First year revision!]



Amperian loop radius r:

$$\oint \vec{B}.d\vec{l} = B.2\pi r = \mu_0 I_{enc} = \mu_0 I + \mu_0\varepsilon_0 \cancel{\frac{\partial}{\partial t}\int \vec{E}.d\vec{a}} \quad 0 \text{ (constant)}$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

[$\vec{E}$ negligible anyway for very good conductor]

(f) Use Gauss' law in integral form to derive a formula for the electric field between the capacitor plates. Be specific about the Gaussian surface used, and write the answer in terms of σ, the charge per unit area on the plates. [First year revision!]

Gaussian surface: cylinder, axis ⊥ plates, area A.
One end in LH plate, other end in space between plates
$\vec{E} = 0$ at LH end, $\vec{E}$ // sides so no flux.

$$\Rightarrow \Phi_E = \int \vec{E}.d\vec{a} = EA = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \qquad \Rightarrow E = \frac{\sigma}{\varepsilon_0}$$

(flux through RH end only)

(k) Use the Maxwell-Ampere Law to derive a formula for the magnetic field at a distance $r > R$ from the center of the plates. Express your final answer in terms of the current I flowing in the wires.

As for (j), except now we include the total area of the capacitor for the electric flux :

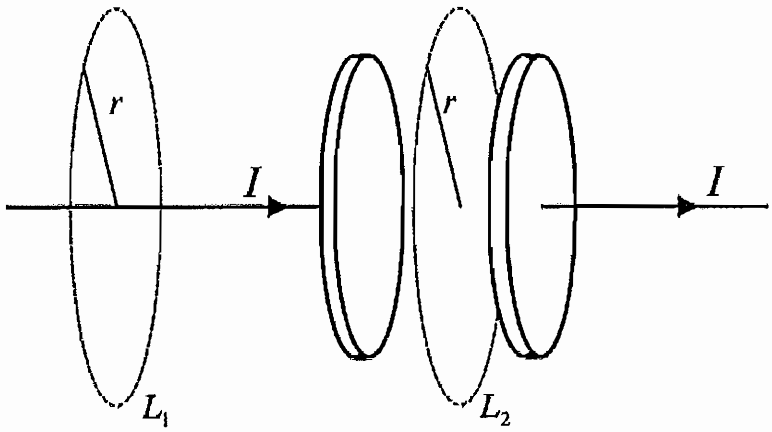
$$B \cdot 2\pi r = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \cdot \pi R^2$$

$$\Rightarrow \quad B = \frac{\mu_0 \varepsilon_0}{2\pi r} \cdot \frac{I}{\varepsilon_0 \pi R^2} \cdot \pi R^2$$

$$\Rightarrow \quad B = \frac{\mu_0 I}{2\pi r}$$

(because we enclose all the equivalent of the current)

(l) Consider the two line integrals of the magnetic field $\oint_{L1} \vec{B} \cdot d\vec{l}$ and $\oint_{L2} \vec{B} \cdot d\vec{l}$ for the loops $L1$ and $L2$ shown in the diagram below; both loops have the same radius r. How do the values of these two loop integrals compare? Is one larger than the other, or are they equal in magnitude? Explain your answer using the formulas you derived.



They are equal.

$$\oint_{L1} \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 I \cdot \quad (\text{see (e)})$$

$$\oint_{L2} \vec{B} \cdot d\vec{l} = B \cdot 2\pi r = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \cdot \pi R^2 \quad (\text{as in (k)})$$

$$= \mu_0 \varepsilon_0 \frac{I \cdot \pi R^2}{\varepsilon_0 \pi R^2} = \mu_0 I$$