NASSP Honours Electrodynamics Part 1

Tutorial Problem Set 1: Electrostatics SOLUTIONS

Q.1.* If the electric field points in the z-direction <u>everywhere</u>, then this means there is no variation with x or y, so we must have $\frac{\partial E_z}{\partial x} = 0$ and $\frac{\partial E_z}{\partial y} = 0$.

It also implies that the x and y components are zero: $E_x = 0$ and $E_y = 0$.

(i) if $\rho = 0$ then Gauss's law gives us $\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0} = 0$

since $E_x = 0$ and $E_y = 0$, this gives $\frac{\partial E_z}{\partial z} = 0$

(ii) if $\rho \neq 0$ then $\frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}$

Q.2. (a) By def. $\vec{P} = n\vec{p}$ where $\vec{p}$ is the dipole moment per atom (we want the magnitude).

So $p = P/n$ where no. of atoms per unit vol. $n = \frac{\rho}{M} N_0$ $\left[= \frac{\text{mass/volume}}{\text{mass/mole}} \times \frac{\text{atoms}}{\text{mole}} \right]$

where N_0 is Avogadro's number $= 6.02 \times 10^{23}$ and M is the molar mass in kg.

For C_6^{12} , $M = 12 \times 10^{-3}$ kg, so $n = \frac{3.5\times10^3}{12\times10^{-3}} \times 6.02 \times 10^{23} = 1.76 \times 10^{29}\ \text{m}^{-3}$.

Then average dipole moment $p = \frac{P}{n} = \frac{1.0\times10^{-7}}{1.76\times10^{29}} = 5.7 \times 10^{-37}\ \text{C.m}$

(b) By def. $p = qs$ where charge $q = 6e = 6 \times 1.6 \times 10^{-19} = 9.6 \times 10^{-19}$ C here

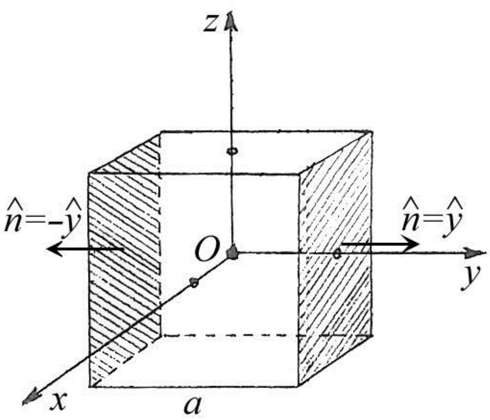
Then separation $s = \frac{p}{q} = \frac{5.7\times10^{-37}}{9.6\times10^{-19}} = 5.9 \times 10^{-19}$ m

This is VERY small, even on the scale of the atom ($\sim 10^{-10}$ m), or even the nucleus ($\sim 10^{-15}$ m).

Q.3.* Given $\vec{P} = k\vec{r} = k(x\hat{x} + y\hat{y} + z\hat{z})$ where k is a constant.

We know volume bound charge density $\rho_b = -\nabla \cdot \vec{P}$

and surface bound charge density $\sigma_b = \vec{P} \cdot \hat{n}$



(a) $\rho_b = -\nabla \cdot \vec{P} = -k\left(\frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z\right)$

$= -k(1 + 1 + 1) = -3k$

The volume of the cube is a^3, so the total volume bound charge is $Q_{b\mathcal{V}} = \rho_b \mathcal{V} = -3ka^3$

(b) At e.g. the right hand shaded surface shown, $y = a/2$ (O is at the centre of the cube) and the normal to the surface is $\hat{y}$. Thus $\sigma_b = \vec{P} \cdot \hat{n} = k\,\vec{r} \cdot \hat{y} = k(x\hat{x} + y\hat{y} + z\hat{z}) \cdot \hat{y} = ky = ka/2$.

The area is a^2 and so the bound charge on this face is $\sigma_b \cdot A = (ka/2)\, a^2 = ka^3/2$

Now for each face we will have the same situation, e.g. at the left hand shaded surface,

$y = -a/2$, $\hat{n} = -\hat{y}$ so $\sigma_b = k\,\vec{r}\cdot(-\hat{y}) = k(x\hat{x} + y\hat{y} + z\hat{z})\cdot(-\hat{y}) = -ky = -k(-a/2)$

and the bound charge on this face is again $\sigma_b \cdot A = (ka/2)a^2 = ka^3/2$.

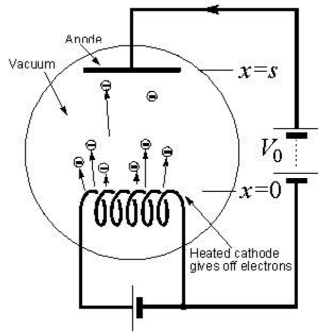
This will be the same for the faces of the cube at $x = \pm a/2$ and $z = \pm a/2$

so the total surface bound charge is $Q_{bS} = 6\,(ka^3/2) = +3ka^3$

(c) The total bound charge is $Q_{bV} + Q_{bS} = -3ka^3 + 3ka^3 = 0$ as it must be.

Q.4. (a) We must solve 1D Poisson's equation $\nabla^2 V = \frac{d^2V}{dx^2} = -\rho/\varepsilon_0$ with $\rho = -\frac{4\varepsilon_0 V_0}{9s^{4/3}x^{2/3}}$

and boundary conditions $V = 0$ at $x = 0$ and $V = V_0$ at $x = s$.



i.e. $\frac{d^2V}{dx^2} = \frac{4V_0}{9s^{4/3}}x^{-2/3}$ (note ε_0 cancels)

Integrate: $\frac{dV}{dx} = \left(\frac{3}{1}\right)\frac{4V_0}{9s^{4/3}}x^{1/3} + A$ (constant of integration)

Integrate: $V = \left(\frac{3}{4}\right)\frac{4V_0}{3s^{4/3}}x^{4/3} + Ax + B$

$$= \frac{V_0}{s^{4/3}}x^{4/3} + Ax + B$$

BC 1: $0 = \frac{V_0}{s^{4/3}}.0 + A.0 + B \quad \Longrightarrow \quad B = 0$

BC 2: $V_0 = \frac{V_0}{s^{4/3}}s^{4/3} + As = V_0 + As \quad \Longrightarrow \quad A = 0$

Thus solution: $V = V_0\left(\frac{x}{s}\right)^{4/3}$

(b) At $x = s$, $\rho = -\frac{4\varepsilon_0 V_0}{9s^{4/3}s^{2/3}} = -\frac{4\varepsilon_0 V_0}{9s^2}$

Assume $v = 0$ at $x = 0$ then at $x = s$ the electrons have KE $\frac{1}{2}m_e v^2 = qV_0 = eV_0$

So $v = \sqrt{\frac{2eV_0}{m_e}}$ and then $J = \rho v = -\frac{4\varepsilon_0 V_0}{9s^2}\left(\frac{2eV_0}{m_e}\right)^{1/2} = -\frac{4\varepsilon_0}{9s^2}\sqrt{\frac{2e}{m_e}}\,{V_0}^{3/2}$

[The proportionality of the current in a vacuum tube to ${V_0}^{3/2}$ is the "Child-Langmuir law".]

Q.5. Circulation $\oint_C \vec{A}\cdot d\vec{l} = \int_S \left(\nabla\times\vec{A}\right)\cdot d\vec{a}$ by Stokes's theorem,

where S is any surface bounded by the closed loop C . Now $\nabla\times\vec{A} = \vec{B}$,

so $\oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot d\vec{a} = \Phi_m$, the magnetic flux through S . Q.E.D.