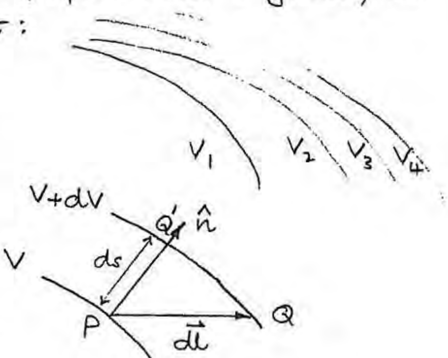
ELECTROSTATICS - BASIC IDEAS

Scalar and Vector Fields ; Gradient

A scalar or vector function of position is called a <u>field</u> ; a scalar field has a (scalar) value at each point in space , a vector field has a vector value (i.e. magnitude and direction) at each point in space.
These fields can be operated on by differential operators, the gradient, divergence, curl and Laplacian, which generate new fields.

The <u>gradient</u> operator acts on a scalar field and the result is a vector field. We will apply it in particular to the scalar field V (electric potential).

Suppose V varies in such a way that we can draw a series of 3-dimensional equipotential surfaces, on each of which V is constant:



Consider two neighbouring surfaces having potentials V and $V+dV$:
Let $\hat{n}$ be a unit vector normal to the V-equipotential, and the shortest distance to $V+dV$ be ds (from point P).
Consider a vector $\vec{dl}$ originating at point P such that its tip is on the surface $V+dV$. Clearly

$$(V+dV) - V = dV = -\vec{E}\cdot\vec{dl}$$

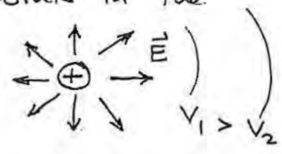
Now $\vec{E}$ is in the direction of $-\hat{n}$, so $\vec{E}\cdot\vec{dl} = E\,ds$ and the magnitude of E is given by $\dfrac{dV}{ds}$,

where dV/ds is the <u>maximum spatial rate of change</u> of V. We can then write

$$\vec{E} = -\hat{n}\,\frac{dV}{ds} \quad \text{or} \quad \vec{E} = -\,\text{grad}\,V$$

$$\text{or} \quad \underline{\vec{E} = -\nabla V}$$

where "grad" is the gradient operator, ∇ (= "del") denotes this operation, and we say "E equals minus grad V". In one dimension this is simply $E = -dV/dx$ which we saw in first year. The minus sign is to get the sense right, so that the field points in the direction of <u>de</u>creasing potential:
(grad points in the direction of increase of the function).



Grad in Cartesian Coordinates

Suppose $\vec{dl}$ has components dx, dy, dz.

Then $\quad dV = -\vec{E}\cdot\vec{dl} = -E_x\,dx - E_y\,dy - E_z\,dz$

But $\quad dV = \dfrac{\partial V}{\partial x}dx + \dfrac{\partial V}{\partial y}dy + \dfrac{\partial V}{\partial z}dz$

Comparing, we see $\quad E_x = -\dfrac{\partial V}{\partial x}, \quad E_y = -\dfrac{\partial V}{\partial y}, \quad E_z = -\dfrac{\partial V}{\partial z}$

i.e. $$\vec{E} = -\hat{x}\frac{\partial V}{\partial x} - \hat{y}\frac{\partial V}{\partial y} - \hat{z}\frac{\partial V}{\partial z} = -\nabla V$$

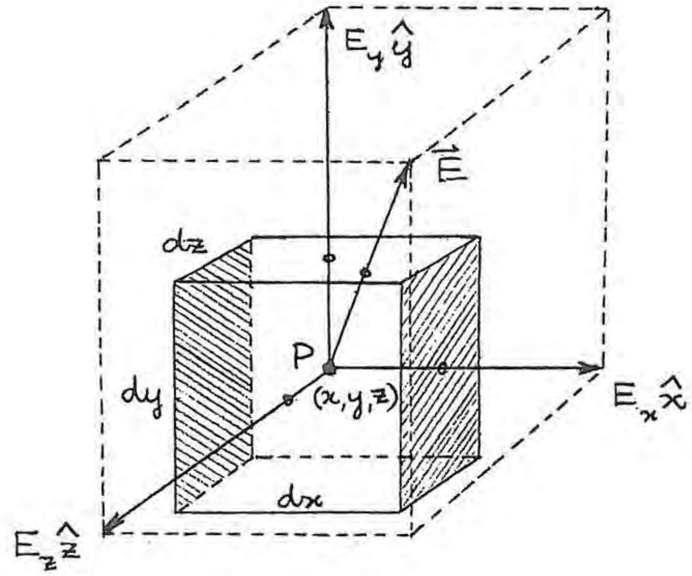
so $$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

In spherical polar coords. $\quad \nabla = \hat{r}\dfrac{\partial}{\partial r} + \hat{\theta}\dfrac{1}{r}\dfrac{\partial}{\partial\theta} + \hat{\phi}\dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}$

In cylindrical coords $\quad \nabla = \hat{r}\dfrac{\partial}{\partial r} + \hat{\phi}\dfrac{1}{r}\dfrac{\partial}{\partial\phi} + \hat{z}\dfrac{\partial}{\partial z}$

FLUX AND DIVERGENCE

(Wangsness § 1-10 + 1-14)



Consider a vector field $\vec{E}$ with Cartesian components (E_x, E_y, E_z) at some point P(x,y,z), and a box with dimensions dx, dy, dz around the point.

Flux of $\vec{E}$ thru RH (shaded) face

$$d\Phi_R = \left(E_x + \frac{\partial E_x}{\partial x}\cdot\frac{dx}{2}\right)\underbrace{dydz}_{\text{area}} \qquad \text{(outwards)}$$

and thru LH (shaded) face

$$d\Phi_L = -\left(E_x - \frac{\partial E_x}{\partial x}\cdot\frac{dx}{2}\right) dydz \qquad \text{(inwards)}$$

⇒ Net outward flux in x-direction

$$d\Phi_x = d\Phi_R + d\Phi_L = \frac{\partial E_x}{\partial x}\cdot dxdydz = \frac{\partial E_x}{\partial x}\cdot d\mathcal{V}$$

(Volume element)

⇒ Net total outward flux

$$d\Phi = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)d\mathcal{V}$$

$$\Rightarrow \quad d\Phi = \nabla\cdot\vec{E}\, d\mathcal{V},$$

from the definition of ∇ ("del") as the "vector" $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$; $\nabla\cdot\vec{E}$ ("del dot E") is then the "dot product" of del and the vector $\vec{E}$.

Define divergence of $\vec{E}$ = $\underline{\text{div}\,\vec{E} = \nabla\cdot\vec{E} = \frac{d\Phi}{d\mathcal{V}}}$ i.e. outgoing flux per unit volume

Total flux out of a finite volume $\mathcal{V}$,

$$\Phi = \int_{\mathcal{V}} (\nabla\cdot\vec{E})\,d\mathcal{V} \qquad \text{(from top of page)}$$

But by definition $\Phi = \int_S \vec{E}\cdot d\vec{A}$ (i.e. surface integral of outward normal component of $\vec{E}$)

where S is the surface enclosing volume $\mathcal{V}$,

$$\Rightarrow \quad \boxed{\int_S \vec{E}\cdot d\vec{A} = \int_{\mathcal{V}} \nabla\cdot\vec{E}\, d\mathcal{V}}$$

i.e. The surface integral of a vector over a closed surface is equal to the volume integral of its divergence over the volume enclosed by that surface.

- The Divergence Theorem *

If $\mathcal{V}\to 0$ so that $\nabla\cdot\vec{E}$ is effectively constant throughout it, we then have

$$\int_S \vec{E}\cdot d\vec{A} = (\nabla\cdot\vec{E})\,\mathcal{V}$$

so that divergence can be defined as

$$\boxed{\text{div}\,\vec{E} = \lim_{\mathcal{V}\to 0}\frac{1}{\mathcal{V}}\int_S \vec{E}\cdot d\vec{A}}$$

(which amounts to the same as above)

In Cartesian coordinates, writing this as

$$\nabla\cdot\vec{E} = \frac{d\Phi}{d\mathcal{V}} \quad \text{from above},$$

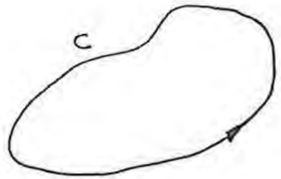
we see that
$$\nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

[i.e. "dot product" of $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and (E_x, E_y, E_z)]

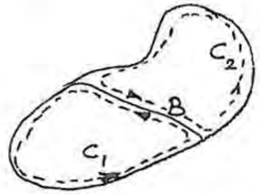
* Also known as Gauss's Theorem

Curl of a Vector Field ; Stokes's Theorem

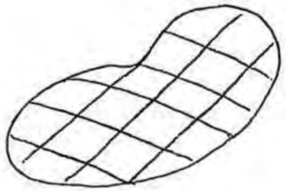
Consider line integrals around closed contours like $\oint \vec{E}.\vec{dl}$, $\oint \vec{B}.\vec{dl}$. Such a line integral is called the <u>circulation</u> of the vector .



Let the circulation of a vector $\vec{A}$ around closed contour C be $\Gamma = \oint_c \vec{A}.\vec{dl}$



Now 'bridge' C with a bridge B :
Clearly $\Gamma = \Gamma_1 + \Gamma_2$
since the two parts of Γ_1 and Γ_2 along B cancel.



We can continue to subdivide, getting $\Gamma = \sum_n \Gamma_n$

i.e. $\oint_c \vec{A}.\vec{dl} = \sum_n \oint_{C_n} \vec{A}.\vec{dl}$

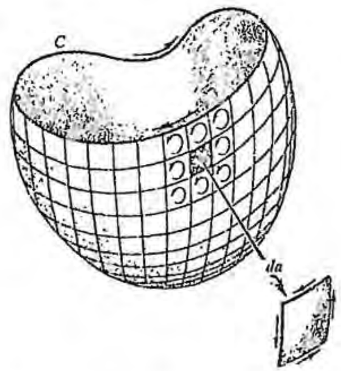
It seems natural to look at the ratio of Γ_n to the loop area ; this can be expected to approach a limit. However, the area is a vector with its direction defined by the normal $\hat{n}$, and the ratio Γ_n / area (size a_n) depends on the orientation of $\hat{n}$.

We define
$$\lim_{\substack{n\to\infty \\ a_n\to 0}} \frac{\oint_{C_n} \vec{A}.\vec{dl}}{a_n} = \hat{n}.\,\mathrm{curl}\,\vec{A}$$

where curl $\vec{A}$ is a vector whose component in the $\hat{n}$ direction is this limiting ratio.

[Read this **after** the following double-page spread]

<u>Stokes' Theorem</u>



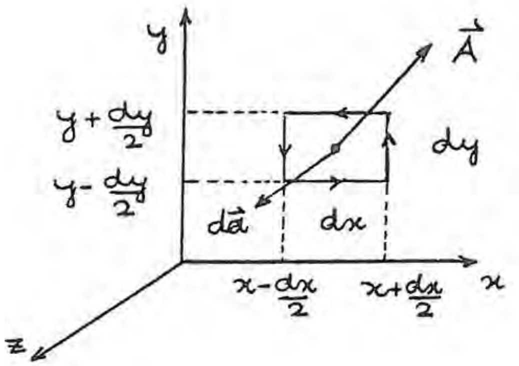
The equation $\oint \vec{A}.\vec{d\ell} = (\nabla \times \vec{A}) \cdot \vec{da}$ is true only for a path so small that $\nabla \times \vec{A}$ can be considered constant over the surface $\vec{da}$ bounded by the path. For a longer arbitrary path we can divide the surface into small elements of area $\vec{da}_1, \vec{da}_2$ etc., as in the above figure, for each of which the above equation holds. The sum of the left hand sides of these equations is simply the line integral around the external boundary (C in the above figure), since there are always two equal and opposite contributions to the sum along every common side between adjacent $\vec{da}$'s. The sum of the right hand sides is the integral of $(\nabla \times \vec{A}) \cdot \vec{da}$ over the finite surface S bounded by C. Thus for an arbitrary path:

$$\boxed{\oint_C \vec{A} \cdot \vec{d\ell} = \int_S (\nabla \times \vec{A}) \cdot \vec{da}}$$

- <u>Stokes' Theorem</u>

i.e. the line integral around any closed path C is equal to the surface integral of the curl over *any* surface S bounded by C.

CURL :*



A vector field $\vec{A}$ is said to be *conservative* if the line integral of $\vec{A}.d\ell$ around any closed curve is zero, i.e. $\oint \vec{A}.\vec{d\ell} = 0$.

Let us calculate the value of this integral in the more general case where it is *not* equal to zero. For an infinitesimal element of path $\vec{d\ell}$ in the *xy-plane* and from the definition of the scalar product,

$$\vec{A}.\vec{d\ell} = A_x\ dx + A_y\ dy$$

Thus for any closed path in the xy-plane and for any $\vec{A}$,

$$\oint \vec{A}.\vec{d\ell} = \oint A_x\ dx + \oint A_y\ dy$$

Consider now the rectangular closed path with sides dx, dy in the xy-plane around the point (x,y,0) where the vector A has a particular value as shown above. Here

$$\oint A_x\ dx = \underbrace{(A_x - \frac{\partial A_x}{\partial y}\frac{dy}{2})\ dx}_{\text{(bottom side)}} - \underbrace{(A_x + \frac{\partial A_x}{\partial y}\frac{dy}{2})\ dx}_{\text{(top side)}}$$

$$= - \frac{\partial A_x}{\partial y}\ dx\ dy$$

Similarly $\oint A_y\ dy = \frac{\partial A_y}{\partial x}\ dx\ dy$

(adding right and left sides)

and

$$\oint A.d\ell = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\ dx\ dy$$

which we can write as

$$\oint A.\vec{d\ell} = g_z\ da$$

where

$g_z = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$ is the z component of the vector which we get by considering the symmetrical quantities $(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})$ and $(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})$ (which we would get similarly if we considered similar closed loops in the yz- and xz-planes) as the x and y components respectively. We can write this as:

$$\boxed{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \nabla \times \vec{A}}$$

and we call this vector the *curl* of $\vec{A}$:

$$\text{curl}\ \vec{A} = \nabla \times \vec{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\ \hat{x} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\ \hat{y} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\ \hat{z}$$

Consider the area element da (= dx dy above) as a vector $\vec{da}$ pointing in the direction of advance of a RH screw turned in the direction of the line integral (e.g. above $\oint \vec{A}.\vec{d\ell}$ is anticlockwise and $\vec{da}$ is outwards). For the above case in the xy-plane,

$$\oint \vec{A}.\vec{d\ell} = (\nabla \times \vec{A})\ .\ \vec{da} = (\nabla \times \vec{A})\ .\ \hat{z}\ da$$

and in general for any da, we get:

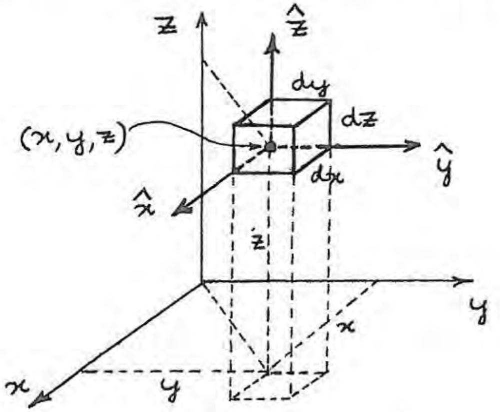
$$\boxed{(\nabla \times \vec{A})\ .\ \hat{n} = \lim_{S \to 0} \frac{1}{S} \oint \vec{A}.\vec{d\ell}}$$

i.e. the *component of the curl of a vector normal to a surface* S is equal to the line integral of the vector around the boundary of S (the "circulation") divided by the area of S when this area approaches zero.

*From Lorrain, Corson & Lorrain §1.8

VECTOR OPERATORS, ETC.

CARTESIAN COORDINATES (x, y, z)



Vol. element: $\quad dV = dx\,dy\,dz$

Grad: $\quad \nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right)$

Div.: $\quad \nabla \cdot \vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

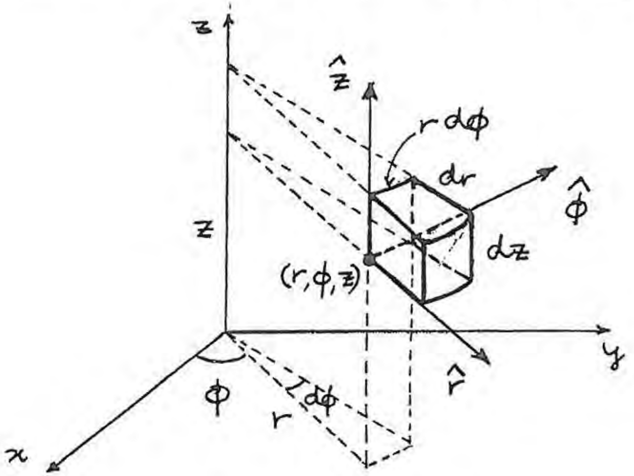
Curl: $\quad \nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

Laplacian: $\quad \nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$

$$\nabla^2 \vec{A} = \left(\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z \right)$$

CYLINDRICAL COORDINATES (r, ϕ, z)



Vol. element: $\quad dV = r\,dr\,d\phi\,dz$

Grad: $\quad \nabla f = \left(\dfrac{\partial f}{\partial r}, \dfrac{1}{r}\dfrac{\partial f}{\partial \phi}, \dfrac{\partial f}{\partial z} \right)$

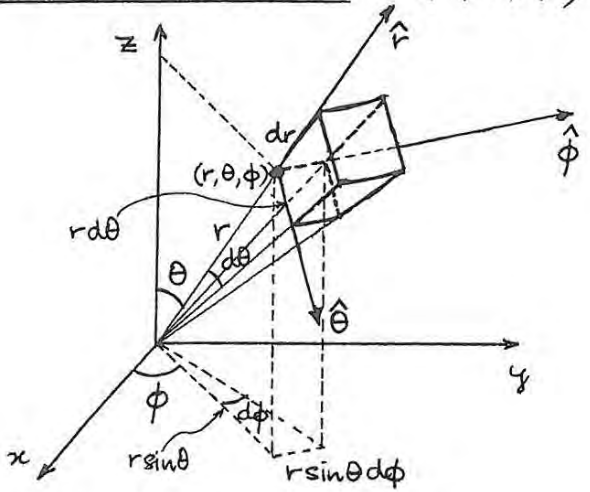
Div.: $\quad \nabla \cdot \vec{A} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rA_r\right) + \dfrac{1}{r}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}$

Curl: $\quad \nabla \times \vec{A} = \dfrac{1}{r} \begin{vmatrix} \hat{r} & r\hat{\phi} & \hat{z} \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ A_r & rA_\phi & A_z \end{vmatrix}$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{\partial A_\phi}{\partial r} + \frac{A_\phi}{r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \right)$$

Laplacian: $\quad \nabla^2 f = \dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial \phi^2} + \dfrac{\partial^2 f}{\partial z^2}$

SPHERICAL POLAR COORDINATES (r, θ, ϕ)



Vol. element : $\quad dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Grad : $\quad \nabla f = \left(\dfrac{\partial f}{\partial r} , \; \dfrac{1}{r}\dfrac{\partial f}{\partial \theta} , \; \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \phi} \right)$

Div. : $\quad \nabla \cdot \vec{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 A_r\right) + \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial \theta}(\sin\theta A_\theta) + \dfrac{\partial A_\phi}{\partial \phi}\right]$

$$= \frac{\partial A_r}{\partial r} + \frac{2}{r}A_r + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\cot\theta}{r}A_\theta + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Curl : $\quad \nabla \times \vec{A} = \dfrac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

$$= \left(\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) - \frac{\partial}{\partial\phi}(rA_\theta)\right], \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r\sin\theta A_\phi)\right], \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] \right)$$

$$= \left(\frac{1}{r}\frac{\partial A_\phi}{\partial\theta} + \frac{\cot\theta}{r}A_\phi - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial A_\phi}{\partial r} - \frac{A_\phi}{r}, \frac{A_\theta}{r} + \frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta} \right)$$

Laplacian: $\quad \nabla^2 f = \dfrac{2}{r}\dfrac{\partial f}{\partial r} + \dfrac{\partial^2 f}{\partial r^2} + \dfrac{\cot\theta}{r^2}\dfrac{\partial f}{\partial\theta} + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2} + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial\phi^2}$

USEFUL VECTOR IDENTITIES

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$$

$$\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f) \qquad *$$

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) + (\nabla f)\times\vec{A}$$

$$\nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + (\nabla\cdot\vec{B})\vec{A} - (\nabla\cdot\vec{A})\vec{B}$$

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \qquad \dagger$$

* div of (scalar × vector) = scalar × div of vector + vector · grad of scalar
(curl similar)

† In non-Cartesian coordinate systems this equation defines $\nabla^2\vec{A}$