

1.2 Differential Calculus

1.2.1 “Ordinary” Derivatives

Question: Suppose we have a function of one variable: $f(x)$. What does the derivative, df/dx , do for us? *Answer:* It tells us how rapidly the function $f(x)$ varies when we change the argument x by a tiny amount, dx :

$$df = \left(\frac{df}{dx} \right) dx. \quad (1.33)$$

In words: If we change x by an amount dx , then f changes by an amount df ; the derivative is the proportionality factor. For example, in Fig. 1.17(a), the function varies slowly with x , and the derivative is correspondingly small. In Fig. 1.17(b), f increases rapidly with x , and the derivative is large, as you move away from $x = 0$.

Geometrical Interpretation: The derivative df/dx is the *slope* of the graph of f versus x .

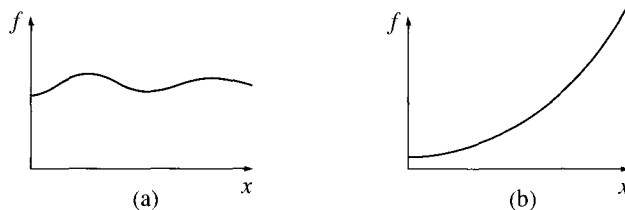


Figure 1.17

1.2.2 Gradient

Suppose, now, that we have a function of *three* variables—say, the temperature $T(x, y, z)$ in a room. (Start out in one corner, and set up a system of axes; then for each point (x, y, z) in the room, T gives the temperature at that spot.) We want to generalize the notion of “derivative” to functions like T , which depend not on *one* but on *three* variables.

Now a derivative is supposed to tell us how fast the function varies, if we move a little distance. But this time the situation is more complicated, because it depends on what *direction* we move: If we go straight up, then the temperature will probably increase fairly rapidly, but if we move horizontally, it may not change much at all. In fact, the question “How fast does T vary?” has an infinite number of answers, one for each direction we might choose to explore.

Fortunately, the problem is not as bad as it looks. A theorem on partial derivatives states that

$$dT = \left(\frac{\partial T}{\partial x} \right) dx + \left(\frac{\partial T}{\partial y} \right) dy + \left(\frac{\partial T}{\partial z} \right) dz. \quad (1.34)$$

This tells us how T changes when we alter all three variables by the infinitesimal amounts dx, dy, dz . Notice that we do *not* require an infinite number of derivatives—*three* will suffice: the *partial* derivatives along each of the three coordinate directions.

Equation 1.34 is reminiscent of a dot product:

$$\begin{aligned} dT &= \left(\frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \right) \cdot (dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}) \\ &= (\nabla T) \cdot (d\mathbf{l}), \end{aligned} \quad (1.35)$$

where

$$\nabla T \equiv \frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \quad (1.36)$$

is the **gradient** of T . ∇T is a *vector* quantity, with three components; it is the generalized derivative we have been looking for. Equation 1.35 is the three-dimensional version of Eq. 1.33.

Geometrical Interpretation of the Gradient: Like any vector, the gradient has *magnitude* and *direction*. To determine its geometrical meaning, let's rewrite the dot product (1.35) in abstract form:

$$dT = \nabla T \cdot d\mathbf{l} = |\nabla T| |d\mathbf{l}| \cos \theta, \quad (1.37)$$

where θ is the angle between ∇T and $d\mathbf{l}$. Now, if we *fix* the *magnitude* $|d\mathbf{l}|$ and search around in various *directions* (that is, vary θ), the *maximum* change in T evidently occurs when $\theta = 0$ (for then $\cos \theta = 1$). That is, for a fixed distance $|d\mathbf{l}|$, dT is greatest when $\mathbf{l}$ move in the *same direction* as ∇T . Thus:

The gradient ∇T points in the direction of maximum increase of the function T .

Moreover:

The magnitude $|\nabla T|$ gives the slope (rate of increase) along this maximal direction.

Imagine you are standing on a hillside. Look all around you, and find the direction of steepest ascent. That is the *direction* of the gradient. Now measure the *slope* in that direction (rise over run). That is the *magnitude* of the gradient. (Here the function we're talking about is the height of the hill, and the coordinates it depends on are positions—latitude and longitude, say. This function depends on only *two* variables, not *three*, but the geometrical meaning of the gradient is easier to grasp in two dimensions.) Notice from Eq. 1.37 that the direction of maximum *descent* is opposite to the direction of maximum *ascent*, while at right angles ($\theta = 90^\circ$) the slope is zero (the gradient is perpendicular to the contour lines). You can conceive of surfaces that do not have these properties, but they always have “kinks” in them and correspond to nondifferentiable functions.

What would it mean for the gradient to vanish? If $\nabla T = 0$ at (x, y, z) , then $dT = 0$ for small displacements about the point (x, y, z) . This is, then, a **stationary point** of the function $T(x, y, z)$. It could be a maximum (a summit), a minimum (a valley), a saddle

point (a pass), or a “shoulder.” This is analogous to the situation for functions of *one* variable, where a vanishing derivative signals a maximum, a minimum, or an inflection. In particular, if you want to locate the extrema of a function of three variables, set its gradient equal to zero.

Example 1.3

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

Solution;

$$\begin{aligned}\nabla r &= \frac{\partial r}{\partial x} \hat{\mathbf{x}} + \frac{\partial r}{\partial y} \hat{\mathbf{y}} + \frac{\partial r}{\partial z} \hat{\mathbf{z}} \\ &= \frac{1}{2} \frac{2x}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{x}} + \frac{1}{2} \frac{2y}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{y}} + \frac{1}{2} \frac{2z}{\sqrt{x^2 + y^2 + z^2}} \hat{\mathbf{z}} \\ &= \frac{x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.\end{aligned}$$

Does this make sense? Well, it says that the distance from the origin increases most rapidly in the radial direction, and that its *rate* of increase in that direction is 1... just what you'd expect.

Problem 1.11 Find the gradients of the following functions:

(a) $f(x, y, z) = x^2 + y^3 + z^4$.

(b) $f(x, y, z) = x^2 y^3 z^4$.

(c) $f(x, y, z) = e^x \sin(y) \ln(z)$.

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

- **Problem 1.13** Let $\mathbf{z}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) , and let z be its length. Show that

(a) $\nabla(z^2) = 2\mathbf{z}$.

(b) $\nabla(1/z) = -\hat{\mathbf{z}}/z^2$.

(c) What is the *general* formula for $\nabla(z^n)$?

- ! **Problem 1.14** Suppose that f is a function of two variables (y and z) only. Show that the gradient $\nabla f = (\partial f/\partial y)\hat{\mathbf{y}} + (\partial f/\partial z)\hat{\mathbf{z}}$ transforms as a vector under rotations, Eq. 1.29. [Hint: $(\partial f/\partial \bar{y}) = (\partial f/\partial y)(\partial y/\partial \bar{y}) + (\partial f/\partial z)(\partial z/\partial \bar{y})$, and the analogous formula for $\partial f/\partial \bar{z}$. We know that $\bar{y} = y \cos \phi + z \sin \phi$ and $\bar{z} = -y \sin \phi + z \cos \phi$; “solve” these equations for y and z (as functions of $\bar{y}$ and $\bar{z}$), and compute the needed derivatives $\partial y/\partial \bar{y}$, $\partial z/\partial \bar{y}$, etc.]

1.2.3 The Operator ∇

The gradient has the formal appearance of a vector, ∇ , “multiplying” a scalar T :

$$\nabla T = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) T. \quad (1.38)$$

(For once I write the unit vectors to the *left*, just so no one will think this means $\partial \hat{\mathbf{x}}/\partial x$, and so on—which would be zero, since $\hat{\mathbf{x}}$ is constant.) The term in parentheses is called “del”:

$$\nabla = \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}. \quad (1.39)$$

Of course, del is *not* a vector, in the usual sense. Indeed, it is without specific meaning until we provide it with a function to act upon. Furthermore, it does not “multiply” T ; rather, it is an instruction to *differentiate* what follows. To be precise, then, we should say that ∇ is a **vector operator** that *acts upon* T , not a vector that multiplies T .

With this qualification, though, ∇ mimics the behavior of an ordinary vector in virtually every way; almost anything that can be done with other vectors can also be done with ∇ , if we merely translate “multiply” by “act upon.” So by all means take the vector appearance of ∇ seriously: it is a marvelous piece of notational simplification, as you will appreciate if you ever consult Maxwell’s original work on electromagnetism, written without the benefit of ∇ .

Now an ordinary vector $\mathbf{A}$ can multiply in three ways:

1. Multiply a scalar a : $\mathbf{A}a$;
2. Multiply another vector $\mathbf{B}$, via the dot product: $\mathbf{A} \cdot \mathbf{B}$;
3. Multiply another vector via the cross product: $\mathbf{A} \times \mathbf{B}$.

Correspondingly, there are three ways the operator ∇ can act:

1. On a scalar function T : ∇T (the gradient);
2. On a vector function $\mathbf{v}$, via the dot product: $\nabla \cdot \mathbf{v}$ (the **divergence**);
3. On a vector function $\mathbf{v}$, via the cross product: $\nabla \times \mathbf{v}$ (the **curl**).

We have already discussed the gradient. In the following sections we examine the other two vector derivatives: divergence and curl.

1.2.4 The Divergence

From the definition of ∇ we construct the divergence:

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot (v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}} + v_z \hat{\mathbf{z}}) \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.\end{aligned}\tag{1.40}$$

Observe that the divergence of a vector function $\mathbf{v}$ is itself a *scalar* $\nabla \cdot \mathbf{v}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation: The name **divergence** is well chosen, for $\nabla \cdot \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ spreads out (diverges) from the point in question. For example, the vector function in Fig. 1.18a has a large (positive) divergence (if the arrows pointed *in*, it would be a large *negative* divergence), the function in Fig. 1.18b has zero divergence, and the function in Fig. 1.18c again has a positive divergence. (Please understand that $\mathbf{v}$ here is a *function*—there's a different vector associated with every point in space. In the diagrams,

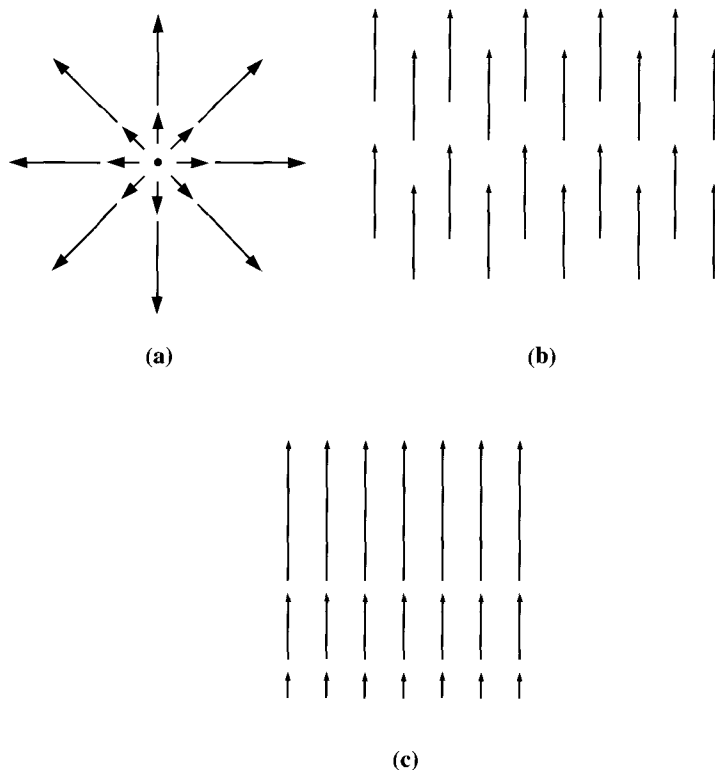


Figure 1.18

of course, I can only draw the arrows at a few representative locations.) Imagine standing at the edge of a pond. Sprinkle some sawdust or pine needles on the surface. If the material spreads out, then you dropped it at a point of positive divergence; if it collects together, you dropped it at a point of negative divergence. (The vector function $\mathbf{v}$ in this model is the velocity of the water—this is a *two-dimensional* example, but it helps give one a “feel” for what the divergence means. A point of positive divergence is a source, or “faucet”; a point of negative divergence is a sink, or “drain.”)

Example 1.4

Suppose the functions in Fig. 1.18 are $\mathbf{v}_a = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$, $\mathbf{v}_b = \hat{\mathbf{z}}$, and $\mathbf{v}_c = z\hat{\mathbf{z}}$. Calculate their divergences.

Solution:

$$\nabla \cdot \mathbf{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3.$$

As anticipated, this function has a positive divergence.

$$\nabla \cdot \mathbf{v}_b = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0,$$

as expected.

$$\nabla \cdot \mathbf{v}_c = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(z) = 0 + 0 + 1 = 1.$$

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$.

- **Problem 1.16** Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you... can you explain it?

- ! **Problem 1.17** In two dimensions, show that the divergence transforms as a scalar under rotations. [Hint: Use Eq. 1.29 to determine $\bar{v}_y$ and $\bar{v}_z$, and the method of Prob. 1.14 to calculate the derivatives. Your aim is to show that $\partial\bar{v}_y/\partial\bar{y} + \partial\bar{v}_z/\partial\bar{z} = \partial v_y/\partial y + \partial v_z/\partial z$.]

1.2.5 The Curl

From the definition of ∇ we construct the curl:

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}} \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right). \quad (1.41)\end{aligned}$$

Notice that the curl of a vector function $\mathbf{v}$ is, like any cross product, a *vector*. (You cannot have the curl of a scalar; that's meaningless.)

Geometrical Interpretation: The name **curl** is also well chosen, for $\nabla \times \mathbf{v}$ is a measure of how much the vector $\mathbf{v}$ “curls around” the point in question. Thus the three functions in Fig. 1.18 all have zero curl (as you can easily check for yourself), whereas the functions in Fig. 1.19 have a substantial curl, pointing in the z -direction, as the natural right-hand rule would suggest. Imagine (again) you are standing at the edge of a pond. Float a small paddlewheel (a cork with toothpicks pointing out radially would do); if it starts to rotate, then you placed it at a point of nonzero *curl*. A whirlpool would be a region of large curl.

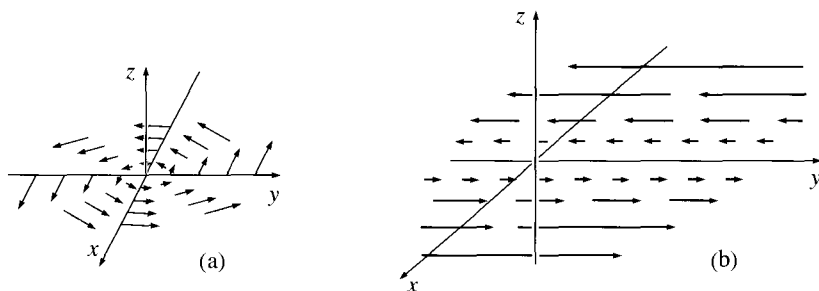


Figure 1.19

Example 1.5

Suppose the function sketched in Fig. 1.19a is $\mathbf{v}_a = -y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$, and that in Fig. 1.19b is $\mathbf{v}_b = x\hat{\mathbf{y}}$. Calculate their curls.

Solution:

$$\nabla \times \mathbf{v}_a = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\hat{\mathbf{z}},$$

and

$$\nabla \times \mathbf{v}_b = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x & 0 \end{vmatrix} = \hat{\mathbf{z}}.$$

As expected, these curls point in the $+z$ direction. (Incidentally, they both have zero divergence, as you might guess from the pictures: nothing is “spreading out”... it just “curls around.”)

Problem 1.18 Calculate the curls of the vector functions in Prob. 1.15.

Problem 1.19 Construct a vector function that has zero divergence and zero curl everywhere. (A *constant* will do the job, of course, but make it something a little more interesting than that!)

1.2.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as the sum rule:

$$\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k \frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f \frac{dg}{dx} + g \frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx} \left(\frac{f}{g} \right) = \frac{g \frac{df}{dx} - f \frac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\nabla(f + g) = \nabla f + \nabla g, \quad \nabla \cdot (\mathbf{A} + \mathbf{B}) = (\nabla \cdot \mathbf{A}) + (\nabla \cdot \mathbf{B}),$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = (\nabla \times \mathbf{A}) + (\nabla \times \mathbf{B}),$$

and

$$\nabla(kf) = k \nabla f, \quad \nabla \cdot (k\mathbf{A}) = k(\nabla \cdot \mathbf{A}), \quad \nabla \times (k\mathbf{A}) = k(\nabla \times \mathbf{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

$$\begin{aligned} fg & \quad (\text{product of two scalar functions}), \\ \mathbf{A} \cdot \mathbf{B} & \quad (\text{dot product of two vector functions}), \end{aligned}$$

and two ways to make a vector:

$$\begin{aligned} f\mathbf{A} & \quad (\text{scalar times vector}), \\ \mathbf{A} \times \mathbf{B} & \quad (\text{cross product of two vectors}). \end{aligned}$$

Accordingly, there are *six* product rules, two for gradients:

$$(i) \quad \nabla(fg) = f\nabla g + g\nabla f,$$

$$(ii) \quad \nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A},$$

two for divergences:

$$(iii) \quad \nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f),$$

$$(iv) \quad \nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

and two for curls:

$$(v) \quad \nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f),$$

$$(vi) \quad \nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$$

You will be using these product rules so frequently that I have put them on the inside front cover for easy reference. The proofs come straight from the product rule for ordinary derivatives. For instance,

$$\begin{aligned} \nabla \cdot (f\mathbf{A}) &= \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) \\ &= \left(\frac{\partial f}{\partial x}A_x + f \frac{\partial A_x}{\partial x} \right) + \left(\frac{\partial f}{\partial y}A_y + f \frac{\partial A_y}{\partial y} \right) + \left(\frac{\partial f}{\partial z}A_z + f \frac{\partial A_z}{\partial z} \right) \\ &= (\nabla f) \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A}). \end{aligned}$$

It is also possible to formulate three quotient rules:

$$\begin{aligned} \nabla \left(\frac{f}{g} \right) &= \frac{g\nabla f - f\nabla g}{g^2}, \\ \nabla \cdot \left(\frac{\mathbf{A}}{g} \right) &= \frac{g(\nabla \cdot \mathbf{A}) - \mathbf{A} \cdot (\nabla g)}{g^2}, \\ \nabla \times \left(\frac{\mathbf{A}}{g} \right) &= \frac{g(\nabla \times \mathbf{A}) + \mathbf{A} \times (\nabla g)}{g^2}. \end{aligned}$$

However, since these can be obtained quickly from the corresponding product rules, I haven't bothered to put them on the inside front cover.

Problem 1.20 Prove product rules (i), (iv), and (v).

Problem 1.21

(a) If $\mathbf{A}$ and $\mathbf{B}$ are two vector functions, what does the expression $(\mathbf{A} \cdot \nabla)\mathbf{B}$ mean? (That is, what are its x , y , and z components in terms of the Cartesian components of $\mathbf{A}$, $\mathbf{B}$, and ∇ ?)

(b) Compute $(\hat{\mathbf{r}} \cdot \nabla)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is the unit vector defined in Eq. 1.21.

(c) For the functions in Prob. 1.15, evaluate $(\mathbf{v}_a \cdot \nabla)\mathbf{v}_b$.

Problem 1.22 (For masochists only.) Prove product rules (ii) and (vi). Refer to Prob. 1.21 for the definition of $(\mathbf{A} \cdot \nabla)\mathbf{B}$.

Problem 1.23 Derive the three quotient rules.

Problem 1.24

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x \hat{\mathbf{x}} + 2y \hat{\mathbf{y}} + 3z \hat{\mathbf{z}}; \quad \mathbf{B} = 3y \hat{\mathbf{x}} - 2x \hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) The same for rule (vi).

1.2.7 Second Derivatives

The gradient, the divergence, and the curl are the only first derivatives we can make with ∇ ; by applying ∇ *twice* we can construct five species of *second* derivatives. The gradient ∇T is a *vector*, so we can take the *divergence* and *curl* of it:

(1) Divergence of gradient: $\nabla \cdot (\nabla T)$.

(2) Curl of gradient: $\nabla \times (\nabla T)$.

The divergence $\nabla \cdot \mathbf{v}$ is a *scalar*—all we can do is take its *gradient*:

(3) Gradient of divergence: $\nabla(\nabla \cdot \mathbf{v})$.

The curl $\nabla \times \mathbf{v}$ is a *vector*, so we can take its *divergence* and *curl*:

(4) Divergence of curl: $\nabla \cdot (\nabla \times \mathbf{v})$.

(5) Curl of curl: $\nabla \times (\nabla \times \mathbf{v})$.

This exhausts the possibilities, and in fact not all of them give anything new. Let's consider them one at a time:

$$\begin{aligned} (1) \quad \nabla \cdot (\nabla T) &= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial T}{\partial x} \hat{\mathbf{x}} + \frac{\partial T}{\partial y} \hat{\mathbf{y}} + \frac{\partial T}{\partial z} \hat{\mathbf{z}} \right) \\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}. \end{aligned} \tag{1.42}$$

This object, which we write $\nabla^2 T$ for short, is called the **Laplacian** of T ; we shall be studying it in great detail later on. Notice that the Laplacian of a *scalar* T is a *scalar*. Occasionally, we shall speak of the Laplacian of a *vector*, $\nabla^2 \mathbf{v}$. By this we mean a *vector* quantity whose x -component is the Laplacian of v_x , and so on:³

$$\nabla^2 \mathbf{v} \equiv (\nabla^2 v_x)\hat{\mathbf{x}} + (\nabla^2 v_y)\hat{\mathbf{y}} + (\nabla^2 v_z)\hat{\mathbf{z}}. \quad (1.43)$$

This is nothing more than a convenient *extension* of the meaning of ∇^2 .

(2) The curl of a gradient is always *zero*:

$$\nabla \times (\nabla T) = 0. \quad (1.44)$$

This is an important fact, which we shall use repeatedly; you can easily prove it from the definition of ∇ , Eq. 1.39. *Beware*: You might think Eq. 1.44 is “obviously” true—isn’t it just $(\nabla \times \nabla)T$, and isn’t the cross product of *any* vector (in this case, ∇) with itself always zero? This reasoning is *suggestive* but not quite *conclusive*, since ∇ is an *operator* and does not “multiply” in the usual way. The proof of Eq. 1.44, in fact, hinges on the equality of cross derivatives:

$$\frac{\partial}{\partial x} \left(\frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left(\frac{\partial T}{\partial x} \right). \quad (1.45)$$

If you think I’m being fussy, test your intuition on this one:

$$(\nabla T) \times (\nabla S).$$

Is *that* always zero? (It *would* be, of course, if you replaced the ∇ ’s by an ordinary vector.)

(3) $\nabla(\nabla \cdot \mathbf{v})$ for some reason seldom occurs in physical applications, and it has not been given any special name of its own—it’s just **the gradient of the divergence**. Notice that $\nabla(\nabla \cdot \mathbf{v})$ is *not* the same as the Laplacian of a vector: $\nabla^2 \mathbf{v} = (\nabla \cdot \nabla)\mathbf{v} \neq \nabla(\nabla \cdot \mathbf{v})$.

(4) The divergence of a curl, like the curl of a gradient, is *always zero*:

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0. \quad (1.46)$$

You can prove this for yourself. (Again, there is a fraudulent short-cut proof, using the vector identity $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.)

(5) As you can check from the definition of ∇ :

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v}. \quad (1.47)$$

So curl-of-curl gives nothing new; the first term is just number (3) and the second is the Laplacian (of a vector). (In fact, Eq. 1.47 is often used to *define* the Laplacian of a vector, in preference to Eq. 1.43, which makes specific reference to Cartesian coordinates.)

Really, then, there are just two kinds of second derivatives: the Laplacian (which is of fundamental importance) and the gradient-of-divergence (which we seldom encounter).

³In curvilinear coordinates, where the unit vectors themselves depend on position, they too must be differentiated (see Sect. 1.4.1).

We could go through a similar ritual to work out *third* derivatives, but fortunately second derivatives suffice for practically all physical applications.

A final word on vector differential calculus: It *all* flows from the operator ∇ , and from taking seriously its vector character. Even if you remembered *only* the definition of ∇ , you should be able, in principle, to reconstruct all the rest.

Problem 1.25 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4.$

(b) $T_b = \sin x \sin y \sin z.$

(c) $T_c = e^{-5x} \sin 4y \cos 3z.$

(d) $\mathbf{v} = x^2 \hat{\mathbf{x}} + 3xz^2 \hat{\mathbf{y}} - 2xz \hat{\mathbf{z}}.$

Problem 1.26 Prove that the divergence of a curl is always zero. Check it for function $\mathbf{v}_a$ in Prob. 1.15.

Problem 1.27 Prove that the curl of a gradient is always zero. Check it for function (b) in Prob. 1.11.

1.3 Integral Calculus

1.3.1 Line, Surface, and Volume Integrals

In electrodynamics we encounter several different kinds of integrals, among which the most important are **line** (or **path**) **integrals**, **surface integrals** (or **flux**), and **volume integrals**.

(a) **Line Integrals.** A line integral is an expression of the form

$$\int_{\mathbf{a}\mathcal{P}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l}, \quad (1.48)$$

where $\mathbf{v}$ is a vector function, $d\mathbf{l}$ is the infinitesimal displacement vector (Eq. 1.22), and the integral is to be carried out along a prescribed path $\mathcal{P}$ from point $\mathbf{a}$ to point $\mathbf{b}$ (Fig. 1.20). If the path in question forms a closed loop (that is, if $\mathbf{b} = \mathbf{a}$), I shall put a circle on the integral sign:

$$\oint \mathbf{v} \cdot d\mathbf{l}. \quad (1.49)$$

At each point on the path we take the dot product of $\mathbf{v}$ (evaluated at that point) with the displacement $d\mathbf{l}$ to the next point on the path. To a physicist, the most familiar example of a line integral is the work done by a force $\mathbf{F}$: $W = \int \mathbf{F} \cdot d\mathbf{l}$.

Ordinarily, the value of a line integral depends critically on the particular path taken from $\mathbf{a}$ to $\mathbf{b}$, but there is an important special class of vector functions for which the line integral is *independent* of the path, and is determined entirely by the end points. It will be our business in due course to characterize this special class of vectors. (A *force* that has this property is called **conservative**.)