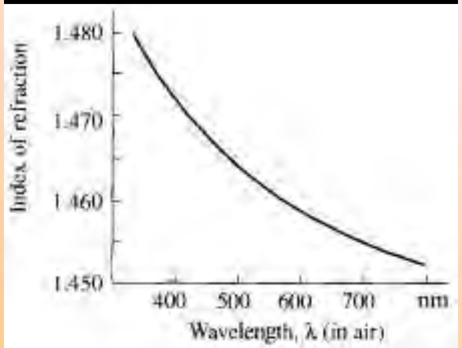
Electromagnetic Waves 3

1. Dispersion and group velocity
2. Wave polarization
3. (EM waves in plasmas)

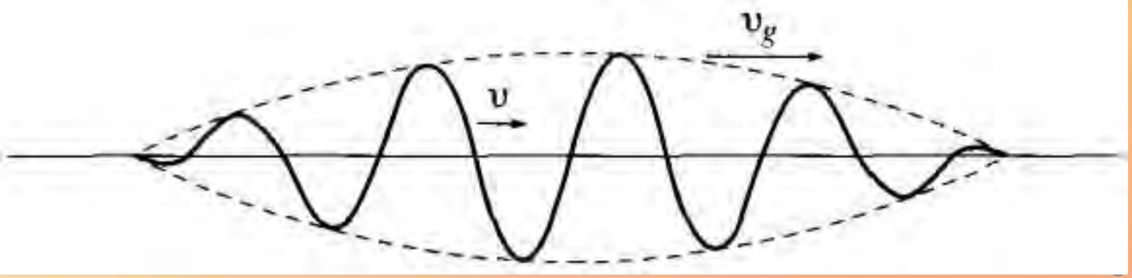
Dispersion



- **Dispersion** is the phenomenon of the **frequency dependence of refractive index**. n is not really a constant, rather $n = n(\omega)$
- A medium in which the wave speed or refractive index depends on frequency is called a **dispersive medium**, e.g. glass $\rightarrow$
- We noted earlier that for transparent media $n \cong \sqrt{K} = \sqrt{\varepsilon_r}$, and the frequency dependence of n can be explained using a model for the behaviour of electrons in dielectrics in terms of damped harmonic motion. This model leads to Cauchy's formula $n = 1 + A(1 + B/\lambda^2)$ where $A =$ coefficient of refraction , $B =$ coefficient of dispersion, accounting for the slow increase of n with frequency (or decrease with increasing wavelength, as in the graph above).

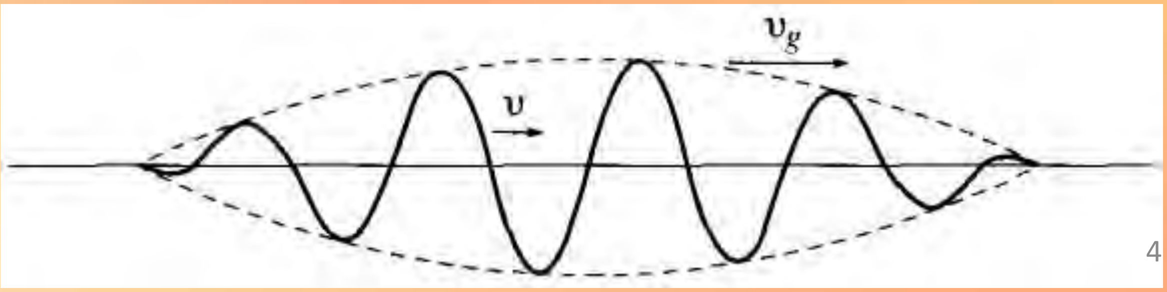
Group Velocity

- In a dispersive medium, the velocity of the wave $\boxed{v = \omega/k}$ i.e. the **phase velocity** ($\vec{v}$ of a surface of constant phase) varies with frequency, i.e. a different ω has a different $\vec{v}$
- A waveform which includes a range of frequencies (and Fourier theory says any wave train of finite length has a range of ω) will change shape as it propagates.
- While each frequency component travels at the phase velocity, the wave group, or wave packet, or envelope, travels at the **group velocity** $\boxed{v_g = \frac{d\omega}{dk}}$ and this is the speed at which energy is transported by the wave.

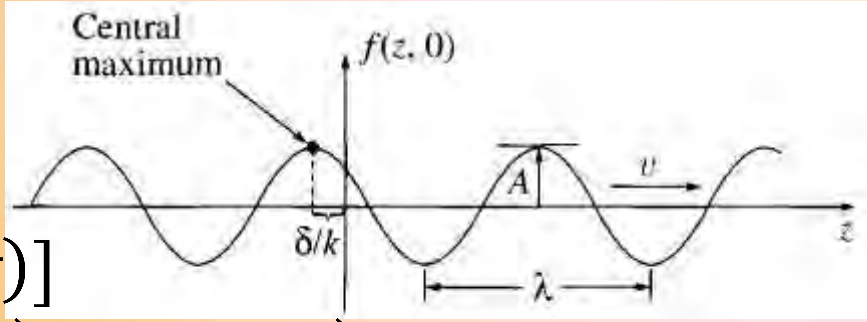


Group Velocity and Dispersion

- Group velocity $v_g = \frac{d\omega}{dk}$ and $\omega = kv$ so $v_g = v + k\frac{dv}{dk}$
- In **non-dispersive media** $dv/dk = 0$ and so $v_g = v$ (all frequencies travel at the same speed).
- In **dispersive media** where $n(k)$ is known, $\omega = k\,c/n$ and so $v_g = \frac{c}{n} - \frac{kc}{n^2}\frac{dn}{dk}$ or $v_g = v\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$
- For optical media with normal dispersion, $dn/dk > 0$ and therefore $v_g < v$: group velocity is less than phase velocity.
- The relationship between ω and k , i.e. $\omega(k)$, is called a **dispersion relation**. e.g. plane EM waves in a conductor obey a dispersion relation (look again at the last slide of ED-08)



A Note on Complex Amplitudes



- We have written a plane wave as e.g. $\vec{E}\,(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$

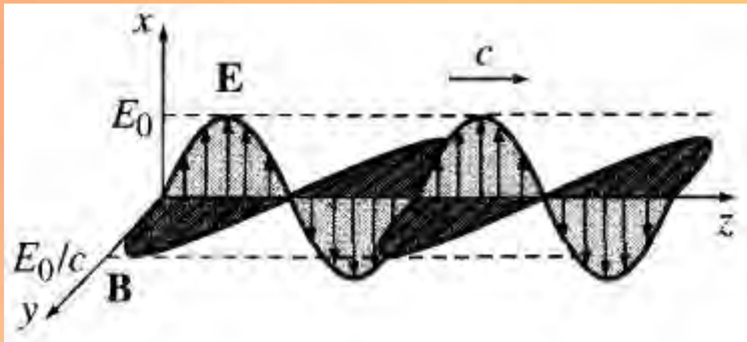
$= \vec{E}_0[\cos(kz-\omega t) + i\sin(kz-\omega t)]$

with real electric field given by $\vec{E}\,(z,t) = \vec{E}_0\cos(kz-\omega t)$ and the amplitude $\vec{E}_0$ appearing in both these eqns. is real.

- In general a wave may have initial phase angle (or "phase constant" or "phase offset") denoted by ϕ or δ, and the wave function is $\vec{E}_0 e^{i(kz-\omega t+\delta)}$ with real part $\vec{E}_0\cos(kz-\omega t+\delta)$. This can be written as $\vec{E}_0 e^{i\delta} e^{i(kz-\omega t)}$. Now we can treat $\vec{E}_0 e^{i\delta}$ as **complex amplitude** $\tilde{E}_0$ with modulus $\vec{E}_0$ = real amplitude, argument δ = phase constant. Then the complex wave fn. is $\vec{E}\,(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$. Ignoring the imaginary part of $\tilde{E}_0$ is equivalent to putting phase constant $\delta = 0$.

Polarization of EM Waves

- So far we have only encoutered linearly polarized waves, where the direction of $\vec{E}$ is constant. We now generalise...
- We can take the direcion of wave propagation to be the z-axis as before, without any loss of generality. Then a plane wave is $\vec{E}(z,t) = \left(\tilde{E}_x\hat{x} + \tilde{E}_y\hat{y}\right)e^{i(kz-\omega t)}$
  where we now use **complex amplitudes** because we can have a **phase difference** between x and y components.
- The **relationship between $\tilde{\boldsymbol{E}}_x$ and $\tilde{\boldsymbol{E}}_y$** describes the **state of polarization**, e.g. $\tilde{E}_y = 0 \Rightarrow$ linearly polarized in $\hat{x}$ dir.
- Often the most convenient form is to have one amplitude real and a *relative phase difference* between the two components.



Circular Polarization

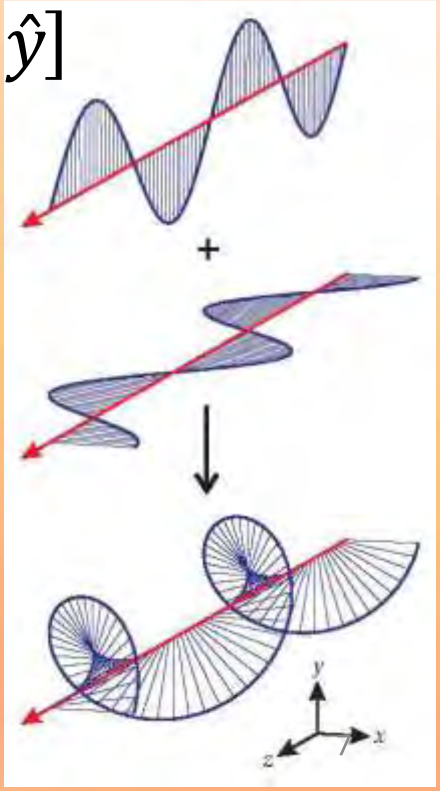
- Consider as an example $\tilde{E}_y = i\tilde{E}_x$ with $\tilde{E}_x$ real. This means $\tilde{E}_y$ is out of phase with $\tilde{E}_x$ by $i = e^{i\pi/2}$ or $\delta = \frac{\pi}{2}$.

The real field of the plane wave is

$$\vec{E}(z,t) = \mathrm{Re}\left[E_x e^{i(kz-\omega t)}\hat{x} + E_x e^{i\pi/2} e^{i(kz-\omega t)}\hat{y}\right]$$

$$= \left[E_x \cos(kz-\omega t)\,\hat{x} + E_x \cos(kz-\omega t+\pi/2)\,\hat{y}\right]$$

$$= E_x\left[\cos(kz-\omega t)\,\hat{x} - \sin(kz-\omega t)\,\hat{y}\right]$$



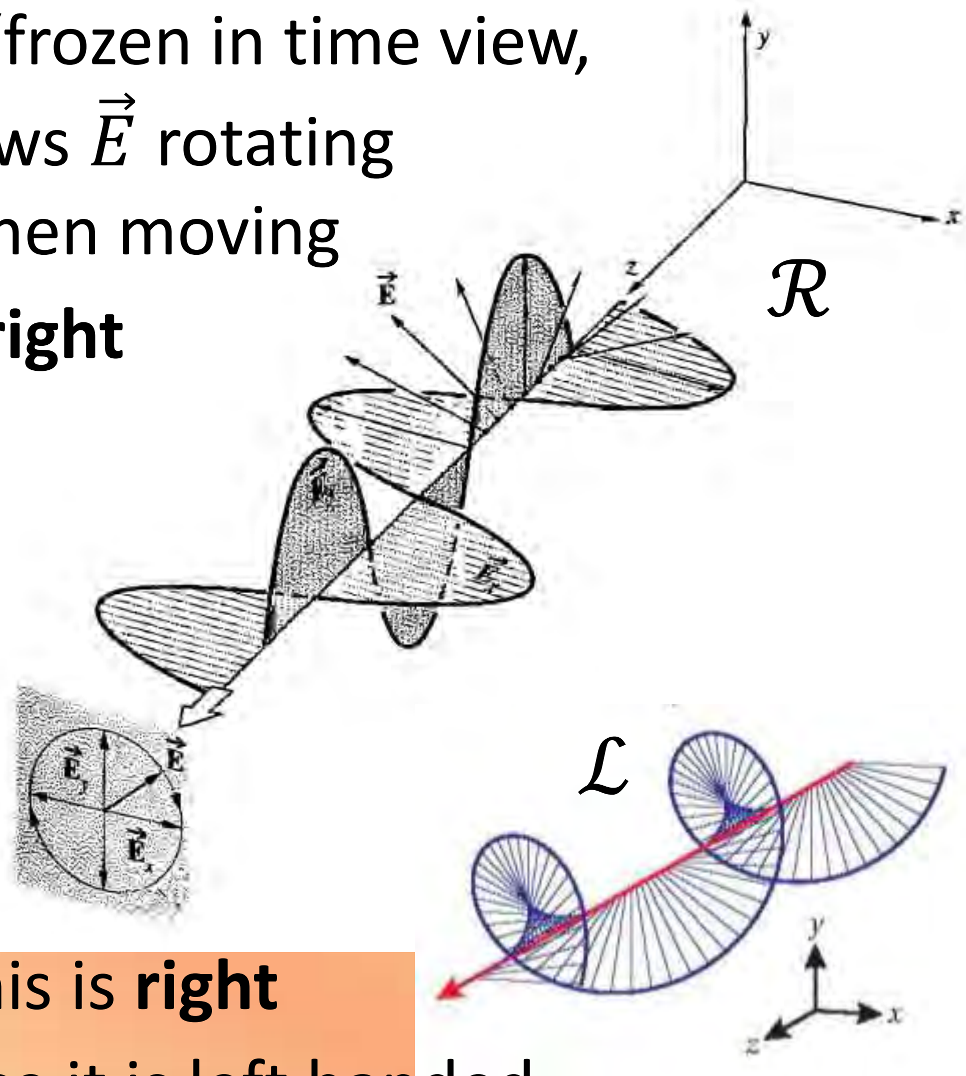
So the field in the y-dirn. lags the x-dirn. by $1/4$ cycle. The magnitude of $\vec{E}$ is constant and $\vec{E}$ rotates in a circle in the xy plane.

This is (left) **circularly polarized light**.

e.g. VLF “whistler mode” waves in the plasmasphere are left circularly polarized.

Left and Right Handedness

- Either: if a “snapshot” or “frozen in time view, as on the previous slide, shows $\vec{E}$ rotating as a “right handed screw” when moving in the direction of $\vec{k}$, this is **right handed**; if in the opposite direction, it is left handed.



- Or: imagine looking along the $\vec{k}$ direction and seeing the light falling on a screen (i.e. fixed z, varying with t); if $\vec{E}$ rotates **anticlockwise**, this is **right handed**, if $\vec{E}$ rotates clockwise it is left handed.

These are “ $\mathcal{R}$-state” and “$\mathcal{L}$-state” polarization respectively.

Left and Right Handedness

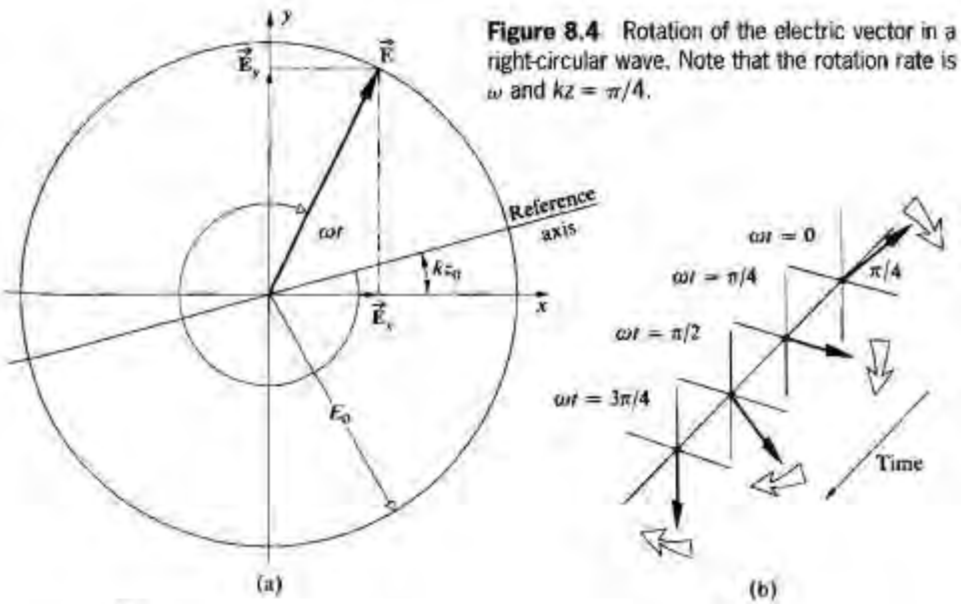
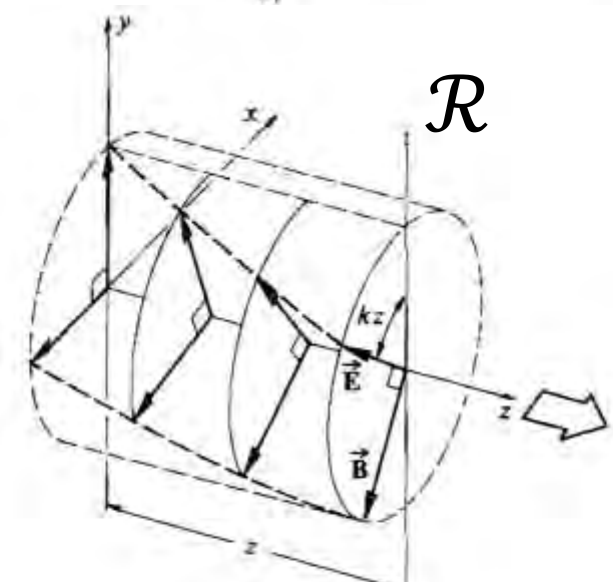
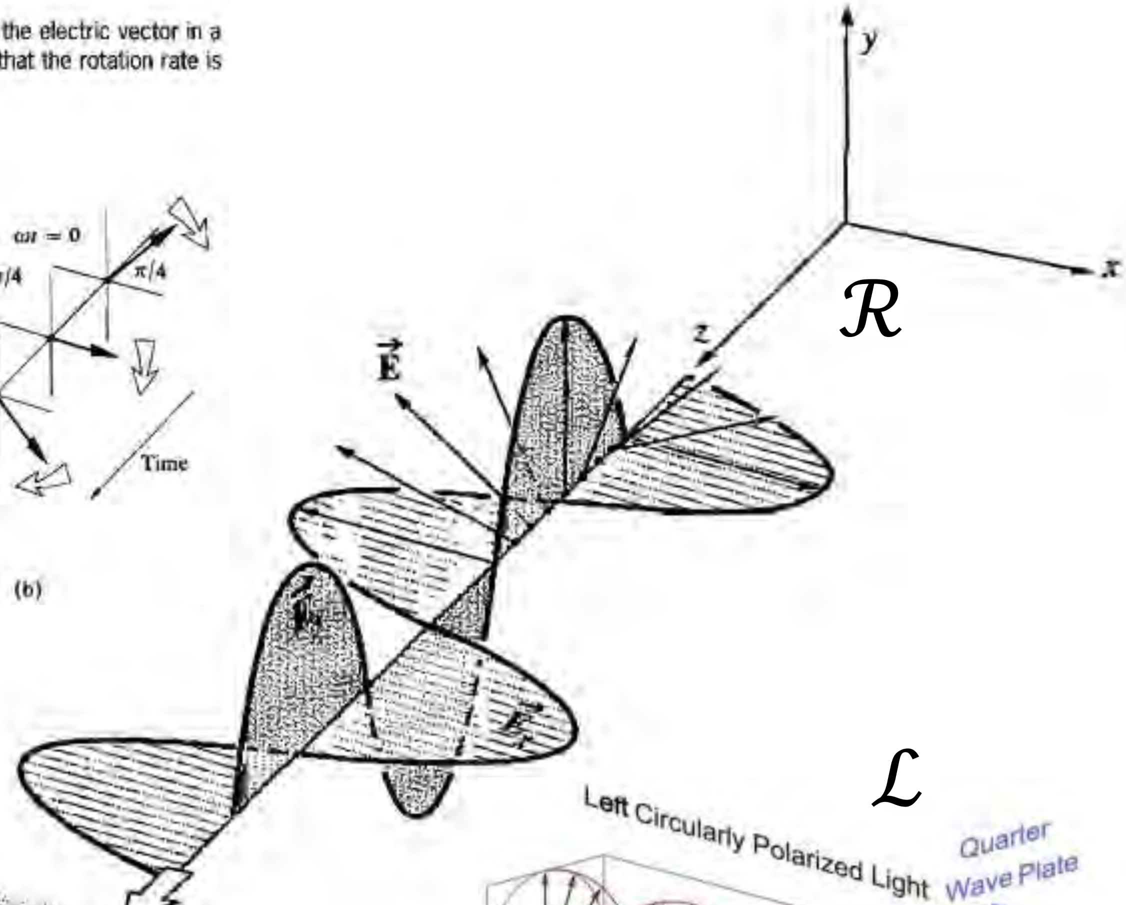
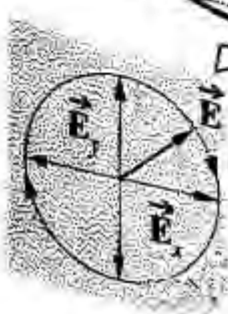
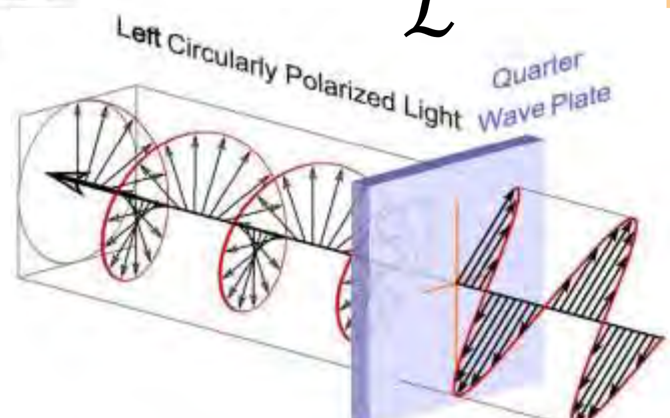


Figure 8.4 Rotation of the electric vector in a right-circular wave. Note that the rotation rate is ω and $kz = \pi/4$.









Elliptical Polarization

- More generally $\vec{E}$ can rotate **and** change in magnitude
- One can show that then the components obey the eqn.

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - 2\left(\frac{E_x}{E_{0x}}\right)\left(\frac{E_y}{E_{0y}}\right)\cos\delta = \sin^2\delta$$

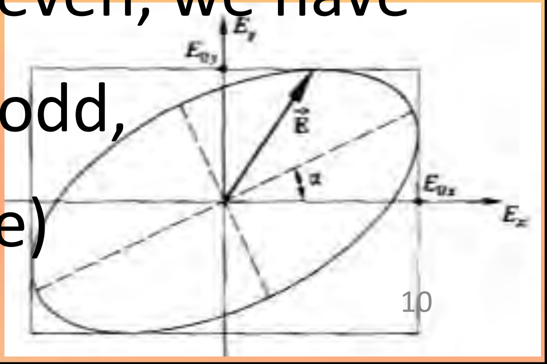
$\Rightarrow$ ellipse at angle α to E_x where $\tan 2\alpha = \frac{2E_{0x}E_{0y}\cos\delta}{{E_{0x}}^2 - {E_{0y}}^2}$

Put $\alpha = 0$ (or $\delta = \pm\pi/2, \pm 3\pi/2, \ldots$) $\rightarrow$ familiar eqn. for an **ellipse** $\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 = 1$ and if $E_{0x} = E_{0y} = E_0$ we get a

circle ${E_x}^2 + {E_y}^2 = {E_0}^2$. If $\delta = n\pi$, n even, we have

linear $E_y = (E_{0y}/E_{0x})E_x$ If $\delta = n\pi$, n odd,

linear $E_y = -(E_{0y}/E_{0x})E_x$ (negative slope)



Elliptical Polarization

- Elliptical polarization is "$\mathcal{E}$-state" (special cases $\mathcal{L}$, $\mathcal{R}$, $\mathcal{P}$)
- In the chart below, δ is the angle "E_x leads E_y by":

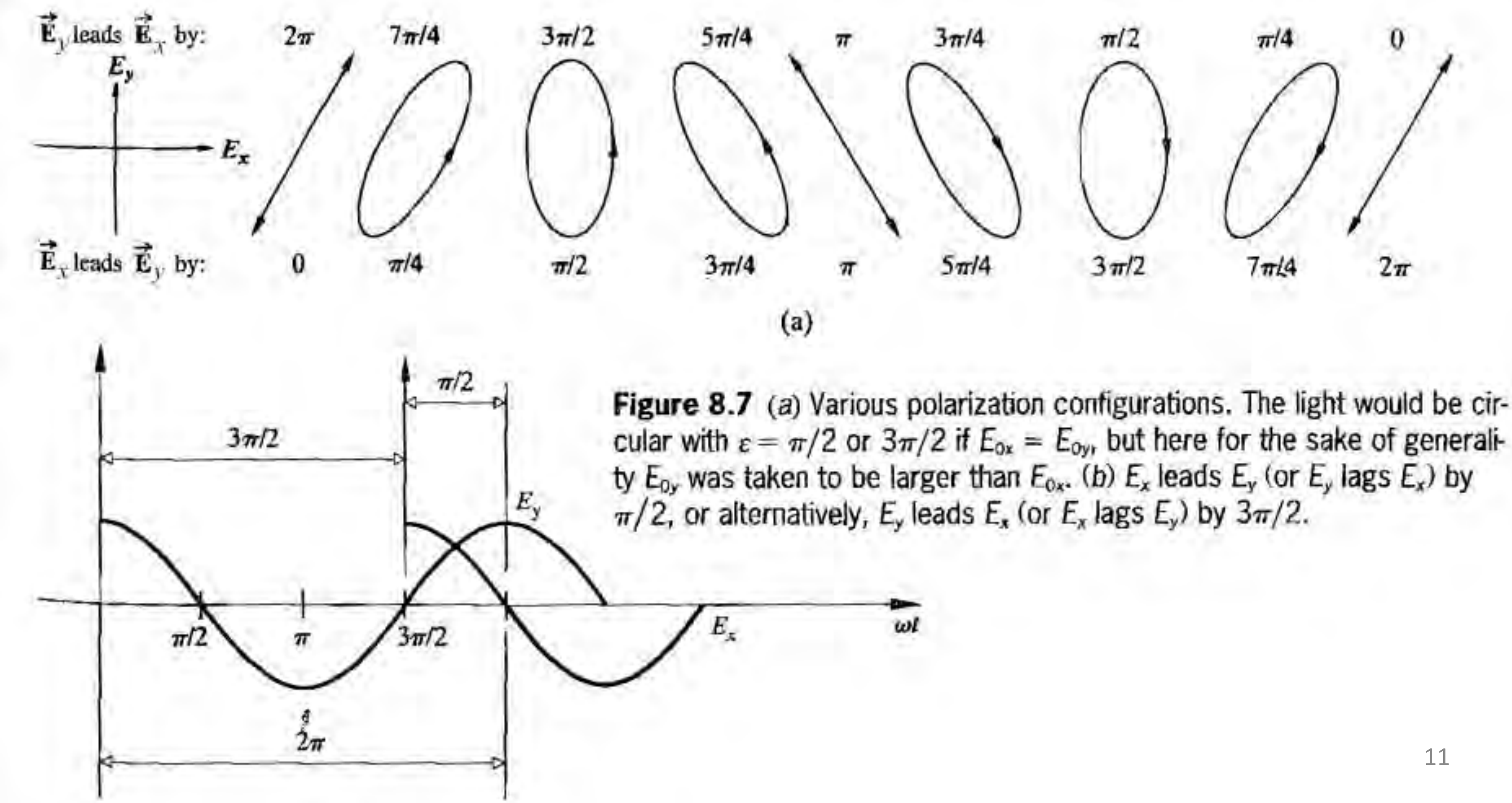


Figure 8.7 (*a*) Various polarization configurations. The light would be circular with $\varepsilon = \pi/2$ or $3\pi/2$ if $E_{0x} = E_{0y}$, but here for the sake of generality E_{0y} was taken to be larger than E_{0x}. (*b*) E_x leads E_y (or E_y lags E_x) by $\pi/2$, or alternatively, E_y leads E_x (or E_x lags E_y) by $3\pi/2$.

Elliptical Polarization

- Determine the major and minor axes of the ellipse by finding the angle (w.r.t. x-axis) at which an intensity max. or min. occurs.
- Define $\tilde{E}_{\text{eff}} = \sqrt{|E_x|^2 + |E_y|^2} e^{-i\phi_x}$ [‘effective’ $\vec{E}$]

$A = |E_x|/|\tilde{E}_{\text{eff}}|$, $B = |E_y|/|\tilde{E}_{\text{eff}}|$, $\delta = \phi_y - \phi_x$

- An axis (may be major or minor) occurs at

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2AB\cos\delta}{A^2 - B^2}\right)$$

- The other axis (minor or major) is at $\alpha \pm \frac{\pi}{2}$
- Find magnitude of $\vec{E}$ at α and at $\alpha \pm \pi/2$:

$$E(\alpha) = |E_{\text{eff}}|\sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha + AB\cos\delta\sin 2\alpha}$$

$$E\left(\alpha \pm \frac{\pi}{2}\right) = |E_{\text{eff}}|\sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha - AB\cos\delta\sin 2\alpha}$$

The larger of these must be the major axis.

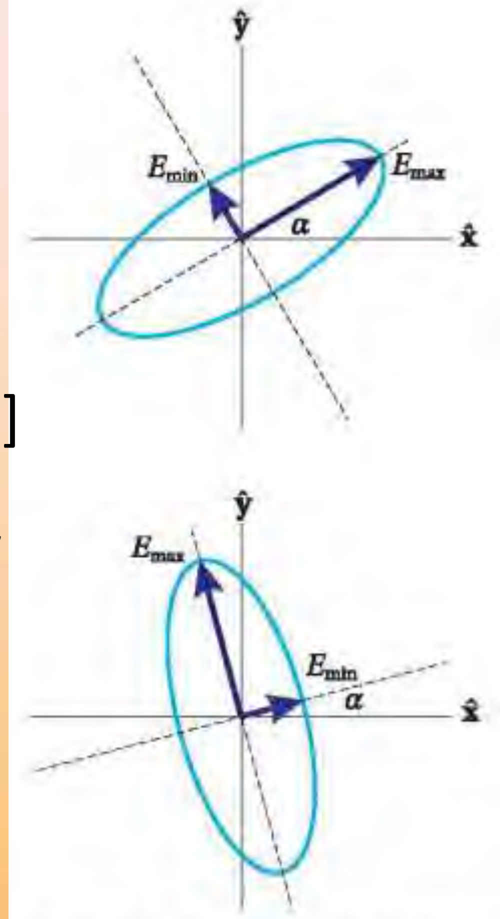
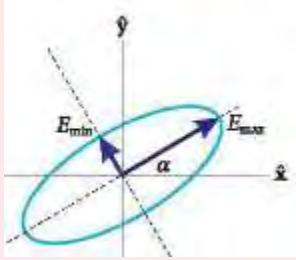


Figure 6.3 The electric field of elliptically polarized light traces an ellipse in the plane perpendicular to its propagation direction. The two plots are for different values of A, B, and δ. The angle α can describe the major axis (top) or the minor axis (bottom), depending on the values of these parameters.

Elliptical Polarization



Elliptically polarized light can be characterized by

- **ellipticity** = ratio of minor axis to major axis: $e = E_{\mathrm{min}}/E_{\mathrm{max}}$ where $0 \leq e \leq 1$ [$0 =$ linear , $1 =$ circular polarization]
- **helicity** or "handedness", determined by the value of δ : $0 < \delta < \pi$ $\rightarrow$ left , $\pi < \delta < 2\pi$ $\rightarrow$ right

Example: Elliptically polarized light with $\vec{E}$-field components

$E_x = |\tilde{E}_{\mathrm{eff}}|/\sqrt{2}$; $E_y = (|\tilde{E}_{\mathrm{eff}}|/\sqrt{2}\,)e^{i\pi/4}$

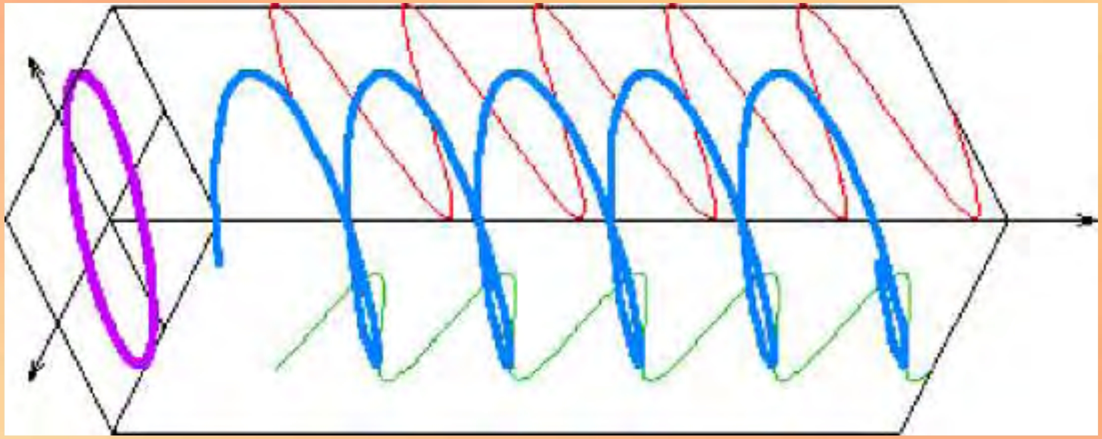
We find $\alpha = \pi/4 = 45°$; calculating E gives

$E(\alpha) = 0.924\,|E_{\mathrm{eff}}|$ (major)

$E\left(\alpha - \frac{\pi}{2}\right) = 0.380\,|E_{\mathrm{eff}}|$

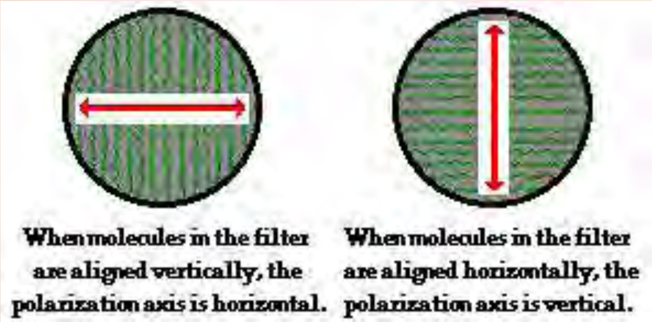
Then the ellipticity is

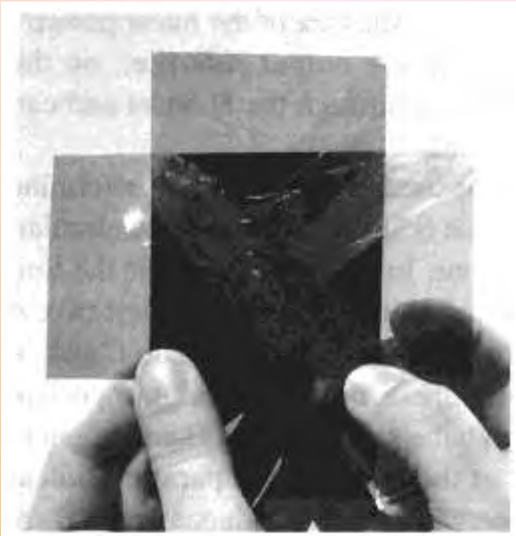
$$e = \frac{E_{\mathrm{min}}}{E_{\mathrm{max}}} = \frac{0.380}{0.924} = 0.41$$

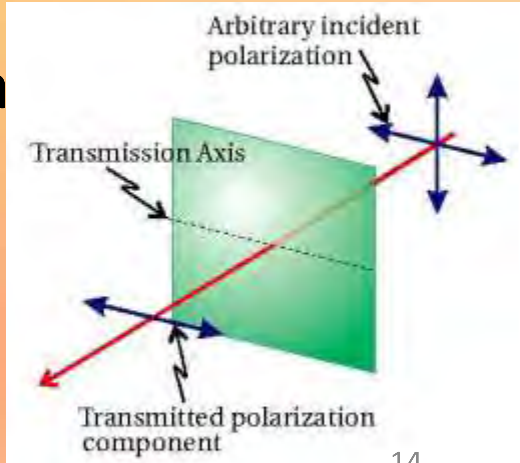


Linear Polarizers

- "Polaroid" is an example of a **linear polarizer**. It has long molecules (polymer chains) in a particular direction. When $\vec{E}$ ∥ molecules, they conduct and light is absorbed; when $\vec{E}$ ⊥ , no current and no absorption: light is **transmitted**.
- Direction in which light transmitted = **transmission axis** (for polaroid this is ⊥ polymer chains; there are many others)
- A polarizer changes the state of polarization by only allowing $\vec{E}$ components ∥ the trans-mission axis. e.g. If transmission axis is in x-direction, only E_x emerges and $E_y \rightarrow 0$.
- Linear ("plane") polarization is "$\mathcal{P}$-state".

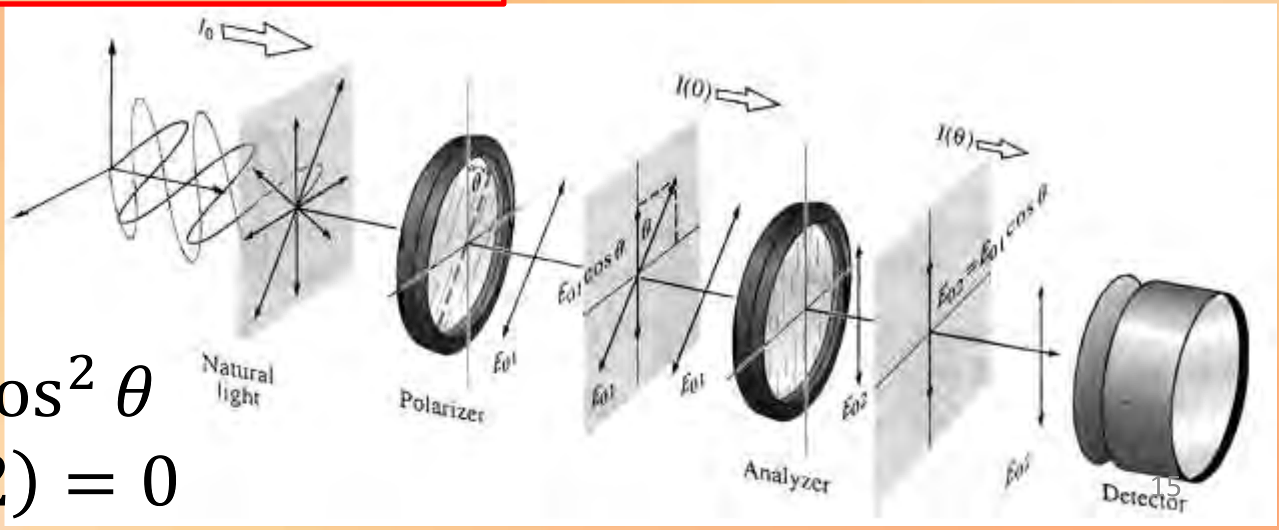






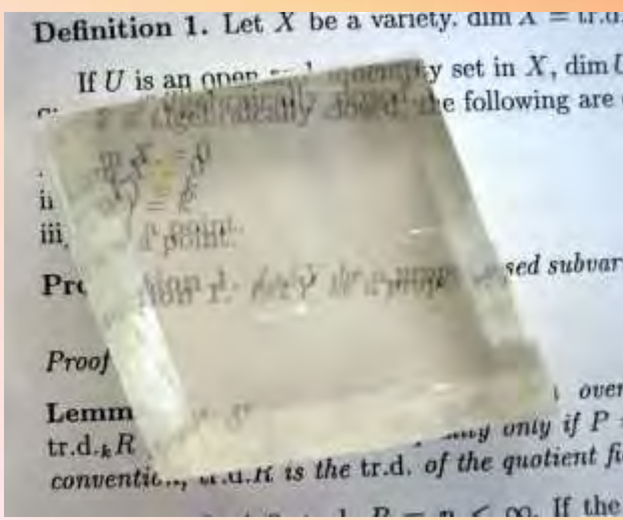
Malus's Law

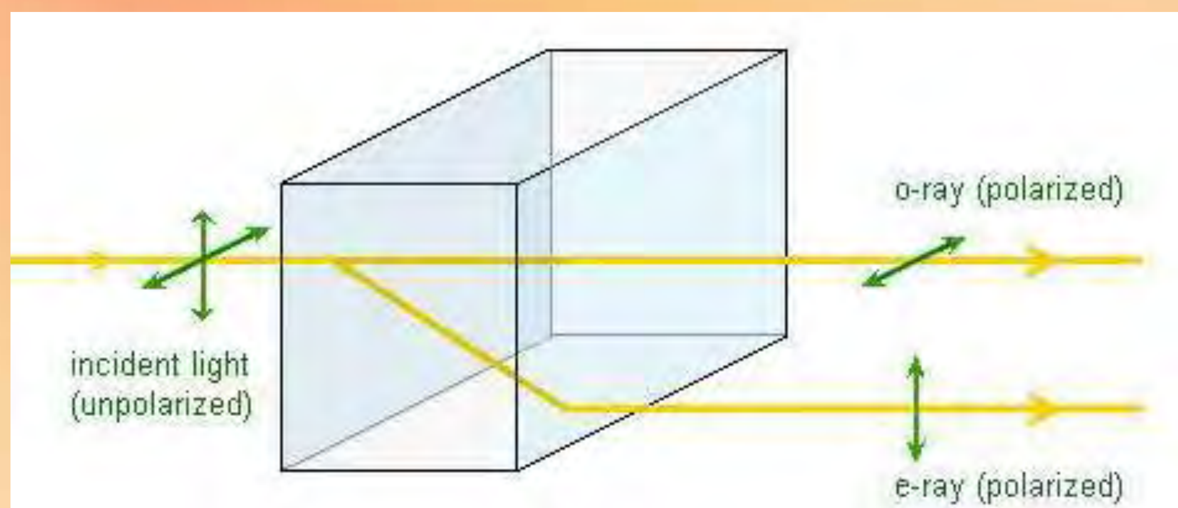
- Suppose "natural light" or "unpolarized light" with intensity I_0 is incident on linear polarizer; we get out light linearly polarized in the direction of the optic axis, intensity $I_0/2$.
- If this light is then incident on another polarizer, the "analyzer", with its axis at angle θ to the first, only the component of $\vec{E}$ ∥ axis is transmitted, i.e. $E\cos\theta$.
- The intensity of the output light is then $\propto \cos^2\theta$; this is **Malus's law**: $\boxed{I(\theta) = I(0)\cos^2\theta}$ with $I(0) =$ maximum.
- Overall, the final output intensity from the two polarizers is $I(\theta) = (I_0/2)\cos^2\theta$
- Of course $I(\pi/2) = 0$



Wave Plates

- A “**wave plate**” or “**retarder**” is typically made from a birefringent crystal, which has two values of n , depending on the direction of polarization (i.e. direction of $\vec{E}$)
- Such a crystal (e.g. calcite, below) has a “**fast axis**” and a “**slow axis**”, such that if $\vec{E}$ ∥ fast axis, $n = n_{\text{fast}}$,
  and if $\vec{E}$ ∥ slow axis (i.e. $\perp$ fast axis), $n = n_{\text{slow}}$,
  where $n_{\text{slow}} > n_{\text{fast}}$ [so that $v_{\text{slow}} < v_{\text{fast}}$ since $n = c/v$]
  e.g. calcite has $n_{\text{slow}} = 1.658$ and $n_{\text{fast}} = 1.486$



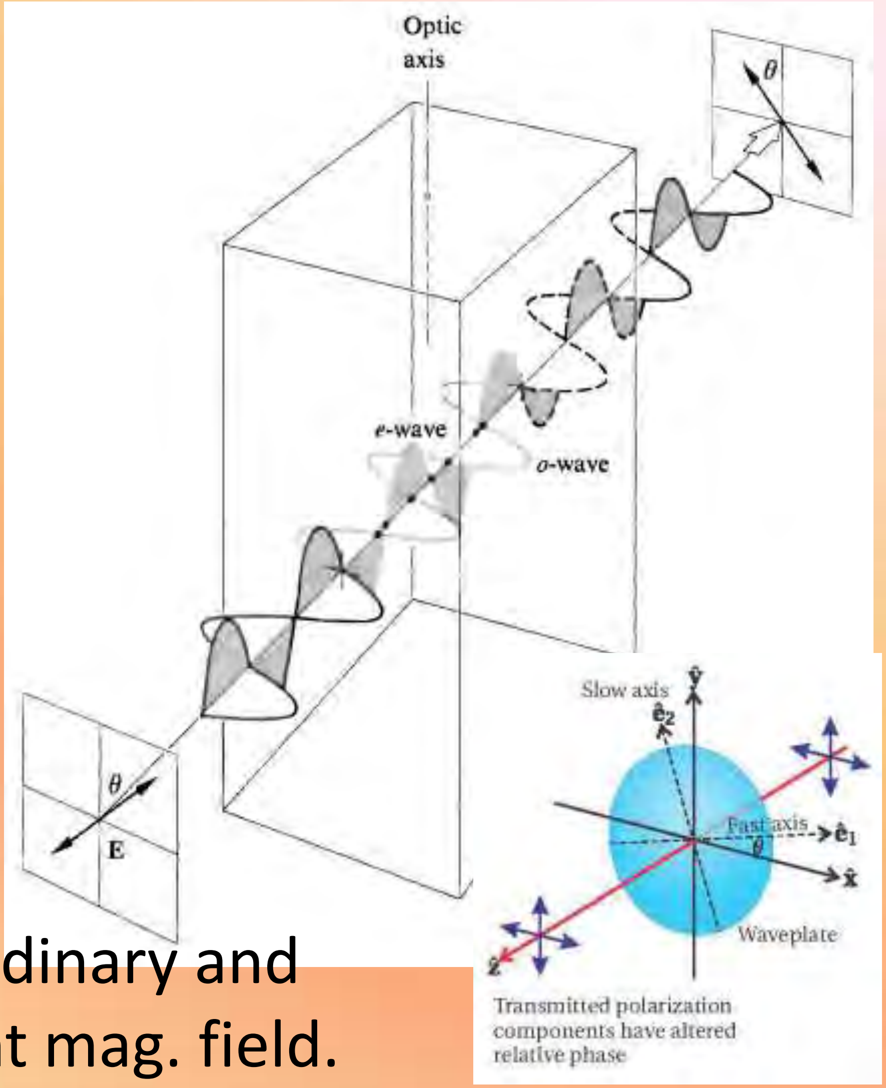


Wave Plates

- For light polarized at some angle θ, the two components (**fast** and **slow** or “ordinary” and “extraordinary” [“o” and “e”]) travel at different speeds, so the effect of a wave plate is to introduce a **phase difference** δ between the two components

$$\delta = k_{\mathrm{slow}} d - k_{\mathrm{fast}} d$$
$$= (2\pi d/\lambda_0)(n_{\mathrm{slow}} - n_{\mathrm{fast}})$$

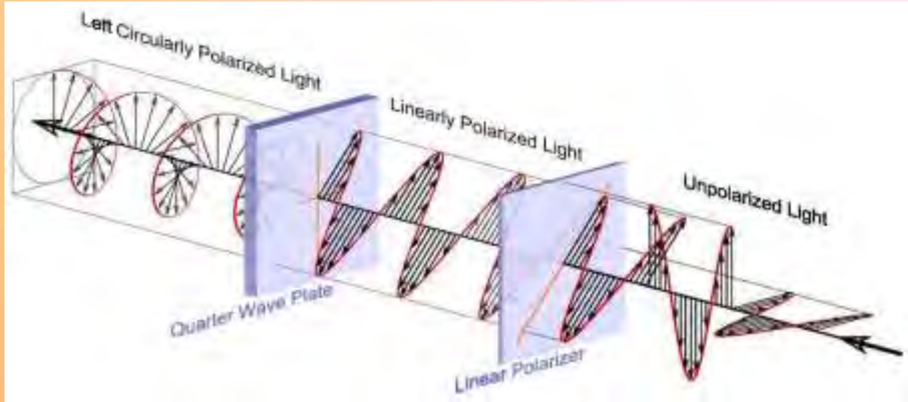
  where d is the thickness of the plate, which can be adjusted for any desired δ , and λ_0 is the wavelength in vacuum.



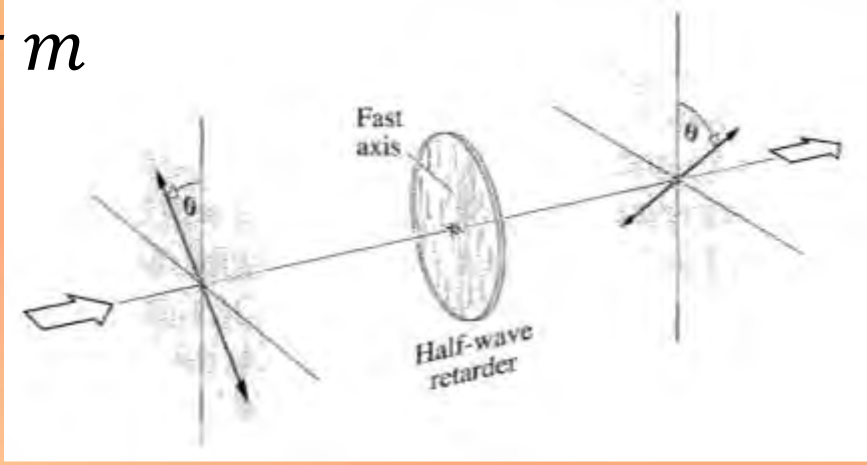
- The ionosphere is birefringent; the phase difference between ordinary and extraordinary waves is ∝ ambient mag. field.

Wave Plates

- **Quarter wave plate** produces a phase difference
  $$\delta = k_{\text{slow}}d - k_{\text{fast}}d = \pi/2 + 2\pi\, m$$
  where m is an integer ($m = 0$ is 'zero order') so the polarization component along the slow axis is delayed by $\lambda/4$ (for $m = 0$).
  linear $\rightarrow$ circular polarization

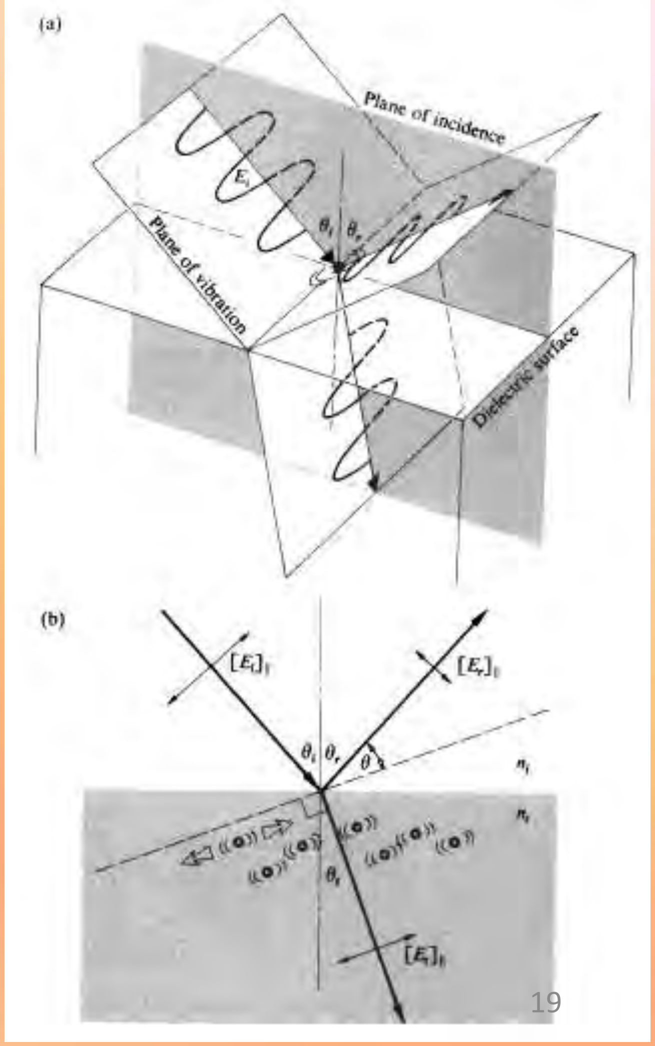


- **Half wave plate** produces a phase difference
  $$\delta = k_{\text{slow}}d - k_{\text{fast}}d = \pi + 2\pi\, m$$
  where m is an integer ($m = 0$ is 'zero order') so the polarization component along the slow axis is delayed by $\lambda/2$ (for $m = 0$).
  linear $\rightarrow$ linear polarization at 2θ



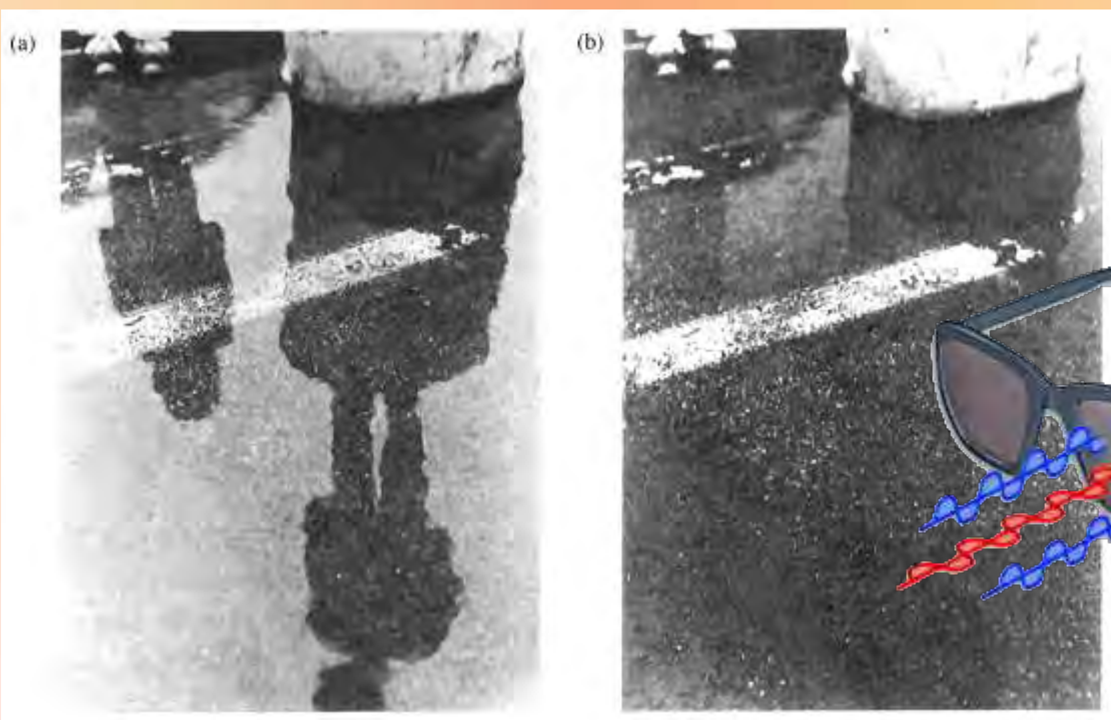
Polarization by Reflection

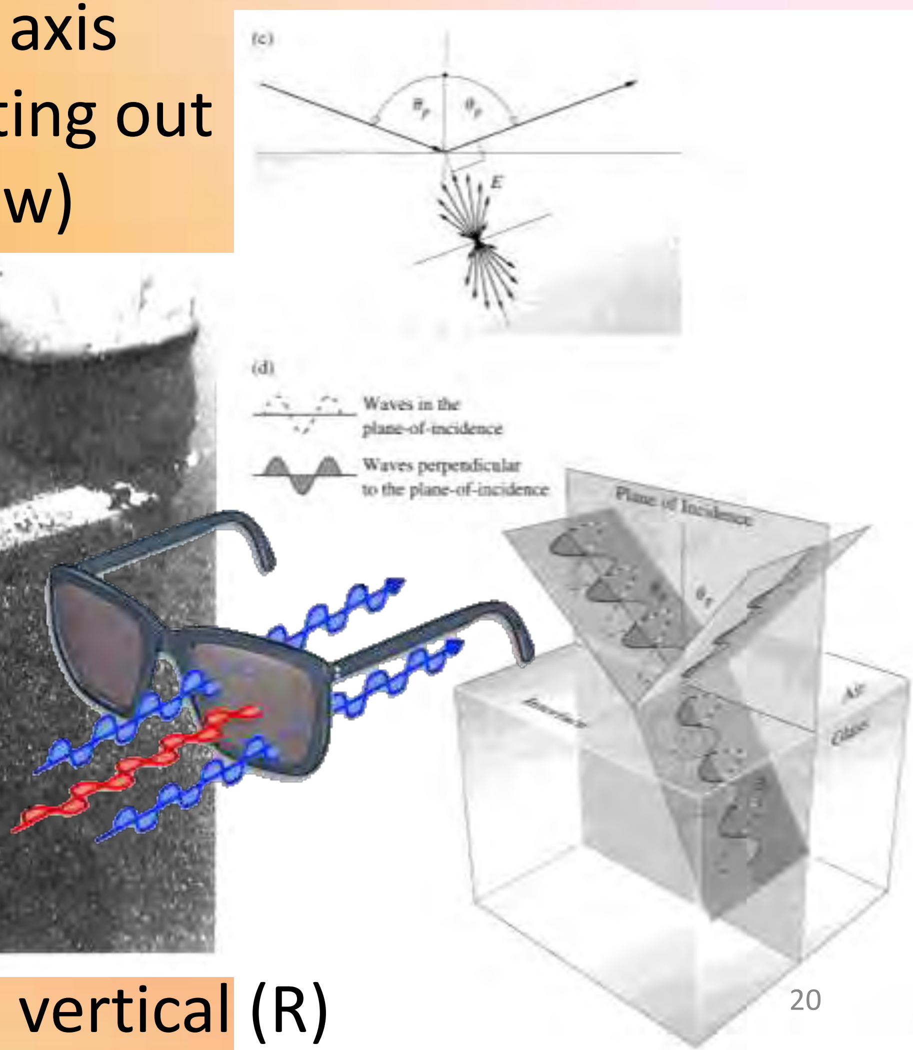
- If $\vec{E}$ is $\perp$ plane of incidence, the bound electrons in the material oscillate and re-radiate, with both reflected and transmitted waves $\perp$ PoI.
- However, if $\vec{E}$ is ∥ plane of incidence, the oscillations of the bound electrons are not as effective in contributing to the reflected wave because of the small angle to the dipole axis.
- At the Brewster angle (only for $\vec{E}$ ∥) of course there is no reflected wave.
- This means that the reflected wave has a smaller proportion ∥ PoI; it is mostly **polarized** $\perp$ plane of incidence, i.e. **parallel to the surface**.



Polarization by Reflection

- So the reflected wave is (at least partially) polarized parallel to the surface. If the surface is horizontal then a polarizer with its transmission axis vertical will be effective in cutting out this reflected "glare" (see below)

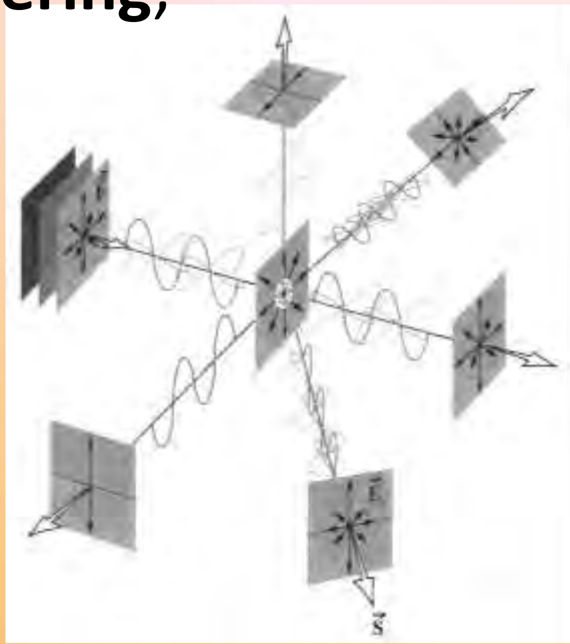




Polaroid axis horizontal (L) and vertical (R)

Polarization by Scattering

- Scattering of light by objects small compared to the wavelength is known as **Rayleigh scattering**; it's what makes the sky appear blue. [scattered intensity $\propto \omega^4$]
- Consider unpolarized light incident on an air molecule; it can be represented by two orthogonal (and incoherent) $\mathcal{P}$-states, both of which result in linearly polarized light scattered **perpendicular to the incident direction** [see figure].



- This can be observed as shown in the photo, looking in a direction at $90°$ to the direction of the Sun. The upper polaroid is darker, indicating partial polarization.

Partial Polarization; Stokes Parameters

- We often have light which is partially polarized, rather than all polarized in a particular way. Stokes parameters describe this.
- Operational definition: imagine 4 filters, each of which will transmit 50% of natural light, as follows (intensity out given):
  (1) isotropic, all polarization states transmitted equally : I_0
  (2) linear polarizer, axis horizontal : I_1
  (3) linear polarizer, axis at $45°$: I_2
  (4) circular polarizer, extinguishing $\mathcal{L}$-states : I_3

- The Stokes parameters are then defined in terms of these I's, and related to the electric fields by

$$\mathcal{S}_0 = 2I_0 \equiv \langle E_{0x}{}^2 \rangle + \langle E_{0y}{}^2 \rangle \quad ;$$
$$\mathcal{S}_1 = 2I_1 - 2I_0 \equiv \langle E_{0x}{}^2 \rangle - \langle E_{0y}{}^2 \rangle$$
$$\mathcal{S}_2 = 2I_2 - 2I_0 \equiv \langle 2E_{0x}E_{0y} \cos\delta \rangle \; ;$$
$$\mathcal{S}_3 = 2I_3 - 2I_0 \equiv \langle 2E_{0x}E_{0y} \sin\delta \rangle$$

Partial Polarization; Stokes Parameters

- Note the "equivalence" symbol in the equations: we have dropped the constant $\varepsilon_0 c/2$; and δ here is the phase difference between $\tilde{E}_x$and $\tilde{E}_y$, i.e. $\delta = \phi_y - \phi_x$
- The Stokes parameters are usually **normalised** by dividing by $\mathcal{S}_0$; this means effectively that we have an incident beam with "unit intensity". With these normalised parameters...
- Natural light (unpolarized) has **Stokes vector** $\begin{bmatrix} \mathcal{S}_0 \\ \mathcal{S}_1 \\ \mathcal{S}_2 \\ \mathcal{S}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- Others: horiz. $\mathcal{P}$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ vert. $\mathcal{P}$ $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ circ. $\mathcal{R}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ circ. $\mathcal{L}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

(usually written in this way as a column vector)

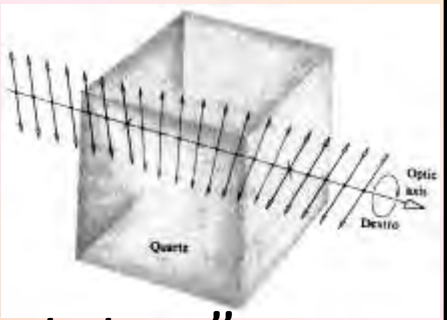
Partial Polarization; Stokes Parameters

- The **degree of polarization** is defined as $V = I_p/(I_p + I_u)$ where I_p and I_u are polarized and unpolarized intensities. It varies between 0 (all unpolarized) and 1 (all polarized).
- V can be obtained from the Stokes parameters as

$$V = \sqrt{\mathcal{S}_1{}^2 + \mathcal{S}_2{}^2 + \mathcal{S}_3{}^2}/\mathcal{S}_0$$

- When light passes through a device which changes its state of polarization (e.g. a quarter wave plate), this can be represented as a matrix operating on the Stokes vector to generate a new vector representing the output. These are known as Mueller matrices. There is also a system of 2D vectors, the Jones vectors, operated on by 2×2 Jones matrices representing the devices, but this system only works for completely polarized light ($V = 1$).

Optical Activity; Faraday Rotation



- There are numerous "**optically active**" materials which cause the direction of polarization of an EM wave to rotate. If the rotation is clockwise looking in the direction of the source, this is "*d-rotatory*" (right handed); if anticlockwise, it is "*l-rotatory*" (left handed).
- Faraday observed the rotation of the plane of polarization when a **strong magnetic field** was applied to a material. If a linearly polarized wave is viewed as the superposition of left and right circularly polarized waves, one finds these travel at slightly different speeds, causing the plane of polarization to rotate. The resulting eqn. is $\Delta\varphi = \frac{e^3}{2\varepsilon_0 {m_e}^2 c\, \omega^2} \int n_e(z) B_0(z) dz$ so the rotation angle depends on the integral of electron density × magnetic field along the path. This is used in astronomy (extensively) and ionospheric physics.

References for Polarization

- Eugene Hecht, *Optics* , 4th Edition, Addison-Wesley, 2002
- Justin Peatross & Michael Ware, *Physics of Light and Optics*, Brigham Young University, free online textbook plus resources (animations etc.) at optics.byu.edu