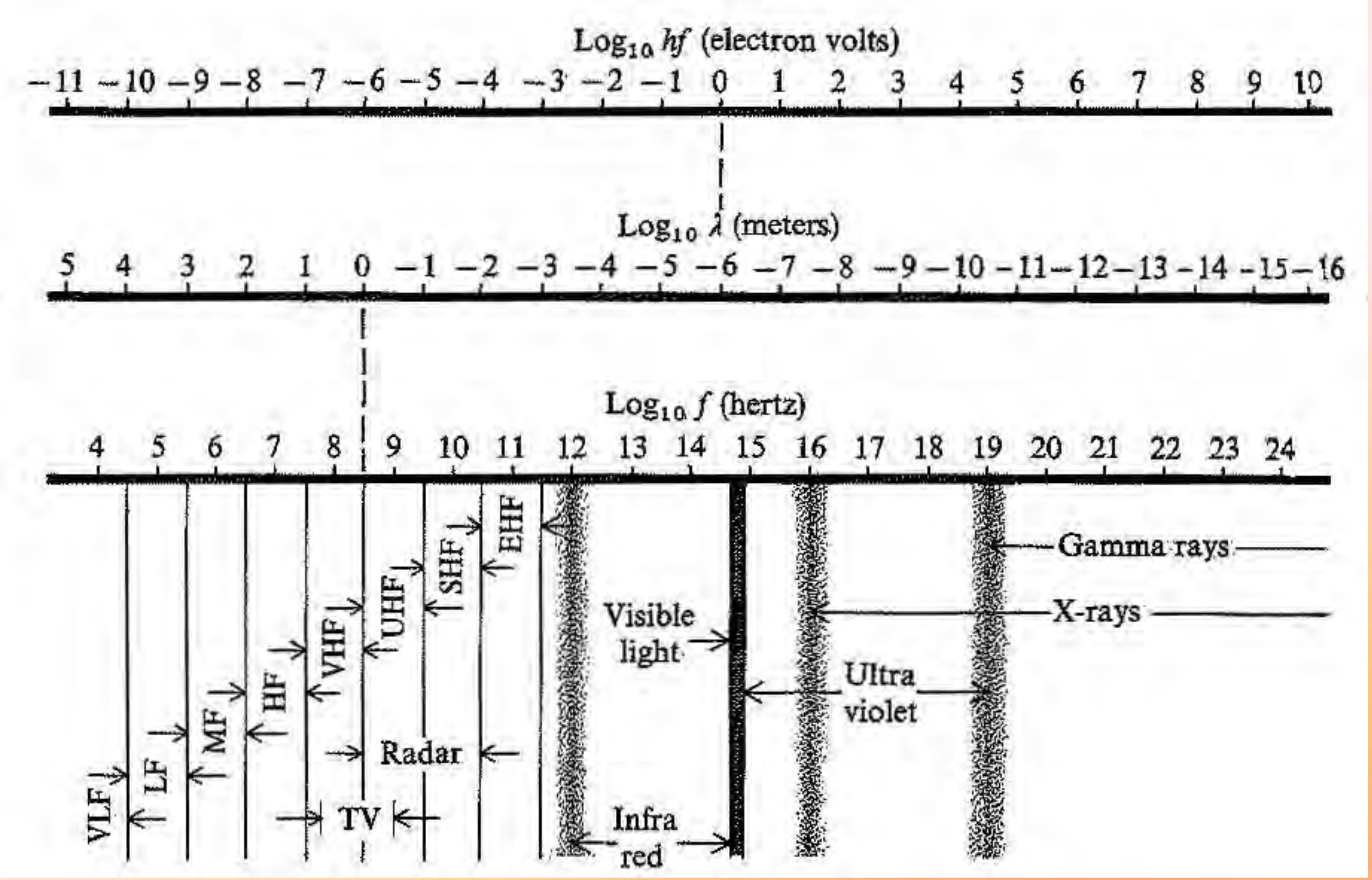
Electromagnetic Waves 2

1. Plane EM waves in linear materials
2. Reflection and refraction at materials interfaces
3. Fresnel equations
4. Plane EM waves in conductors

EM Spectrum



Log$_{10}$ hf (electron volts)
−11 −10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6 7 8 9 10
Log$_{10}$ λ (meters)
5 4 3 2 1 0 −1 −2 −3 −4 −5 −6 −7 −8 −9 −10 −11 −12 −13 −14 −15 −16
Log$_{10}$ f (hertz)
4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
VLF
LF
MF
HF
VHF
UHF
SHF
EHF
Radar
TV
Visible light
Infra red
Ultra violet
X-rays
Gamma rays

EM Waves in l.i.h. Materials

- For l.i.h. materials, the same arguments leading the wave eqns. for $\vec{E}$ and $\vec{B}$ apply as for free space; all we do is replace ε_0 and μ_0 by ε and μ for the material. The wave eqns. are (i) inhomogeneous wave eqns. (with sources)

$$\boxed{\begin{aligned} \nabla^2\vec{E} &= \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\frac{\partial\vec{J}_f}{\partial t} + \nabla\left(\frac{\rho}{\varepsilon_0}\right) \\ \nabla^2\vec{B} &= \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} - \mu\nabla\times\vec{J} \end{aligned}}$$

(ii) homogeneous wave eqns. (no charges or currents)

$$\boxed{\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}}$$

- The wave speed is now $\boxed{v = 1/\sqrt{\mu\varepsilon} < c}$

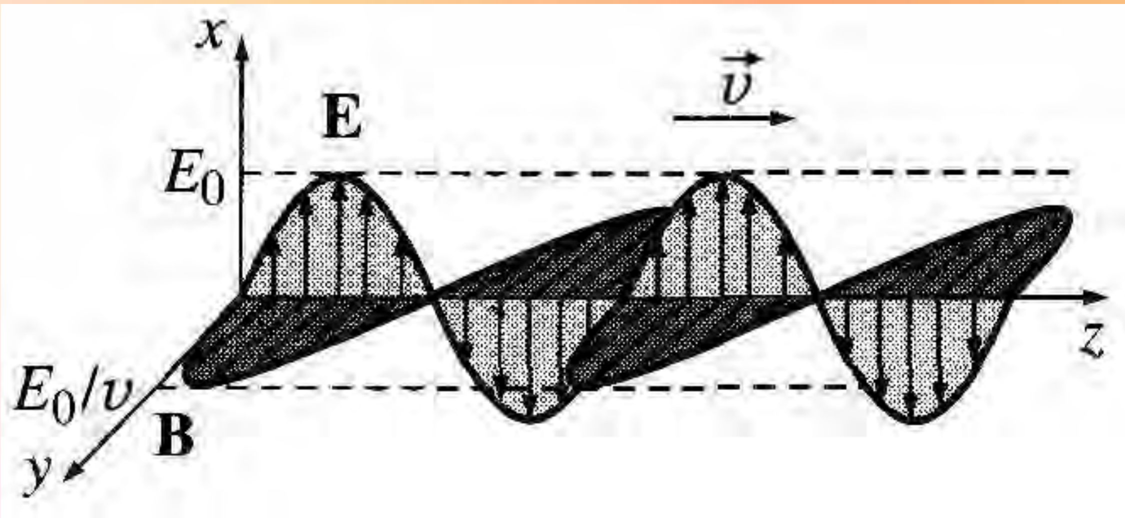
Refractive Index

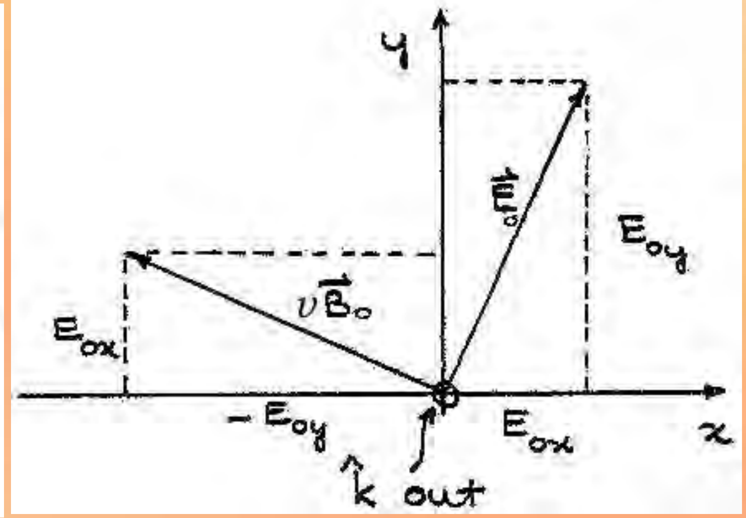
- Define refractive index (or "index of refraction") of a material: $$\boxed{n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}}$$
- Now for almost all non-ferromagnetic materials $\mu \cong \mu_0$ (or $\mu_r \cong 1$). Also $\varepsilon/\varepsilon_0 = \varepsilon_r = K$ (dielectric constant) so $$\boxed{n \cong \sqrt{K}}$$

  - the refractive index of a material is very nearly equal to the square root of its dielectric constant.
- All the results for free space carry over to l.i.h. materials if we change $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$ and then $c \to v$

Plane EM Waves in l.i.h. Materials

- So we have plane wave solutions as before
  $\vec{E}(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$; $\vec{B}(z,t) = \vec{B}_0 e^{i(kz-\omega t)}$
  but now $\boxed{\omega/k = v}$, the **wave speed in the material**.
- The wave vectors are now related by (with $v = 1/\sqrt{\mu\varepsilon}$)
  $\boxed{\vec{B} = (\hat{k} \times \vec{E})/v \quad \text{or} \quad \vec{v} = (\vec{E} \times \vec{B})/B^2}$
  e.g. Glass has $n = 1.5$ so $v_{\text{glass}} = c/n = 2.0 \times 10^8$ m/s





Wave Energy in Materials

- Continuing transcribing $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$ and $c \to v$, we get the energy density of the fields in l.i.h. materials:

$$\boxed{u_e = \tfrac{1}{2}\varepsilon E^2 \;\; ; \;\; u_m = \frac{B^2}{2\mu}}$$

- The Poynting vector is $\vec{S} = \vec{E} \times \vec{H} = (\vec{E} \times \vec{B})/\mu$

  and the wave intensity is $I = \langle S \rangle = \tfrac{1}{2} v\, \varepsilon {E_0}^2$

- Notice here that $v = c/n$ and $n = \sqrt{\varepsilon_r}$ or $\varepsilon_r = n^2$. This means that if a wave passes from vacuum into a material, since $\langle S \rangle$ must be the same (conservation of energy), E_0 will be smaller in the material and $B_0 = E_0/v$ will be larger. This is illustrated in the following example...

Wave Energy in Materials: Example

Example : A laser beam in vacuum has a power of 20 MW and a radius of 1 mm. (a) Find the magnitudes of $\vec{E}$ and $\vec{B}$.

(b) The beam then travels in glass with refractive index 1.6. Find the magnitudes of $\vec{E}$ and $\vec{B}$ now.

Solution : (a) The beam intensity is [in vacuum $\langle S\rangle = \frac{1}{2}c\ \varepsilon_0 {E_0}^2$]

$I = \langle S\rangle = P/A = 20 \times 10^6/[\pi(10^{-3})^2] = 6.4 \times 10^{12}\ \mathrm{W/m^2}$

$$\Rightarrow \quad E_0 = \sqrt{\frac{2\langle S\rangle}{c\varepsilon_0}} = \sqrt{\frac{2\times 6.4\times 10^{12}}{3\times 10^8\times 8.85\times 10^{-12}}} = 6.9 \times 10^7\ \mathrm{V/m}$$

$$B_0 = E_0/c = 6.9 \times 10^7/3 \times 10^8 = 0.23\ \mathrm{T}$$

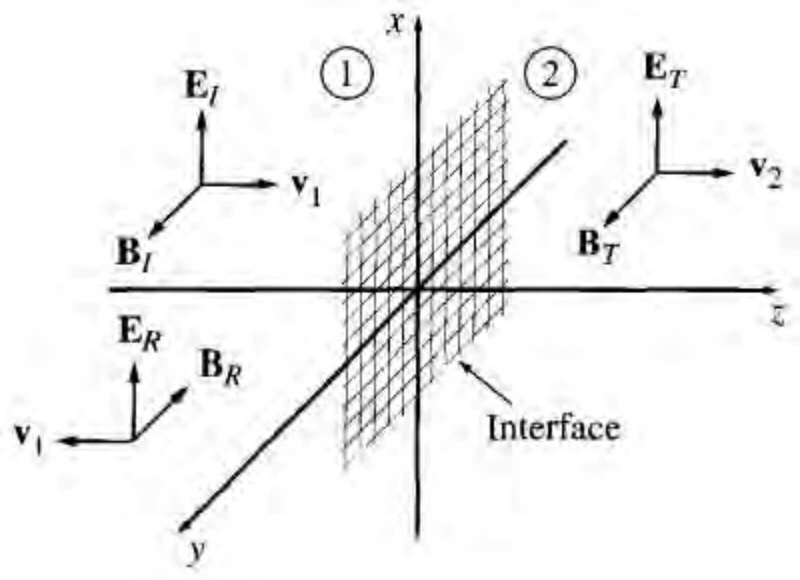
(b) $n = 1.6$ so $v = c/n = 3 \times 10^8/1.6 = 1.88 \times 10^8\ \mathrm{m/s}$

and $\varepsilon_r = n^2 = 1.6^2 = 2.56$ so $\varepsilon = \varepsilon_r\varepsilon_0 = 2.27 \times 10^{-11}\ \mathrm{F/m}$

$$E_0 = \sqrt{\frac{2\langle S\rangle}{v\,\varepsilon}} = \sqrt{\frac{2\times 6.4\times 10^{12}}{1.88\times 10^8\times 2.27\times 10^{-11}}} = 5.5 \times 10^7\ \mathrm{V/m}\ ;\ B_0 = 0.28\ \mathrm{T}$$

Materials Interfaces: Normal Incidence

- xy-plane is boundary between materials 1 and 2 (n_1, n_2)
- Plane wave in z-direction, polarized in x-direction, normal incidence from left:



$$\vec{E}_I(z,t) = E_{0I} e^{i(k_1 z - \omega t)} \hat{x}$$
$$\vec{B}_I(z,t) = (E_{0I}/v_1) e^{i(k_1 z - \omega t)} \hat{y}$$

- There is a reflected wave and a transmitted wave :

$\vec{E}_R(z,t) = E_{0R} e^{i(-k_1 z - \omega t)} \hat{x}$ (in 1, to left)

$\vec{B}_R(z,t) = -(E_{0R}/v_1) e^{i(-k_1 z - \omega t)} \hat{y}$ ($\vec{B}$ reverses)

$\vec{E}_T(z,t) = E_{0T} e^{i(k_2 z - \omega t)} \hat{x}$ (in 2, to right)

$\vec{B}_T(z,t) = (E_{0T}/v_2) e^{i(k_2 z - \omega t)} \hat{y}$

Materials Interfaces: Normal Incidence

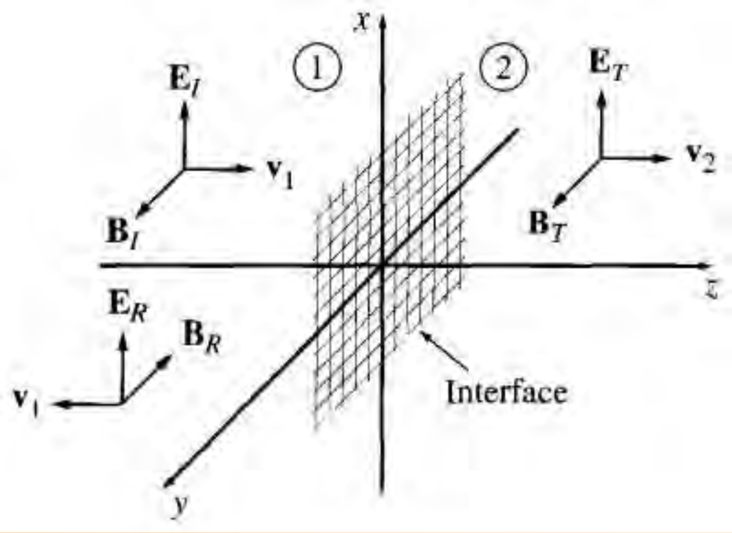
- We apply the boundary conditions

  $\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$; $E_{1t} = E_{2t}$

  $B_{1n} = B_{2n}$; $B_{1t}/\mu_1 = B_{2t}/\mu_2$

  [$\perp$ comps. $\vec{D}, \vec{B}$; // comps. $\vec{E}, \vec{H}$]

  Here we have no $\perp$ components.



- In medium 1 we have $I + R$ so

(1) $E_{0I} + E_{0R} = E_{0T}$ and $(B_{0I} - B_{0R})/\mu_1 = B_{0T}/\mu_2$ or

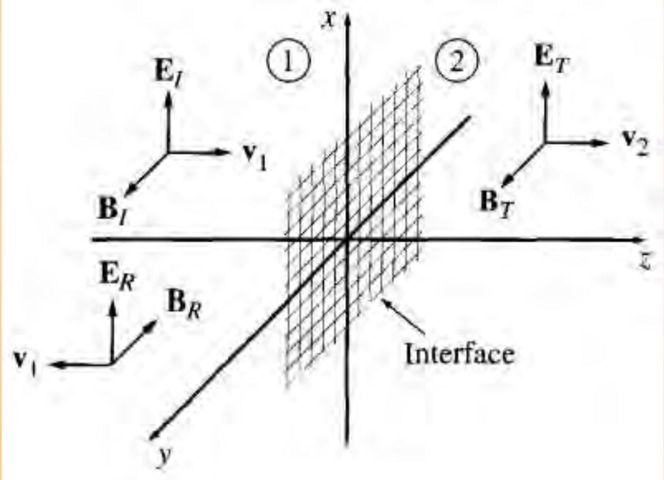
$(E_{0I} - E_{0R})/(\mu_1 v_1) = E_{0T}/(\mu_2 v_2)$ [since $B = E/v$] i.e.

(2) $E_{0I} - E_{0R} = \beta E_{0T}$ where $\boxed{\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}}$

- Solve (1) and (2) to give E_{0R} and E_{0T} in terms of E_{0I}:

$$\boxed{E_{0R} = \left(\frac{1-\beta}{1+\beta}\right) E_{0I} \quad \text{and} \quad E_{0T} = \left(\frac{2}{1+\beta}\right) E_{0I}}$$

Materials Interfaces: Normal Incidence



- We can usually make the very close approximation $\mu_1 = \mu_2 = \mu_0$, then

$$\boxed{\beta = \frac{n_2}{n_1} = \frac{v_1}{v_2}}$$

and we get for the real amplitudes in terms of the refractive indices

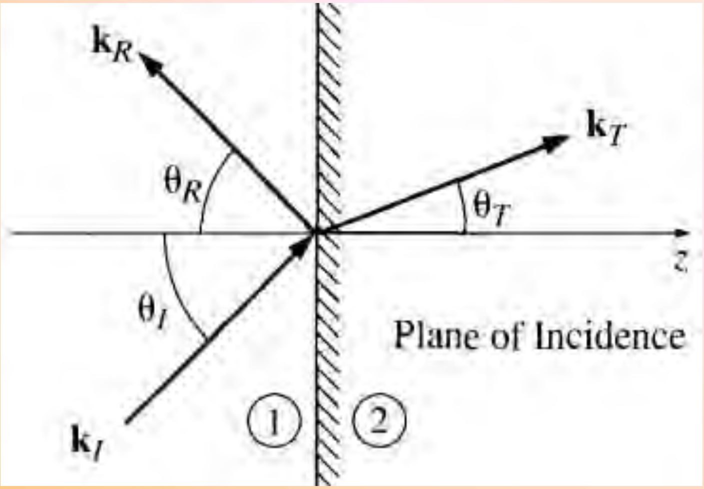
$$\boxed{E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I} \quad \text{and} \quad E_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I}}$$

- The ratios are often written as r and t:

$$r = \frac{E_{0R}}{E_{0I}} = \left(\frac{1-\beta}{1+\beta}\right) = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| \;\; \& \;\; t = \frac{E_{0T}}{E_{0I}} = \left(\frac{2}{1+\beta}\right) = \left(\frac{2n_1}{n_1 + n_2}\right)$$

[lower case for the “amplitude reflection and transmission coefficients”, upper case R and T for the reflection and transmission coefficients (ratios of intensities)...]

Materials Interfaces: Oblique Incidence



- Incident wave, wave vector $\vec{k}_I$, at arbitrary **angle of incidence** θ_I , etc.

$$\vec{E}_I(\vec{r},t) = \vec{E}_{0I}e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}$$

$$\vec{B}_I(\vec{r},t) = (\hat{k}_I \times \vec{E}_I)/v_1$$

- Reflected and transmitted waves

$$\vec{E}_R(\vec{r},t) = \vec{E}_{0R}e^{i(\vec{k}_R\cdot\vec{r}-\omega t)} \qquad \vec{B}_R(\vec{r},t) = (\hat{k}_R \times \vec{E}_R)/v_1$$

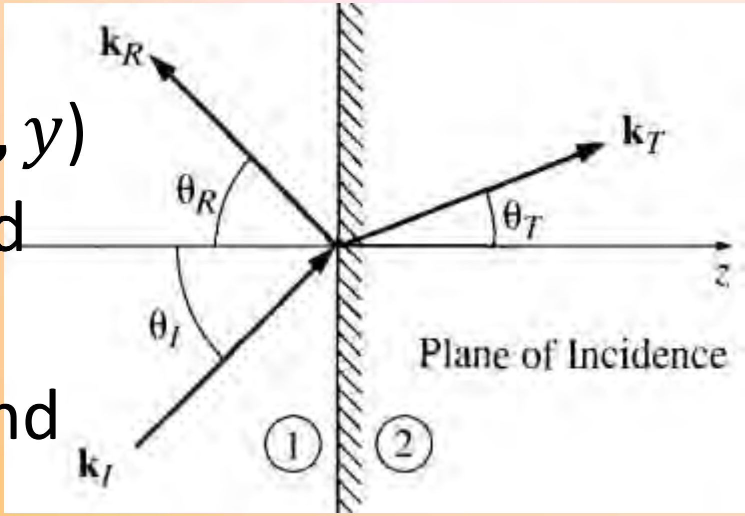
$$\vec{E}_T(\vec{r},t) = \vec{E}_{0T}e^{i(\vec{k}_T\cdot\vec{r}-\omega t)} \qquad \vec{B}_T(\vec{r},t) = (\hat{k}_T \times \vec{E}_T)/v_2$$

- Frequency ω is the same for all 3 waves, so

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \quad \Rightarrow \boxed{k_I = k_R = \frac{v_2}{v_1}k_T = \frac{n_1}{n_2}k_T}$$

- Boundary conditions must be satisfied for all $\vec{r}$ and t, so exponentials must all be equal.

Materials Interfaces: Oblique Incidence



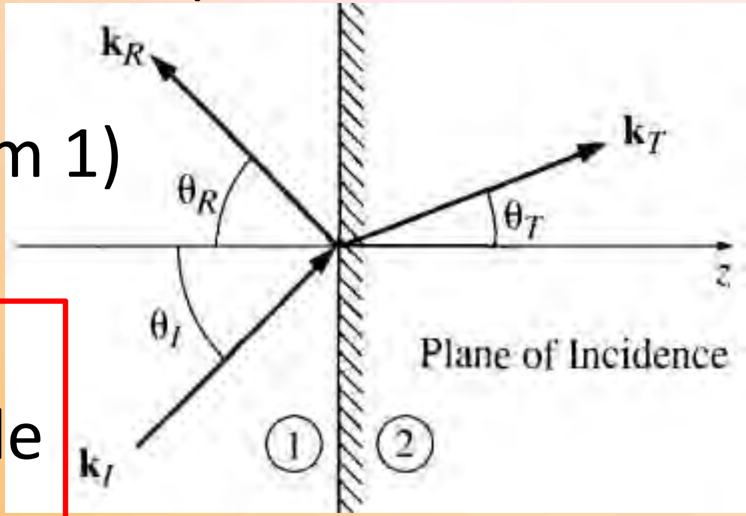
- This means that, at $z = 0$,
  $\vec{k}_I \cdot \vec{r} = \vec{k}_R \cdot \vec{r} = \vec{k}_T \cdot \vec{r}$ (for all x, y)
- This must be satisfied by the x and y components separately, i.e.

★ $k_{Ix} = k_{Rx} = k_{Tx}$ (when $y = 0$) and
$k_{Iy} = k_{Ry} = k_{Ty}$ (when $x = 0$)

- So we can arrange the axes so that $\vec{k}_I$ is in the xz-plane; i.e. we put $k_{Iy} = 0$; then we must have $k_{Ry} = k_{Ty} = 0$ also, i.e. $\vec{k}_R$ and $\vec{k}_I$ will also be in the xz-plane.
- This is the **first law of reflection**: the incident, reflected and transmitted wave vectors all lie in the same plane (which we call the **plane of incidence**).

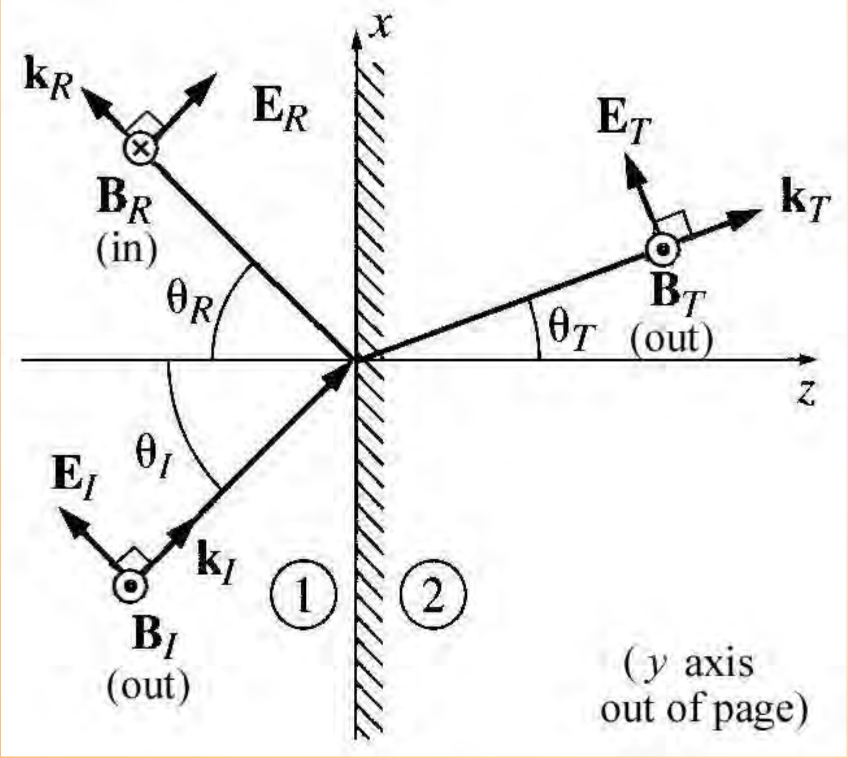
Materials Interfaces: Oblique Incidence

- From eqn. ✶ (x-components // interface)
  $k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$
- But $k_I = k_R = k_1$ (both in medium 1)
  so $\sin\theta_I = \sin\theta_R$, i.e.



- The **second law of reflection**: the angle of incidence equals the angle of reflection: $\theta_I = \theta_R$
- From the first equation above, $\dfrac{\sin\theta_T}{\sin\theta_I} = \dfrac{k_I}{k_T} = \dfrac{n_1}{n_2}$ [slide 11]
- The **law of refraction (Snell's law)**: the angle of incidence and the angle of refraction (transmission) are related by $n_1 \sin\theta_I = n_2 \sin\theta_T$.

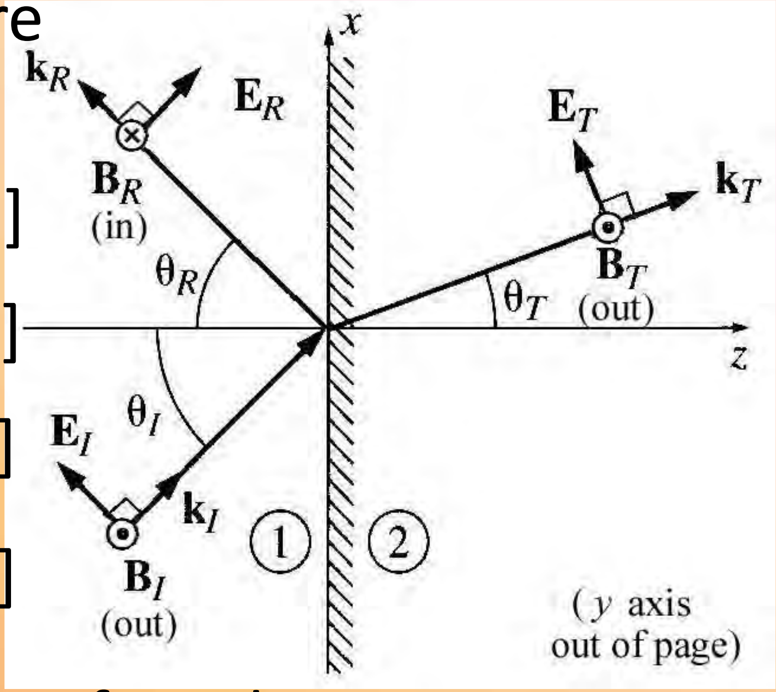
Fresnel's Equations



- In general the electric field of the wave will have components parallel to and perpendicular to the plane of incidence.
- Here we consider $\vec{\boldsymbol{E}}$ **parallel to the plane of incidence**; we derive the relationship between the I, R, T wave **amplitudes**.
- Axes as before: interface is xy-plane (i.e. $\hat{z} \perp$ interface); plane of incidence ("PoI") is xz-plane (so $\hat{y} \perp \mathrm{PoI}$)
- We apply the boundary conditions for $\vec{E}$ and $\vec{B}$; we know that the exponential factors $e^{i(\vec{k}.\vec{r}-\omega t)}$ cancel, so we can apply the BCs to the (vector) amplitudes $\vec{E}_0$ and $\vec{B}_0$.

Fresnel's Equations (for $\vec{E} \parallel$ PoI)

- The BCs for the general case (where $\vec{E}$ has x and y components) are:



1. $\varepsilon_1 \left(\vec{E}_{0I} + \vec{E}_{0R}\right)_z = \varepsilon_2 \left(\vec{E}_{0T}\right)_z$ $[D_\perp]$
2. $\left(\vec{B}_{0I} + \vec{B}_{0R}\right)_z = \left(\vec{B}_{0T}\right)_z$ $[B_\perp]$
3. $\left(\vec{E}_{0I} + \vec{E}_{0R}\right)_{x,y} = \left(\vec{E}_{0T}\right)_{x,y}$ $[E_\parallel]$
4. $\frac{1}{\mu_1}\left(\vec{B}_{0I} + \vec{B}_{0R}\right)_{x,y} = \frac{1}{\mu_2}\left(\vec{B}_{0T}\right)_{x,y}$ $[H_\parallel]$

- For polarization (i.e. $\vec{E}$) in the plane of incidence:

1. $\varepsilon_1 \left(-E_{0I} \sin\theta_I + E_{0R} \sin\theta_R\right) = \varepsilon_2\left(-E_{0T} \sin\theta_T\right)$ 2. $0 = 0$
3. $E_{0I} \cos\theta_I + E_{0R} \cos\theta_R = E_{0T} \cos\theta_T$
4. $\frac{1}{\mu_1 v_1}\left(E_{0I} - E_{0R}\right) = \frac{1}{\mu_2 v_2} E_{0T}$ [no cos or sin as $\vec{B} \parallel$ interface]

Applying $\theta_I = \theta_R$ and Snell's law to 1. gives the same as 4.

Fresnel's Equations (for $\vec{E} \parallel$ PoI)

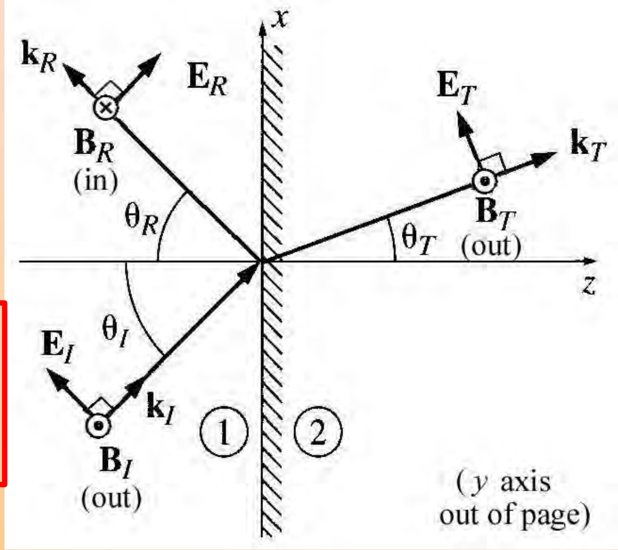
- We have two eqns. to solve:

$$E_{0I} - E_{0R} = (\mu_1 v_1 / \mu_2 v_2) E_{0T} = \beta E_{0T}$$

$$E_{0I} + E_{0R} = (\cos\theta_T / \cos\theta_I) E_{0T} = \alpha E_{0T}$$

where $\boxed{\alpha = \dfrac{\cos\theta_T}{\cos\theta_I}, \ \beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}}$

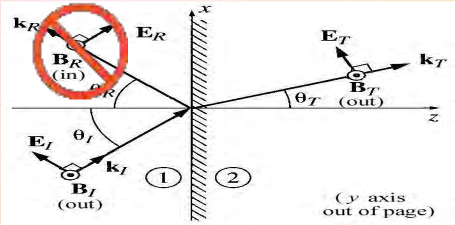
for the ratios of the fields. Solution:



$$\boxed{r_\parallel = \frac{E_{0R}}{E_{0I}} = \frac{\alpha - \beta}{\alpha + \beta} \quad \text{and} \quad t_\parallel = \frac{E_{0T}}{E_{0I}} = \frac{2}{\alpha + \beta}}$$

These are the **Fresnel equations** for polarization **parallel** to the plane of incidence, denoted by the $\parallel$ subscript on r and t, which are the **amplitude reflection coefficient** and the **amplitude transmission coefficient** respectively. [There are also $r_\perp$ and $t_\perp$ for polarization perpendicular to the PoI.]

Brewster's Angle



- α is a function of the angle of incidence:

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-(n_1/n_2)^2\sin^2\theta_I}}{\cos\theta_I} = \sqrt{\frac{1-(n_1/n_2)^2\sin^2\theta_I}{1-\sin^2\theta_I}}$$

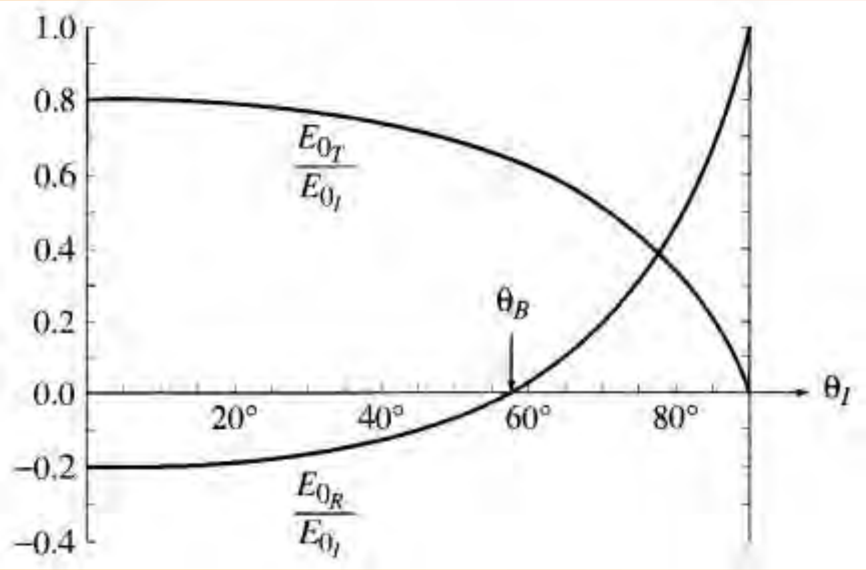
- When $\theta_I = 0$ (normal incidence), $\alpha = 1$ [see slide 9]
- When $\theta_I \to 90°$ ("grazing incidence"), α diverges and $t_\parallel = 0$ i.e. the wave is totally reflected (e.g. car headlights on wet road)
- From $r_\parallel = \frac{E_{0R}}{E_{0I}} = \frac{\alpha-\beta}{\alpha+\beta}$, we see that $r_\parallel = 0$ when $\alpha = \beta$, i.e. there is **no reflected wave**. This occurs when $\theta_I = \theta_B$, the **Brewster angle** , which is the solution of

$$\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2} \quad \Rightarrow \quad \boxed{\theta_B = \tan^{-1}(n_2/n_1)}$$

- Note this only occurs for polarization $\parallel$ PoI ; there is no Brewster angle for polarization $\perp$ PoI .

Fresnel's Equations (for $\vec{E} \parallel \mathrm{PoI}$)

- The graph shows $t_\parallel$ and $r_\parallel$ for air–glass ($n_1 = 1.0$, $n_2 = 1.5$) as functions of θ_I ($\beta = 1.5$)



- At normal incidence ($\theta_I = 0$)

$$r_\parallel = \frac{1-\beta}{1+\beta} = -0.2\,, t_\parallel = \frac{2}{1+\beta} = 0.8$$

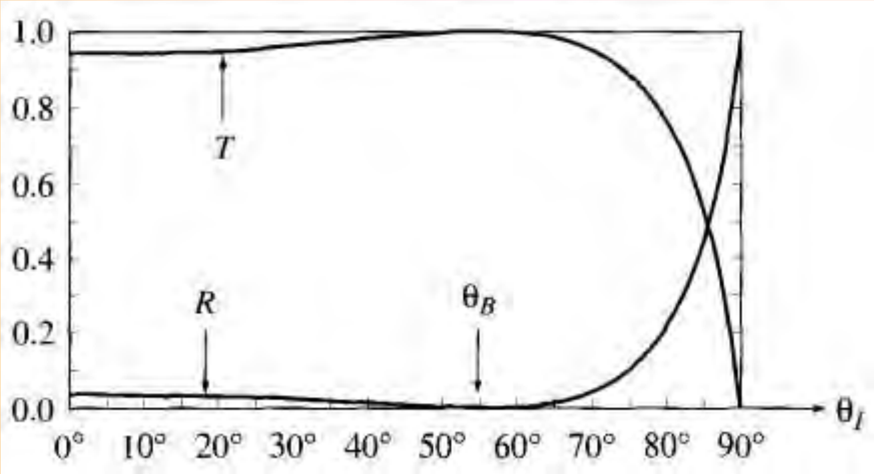
- Brewster's angle ($r_\parallel = 0$) is $\theta_B = \tan^{-1}(n_2/n_1) = \tan^{-1} 1.5 = 56°$
- At the "crossover angle", $r_\parallel = t_\parallel$ so $\alpha - \beta = 2$

- Fresnel's eqns. for $\vec{E} \perp \mathrm{PoI}$: $r_\perp = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right|$, $t_\perp = \frac{2}{1+\alpha\beta}$

  and there is no Brewster's angle for polarization perpendicular to the plane of incidence.

Transmission & Reflection Coefficients

- What about the intensities? Power/area on interface is $I = \langle\vec{S}\rangle.\hat{z}$ so intensities are



$I_I = \frac{1}{2} v_1 \varepsilon_1 {E_{0I}}^2 \cos\theta_I$ incident

$I_R = \frac{1}{2} v_1 \varepsilon_1 {E_{0R}}^2 \cos\theta_R$ reflected

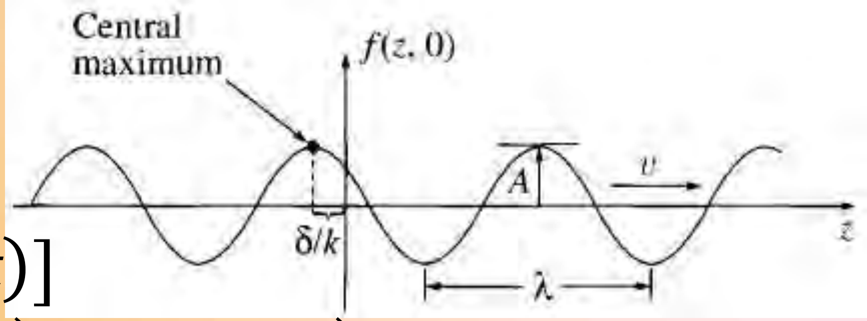
$I_T = \frac{1}{2} v_2 \varepsilon_2 {E_{0T}}^2 \cos\theta_T$ transmitted [cos factor for $\perp$ comps.]

- The **reflection and transmission coefficients** R and T are the ratios of intensities; $R + T = 1$ [energy conservation]

$$\boxed{R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2}$$ [$\theta_R = \theta_I$, same v , ε]

$$\boxed{T = \frac{I_T}{I_I} = \left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{v_2\varepsilon_2}{v_1\varepsilon_1}\frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2}$$

A Note on Complex Amplitudes



- We have written a plane wave as e.g. $\vec{E}\,(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$

$= \vec{E}_0[\cos(kz-\omega t) + i\sin(kz-\omega t)]$

with real electric field given by $\vec{E}\,(z,t) = \vec{E}_0\cos(kz-\omega t)$ and the amplitude $\vec{E}_0$ appearing in both these eqns. is real.

- In general a wave may have initial phase angle (or "phase constant" or "phase offset") denoted by ϕ or δ, and the wave function is $\vec{E}_0 e^{i(kz-\omega t+\delta)}$ with real part $\vec{E}_0\cos(kz-\omega t+\delta)$. This can be written as $\vec{E}_0 e^{i\delta} e^{i(kz-\omega t)}$. Now we can treat $\vec{E}_0 e^{i\delta}$ as **complex amplitude** $\tilde{E}_0$ with modulus $\vec{E}_0$ = real amplitude, argument δ = phase constant. Then the complex wave fn. is $\vec{E}\,(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}$. Ignoring the imaginary part of $\tilde{E}_0$ is equivalent to putting phase constant $\delta = 0$ (as we have done)

EM Waves in Conductors

- In conductors we can no longer assume $\vec{J}_f = 0$ as we did before, and $\vec{J}_f$ is related to $\vec{E}$ by Ohm's law: $\vec{J}_f = \sigma\vec{E}$
- Substitute in Maxwell-Ampere law:

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\varepsilon\,\partial\vec{E}/\partial t$$

- Now in continuity eqn. $\nabla \cdot \vec{J}_f = -\,\partial\rho_f/\partial t$ substitute for $\vec{J}_f$ from Ohm's law, then use Gauss's law:

$$\frac{\partial\rho_f}{\partial t} = -\nabla \cdot \vec{J} = -\nabla \cdot \left(\sigma\vec{E}\right) = -\sigma\left(\nabla \cdot \vec{E}\right) = -\frac{\sigma}{\varepsilon}\rho_f$$

- This solves as $\rho_f(t) = \rho_f(0)e^{-(\sigma/\varepsilon)t}$; this means that any free charge in a conductor will go to zero (the charges will move to the surface) in a characteristic time of ε/σ . Since σ is very large in good conductors, this time is very short. Thus we can assume $\rho_f = 0$ and so $\nabla \cdot \vec{E} = 0$.

EM Waves in Conductors

- Then Maxwell's eqns. are the same as for non-conductors, except for the $\mu\sigma\vec{E}$ term in Maxwell-Ampere:

$$\nabla \cdot \vec{E} = 0 \quad (\rho_f = 0) \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\,\partial\vec{B}/\partial t \qquad \nabla \times \vec{B} = \mu\varepsilon\,\partial\vec{E}/\partial t + \mu\sigma\vec{E}$$

- Take the curl of Faraday's & Ampere's laws as before $\rightarrow$

$$\boxed{\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \quad , \quad \nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2} + \mu\sigma\frac{\partial\vec{B}}{\partial t}}$$

- These 'modified' wave eqns. have **plane wave solutions**

$$\vec{E}(z,t) = \vec{E}_0 e^{i(\tilde{k}z-\omega t)} \quad ; \quad \vec{B}(z,t) = \vec{B}_0 e^{i(\tilde{k}z-\omega t)}$$

but the **wave number is complex**: $\tilde{k} = k + i\kappa$ [$\kappa =$ kappa]

- Substitute plane wave solutions in wave eqns. $\rightarrow$

$\boxed{\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega}$ [note Im part contains σ]

EM Waves in Conductors

- Then we get (where $\tilde{k} = k + i\kappa$)...

Real part $$\boxed{k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2}}$$

Imaginary part $$\boxed{\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}}$$

- Imaginary part $\rightarrow$ exponentially decreasing amplitude:

$$\boxed{\vec{E}(z,t) = \vec{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \;\; ; \;\; \vec{B}(z,t) = \vec{B}_0 e^{-\kappa z} e^{i(kz-\omega t)}}$$

- Amplitude decreases by $1/e$ in distance d = **skin depth** :

  [i.e. $\vec{E}(z) = \vec{E}_0\, e^{-z/d}$] so $\boxed{d = 1/\kappa}$

  e.g. in pure water, $d = 12\ \mathrm{km}$

  See Worked Examples 3, Q.3