Electromagnetic Waves 1

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2. Energy and the Poynting vector
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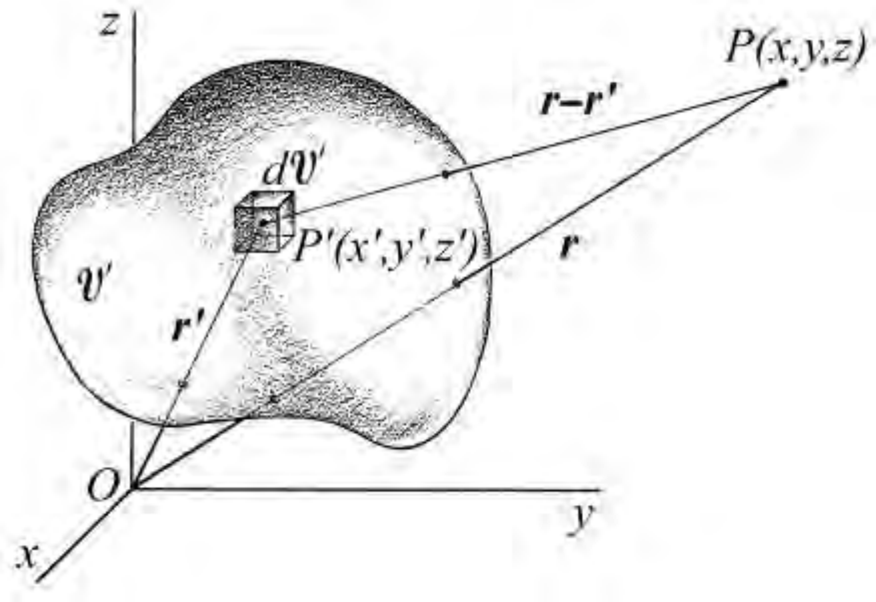


Retarded Potentials

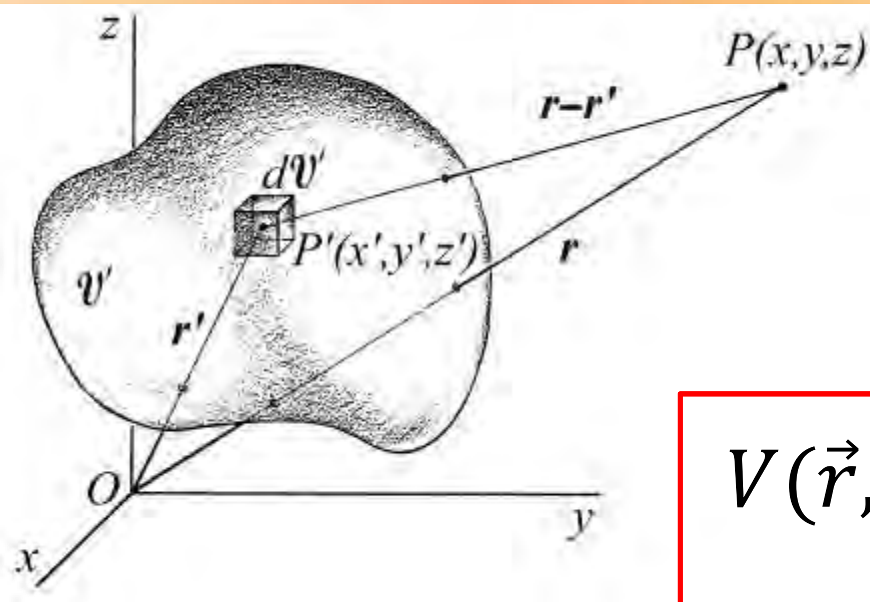
For volume charge & current

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \frac{\rho(\vec{r}\prime)}{R} d\mathcal{V}'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}(\vec{r}\prime)}{R} d\mathcal{V}'$$

where $\vec{R} = \vec{r} - \vec{r}'$ (vector from source to field point)

- In the non-static case, we have to consider the time for the information to travel distance R, at speed c.
- If we are determining the potentials and fields at time t, then we need the state of the source at the earlier time $\boxed{t_r = t - R/c}$, the "**retarded time**".
- (This will be different for different parts of the source; like light from distant stars...)



Retarded Potentials

Thus we should rewrite the equations for the potentials:

$$\boxed{\begin{aligned} V(\vec{r},t) &= \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \frac{\rho(\vec{r}\,',t_r)}{R}\, d\mathcal{V}' \\ \vec{A}(\vec{r},t) &= \frac{\mu_0}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}(\vec{r}\,',t_r)}{R}\, d\mathcal{V}' \end{aligned}}$$

where $\vec{R} = \vec{r} - \vec{r}\,'$ and $t_r = t - R/c$

- These are the **retarded potentials**. The integrals for ρ and $\vec{J}$ are evaluated at the earlier retarded time.
- Griffiths shows these satisfy the inhomogeneous wave eqn. and Lorenz condition, so are real potentials.
- In the static case, we do not have to consider this.

Energy and Poynting's Theorem (1)

Consider work done by Lorentz force moving charge q by $d\vec{l}$ in time dt: $dW = \vec{F}.d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}).\vec{v}\,dt = q\vec{E}.\vec{v}dt$

For distributed charge $q = \rho d\mathcal{V}$ and $\rho\vec{v} = \vec{J}$, so for total charge in volume $\mathcal{V}$, the rate of doing work, i.e. power, is

$P = \frac{dW}{dt} = \int_{\mathcal{V}} (\vec{E}.\vec{J})d\mathcal{V}$ or $\vec{E}.\vec{J}$ is the **power per unit volume**.

Use Maxwell's eqns. to get this in terms of only the fields:

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \;\Rightarrow\; P = \int_{\mathcal{V}} \vec{E}.(\nabla \times \vec{H})d\mathcal{V} - \int_{\mathcal{V}} \vec{E}.\frac{\partial \vec{D}}{\partial t}d\mathcal{V}$$

Vector identity: $\nabla.(\vec{E} \times \vec{H}) = \vec{H}.(\nabla \times \vec{E}) - \vec{E}.(\nabla \times \vec{H}) \;\Rightarrow$

$$P = \int_{\mathcal{V}} \vec{H}.(\nabla \times \vec{E})d\mathcal{V} - \int_{\mathcal{V}} \nabla.(\vec{E} \times \vec{H})d\mathcal{V} - \int_{\mathcal{V}} \vec{E}.\frac{\partial \vec{D}}{\partial t}d\mathcal{V}$$

Now $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\int_{\mathcal{V}} \nabla.(\vec{E} \times \vec{H})d\mathcal{V} = \oint_{\mathcal{S}} (\vec{E} \times \vec{H}).d\vec{a}$

Energy and Poynting's Theorem (2)

Then, rearranging, we get

$$\boxed{\int_{\mathcal{V}} \left(\vec{E}.\frac{\partial \vec{D}}{\partial t} + \vec{H}.\frac{\partial \vec{B}}{\partial t}\right) d\mathcal{V} + \oint_{\mathcal{S}} \left(\vec{E} \times \vec{H}\right).d\vec{a} + P = 0}$$

This result is known as **Poynting's Theorem**, but it is really just a statement of **conservation of energy.**

In l.i.h. materials $\vec{E}.\frac{\partial \vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon E^2\right)$ and $\vec{H}.\frac{\partial \vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2\mu}B^2\right)$:

$$\boxed{\frac{\partial}{\partial t}\int_{\mathcal{V}} \left(\frac{1}{2}\varepsilon E^2 + \frac{B^2}{2\mu}\right) d\mathcal{V} + \oint_{\mathcal{S}} \left(\vec{E} \times \vec{H}\right).d\vec{a} + P = 0}$$

We recognise $\frac{1}{2}\varepsilon E^2 = u_e$ and $\frac{B^2}{2\mu} = u_m$, the energy density of the electric and magnetic fields, so the first term is the **rate of change of electromagnetic energy in the volume**.

Energy Flow and the Poynting Vector

$$\frac{\partial}{\partial t}\int_{\mathcal{V}}\left(\frac{1}{2}\varepsilon E^2+\frac{B^2}{2\mu}\right)d\mathcal{V}+\oint_{\mathcal{S}}\left(\vec{E}\times\vec{H}\right).d\vec{a}+P=0$$

The third term, originally $P=\int_{\mathcal{V}}\left(\vec{E}.\vec{J}\right)d\mathcal{V}$, represents the **rate of energy dissipation (loss) in the volume**.

The second term is the **flux of the vector $\vec{\boldsymbol{E}}\times\vec{\boldsymbol{H}}$** through the surface bounding the volume. This integral represents the total **rate of energy outflow** from the volume. We define the **Poynting vector**: $\boxed{\vec{S}=\vec{E}\times\vec{H}\ \text{ or }\ \vec{S}=\frac{1}{\mu}\left(\vec{E}\times\vec{B}\right)}$

and in free space $\boxed{\vec{S}=\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)}$ where

$\vec{S}.d\vec{a}$ represents **energy per unit time crossing area** $d\vec{a}$, i.e. $\vec{\boldsymbol{S}}$ **represents the energy per unit time crossing unit area.**

Poynting's Theorem, Poynting Vector

Then Poynting's theorem can be rewritten (more compactly)

$$\boxed{\frac{\partial}{\partial t}(U_e + U_m) + \oint_S \vec{S} . d\vec{a} + P = 0}$$

(rate of energy outflow)

(rate of change of EM energy)   (rate of energy loss in volume)

i.e. **total energy is conserved**. [Note $U_e = \int_{\mathcal{V}} u_e d\mathcal{V}$, $U_m = \cdots$]

- 1st term can be positive or negative (U increase or decrease)
- 2nd term can be positive or negative ($\vec{S}$ outflow or inflow)
- 3rd term can only be ≥ 0 (energy dissipated, i.e. lost)
- Poynting vector $\vec{S} = \vec{E} \times \vec{H}$ has units $\mathrm{V/m \,.\, A/m = W/m^2}$ i.e. energy per unit time per unit area [or power p.u. area]
- $\vec{S}$ can be called the "energy flux density". It is essential for understanding how EM waves transport energy.

Wave Equation for $\vec{E}$ (1)

- Immediate consequence of Maxwell's equations
- Take curl of both sides of eqn. for Faraday's law:

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu_0 \vec{J}_f + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

Substitute $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ (Gauss's law)

$$\nabla^2 \vec{E} - \nabla\left(\frac{\rho}{\varepsilon_0}\right) = \mu_0 \frac{\partial \vec{J}_f}{\partial t} + \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$ or, rearranging,

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} + \mu_0 \frac{\partial \vec{J}_f}{\partial t} + \nabla\left(\frac{\rho}{\varepsilon_0}\right)}$$

- The **inhomogeneous wave equation** $\nabla^2 \varphi = \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} + A$

Wave Equation for $\vec{E}$ (2)

- Away from sources, in free space where $\rho = 0$ and $\vec{J}_f = 0$ this reduces to the **homogeneous wave equation**

$$\boxed{\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \qquad \text{where } \mu_0 \varepsilon_0 = \frac{1}{c^2}$$

- i.e. Wave speed $\boxed{c = 1/\sqrt{\mu_0 \varepsilon_0}}$
- Thus Maxwell's equations predict that the electric field obeys the wave eqn., with wave speed equal to the speed of light.
- Units: ε_0 is usually given in $\mathrm{F/m}$ and μ_0 in $\mathrm{H/m}$.
  [$\varepsilon_0 = 8.85 \times 10^{-12} \mathrm{F/m}$; $\mu_0 = 4\pi \times 10^{-7} \mathrm{H/m}$]
  Exercise: Show that the units of $1/\sqrt{\mu_0 \varepsilon_0}$ are indeed $\mathrm{m/s}$.

Wave Equation for $\vec{B}$ (1)

- Take curl of both sides of eqn. for Ampere's law (in vacuum)

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \vec{E})$$

$$\nabla (\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$ by Faraday's law

But $\nabla \cdot \vec{B} = 0$ so we have (with $\vec{J}_f = 0$)

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}$$ **The homogeneous wave equation**

- Again wave speed $c = 1/\sqrt{\mu_0 \varepsilon_0}$
- Thus Maxwell's equations predict directly that the **electric and magnetic fields obey the wave equation**, with **wave speed equal to the speed of light**.

Wave Equation for $\vec{B}$ (2)

- Including the source term in Ampere's law :

$$\nabla \times (\nabla \times \vec{B}) = \nabla \times (\mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t})$$

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \vec{E})$$

$$-\nabla^2 \vec{B} = \mu_0 \nabla \times \vec{J} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{(Faraday's law and } \nabla \cdot \vec{B} = 0\text{)}$$

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} - \mu_0 \nabla \times \vec{J}}$$

- **Inhomogeneous wave equation,** wave speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$
- We have already seen that the scalar and vector potentials obey the inhomogeneous /homogeneous wave equation, with the same speed, $c = 3.00 \times 10^8$ m/s.

Plane EM Waves in Free Space

- We know that the plane wave $\psi(z,t) = \psi_0 \cos(kz - \omega t)$ is a solution of the wave eqn. $\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$ and we can write this as $\psi(z,t) = \psi_0 e^{i(kz-\omega t)}$ with the understanding that we take the real part to get the physical wave.
- We examine **plane waves** of the form

$$\boxed{\vec{E}\,(z,t) = \vec{E}_0 e^{i(kz-\omega t)} \quad , \quad \vec{B}\,(z,t) = \vec{B}_0 e^{i(kz-\omega t)}}$$

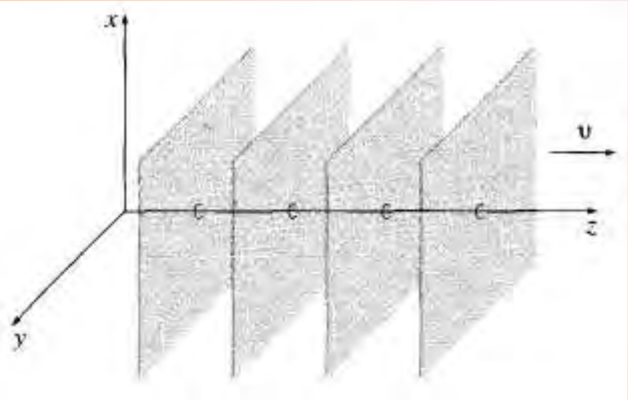
with complex amplitudes $\vec{E}_0$ and $\vec{B}_0$.

- We will show that in order to satisfy Maxwell's equations (which they must if they are real $\vec{E}$ and $\vec{B}$ fields), these waves must be **transverse** ; the waves of the two fields must also be mutually perpendicular and in phase.

Plane EM Waves are Transverse

- Consider a plane wave with $\vec{E}\,(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$ propagating in a charge-free region ($\rho = 0$).
- $\vec{E}$ must satisfy Maxwell's equations, in particular

$\nabla \cdot \vec{E} = 0$, i.e. $\dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z} = 0$



But for plane wave, $\vec{E} = \vec{E}(z)$ only;

$$\partial/\partial x = 0\,,\ \partial/\partial y = 0$$

$$\Rightarrow\ \frac{\partial E_z}{\partial z} = ikE_z = 0 \qquad \Rightarrow \qquad E_z = 0$$

Since $\nabla \cdot \vec{B} = 0$ the same argument also gives $B_z = 0$

There is no component in the direction of propagation: an EM wave is **transverse**, i.e. the oscillation is perpendicular to the direction of propagation

Relationship Between $\vec{E}$ and $\vec{B}$ (1)

- $\vec{E}\,(z,t) = \vec{E}_0 e^{i(kz-\omega t)}$ must also satisfy $-\partial\vec{B}/\partial t = \nabla \times \vec{E}$

$$= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{x} + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\hat{z}$$

But $\partial/\partial x = 0$, $\partial/\partial y = 0$ (plane wave), $E_z = 0$ (transverse) so

$$-\frac{\partial\vec{B}}{\partial t} = -\frac{\partial E_y}{\partial z}\hat{x} + \frac{\partial E_x}{\partial z}\hat{y} = ik\left[-E_{0y}\hat{x} + E_{0x}\hat{y}\right]e^{i(kz-\omega t)}$$

Integration w.r.t. time then gives ($-$ signs cancel)

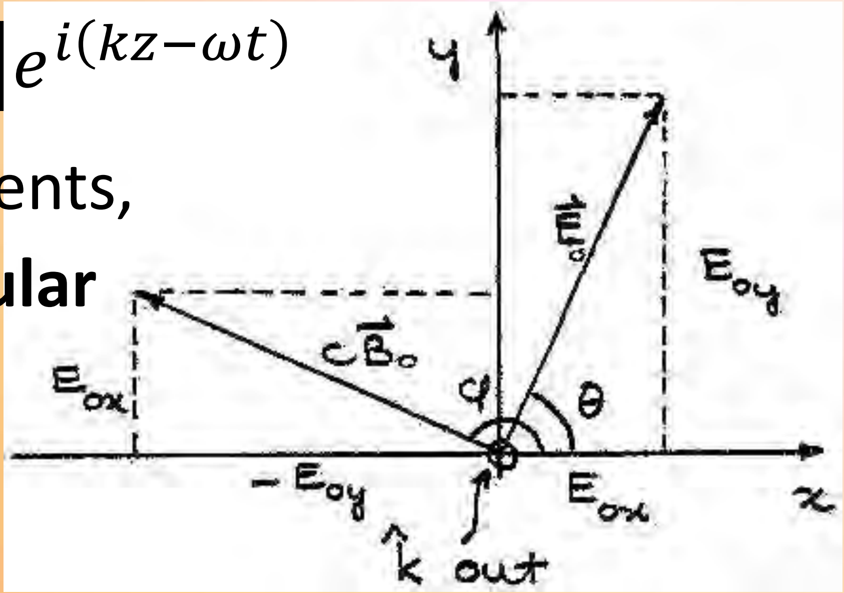
$$\vec{B} = (k/\omega)\left[-E_{0y}\hat{x} + E_{0x}\hat{y}\right]e^{i(kz-\omega t)}$$

Thus $\vec{B}$ is the xy-plane, i.e. **transverse**, and **in phase** with $\vec{E}$.

Now $\left|-E_{0y}\hat{x} + E_{0x}\hat{y}\right| = \left|\vec{E}\right|$ and $\omega/k = c$ so the amplitudes are related by $\boxed{B_0 = E_0/c}$

Relationship Between $\vec{E}$ and $\vec{B}$ (2)

We have $\vec{B} = \frac{k}{\omega}\left[-E_{0y}\hat{x} + E_{0x}\hat{y}\right]e^{i(kz-\omega t)}$



From the relationship of components,

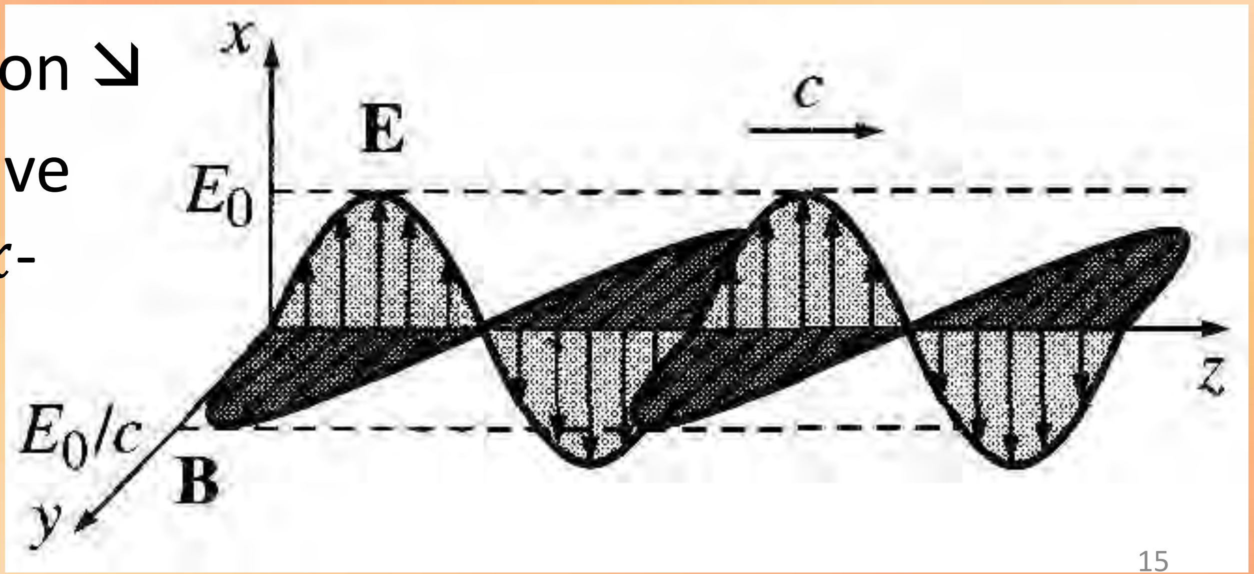
$\vec{B}$ and $\vec{E}$ are **mutually perpendicular**

and $$\boxed{\vec{B} = \frac{k}{\omega}\left(\hat{z} \times \vec{E}\right)}$$

e.g. if $\vec{E}$ is in the x-direction then

$\vec{B}$ is in the y-direction ↘

Here we say the wave is **polarized** in the x-direction (we use the direction of $\vec{E}$.)



Relationship Between $\vec{E}$ and $\vec{B}$ (3)

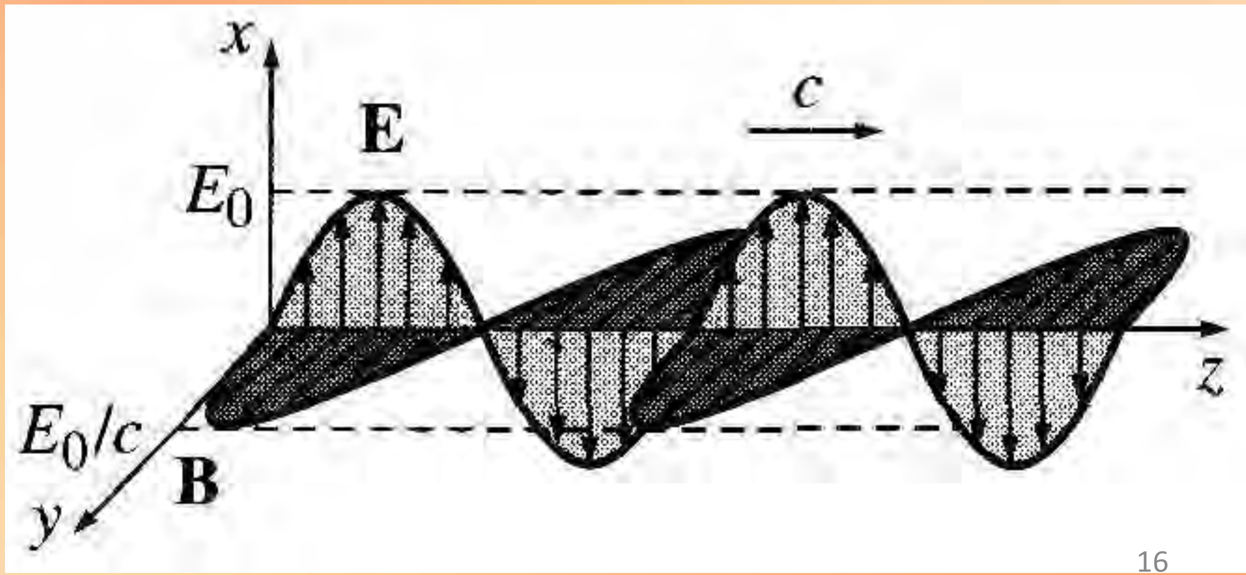
The **wave vector** $\vec{k}$ is defined as the vector with **magnitude** k (the wave number $= 2\pi/\lambda$) pointing in the **direction of propagation** of the wave. In this case $\hat{k} = \hat{z}$.

We could also write the previous eqn. as $\boxed{\vec{B} = (\hat{k} \times \vec{E})/c}$

Write the wave velocity as a vector $\vec{c} = c\hat{k}$, then the relationship between the three vectors can be written

$$\boxed{\vec{c} = \frac{\vec{E}\times\vec{B}}{B^2}}$$

(the velocity is in the direction of $\vec{E} \times \vec{B}$ and has magnitude E/B .)



Relationship Between $\vec{E}$ and $\vec{B}$ (4)

The fields in the wave shown in the figure below (polarized in the x-direction) are written (with $B_0 = E_0/c$)

$\vec{E}\,(z,t) = E_0\hat{x}\, e^{i(kz-\omega t)}$ and $\vec{B}\,(z,t) = B_0\hat{y}\, e^{i(kz-\omega t)}$

with the **real fields** being the real parts:

$\vec{E}\,(z,t) = E_0\hat{x}\ \cos(kz-\omega t)$, $\vec{B}\,(z,t) = B_0\hat{y}\ \cos(kz-\omega t)$

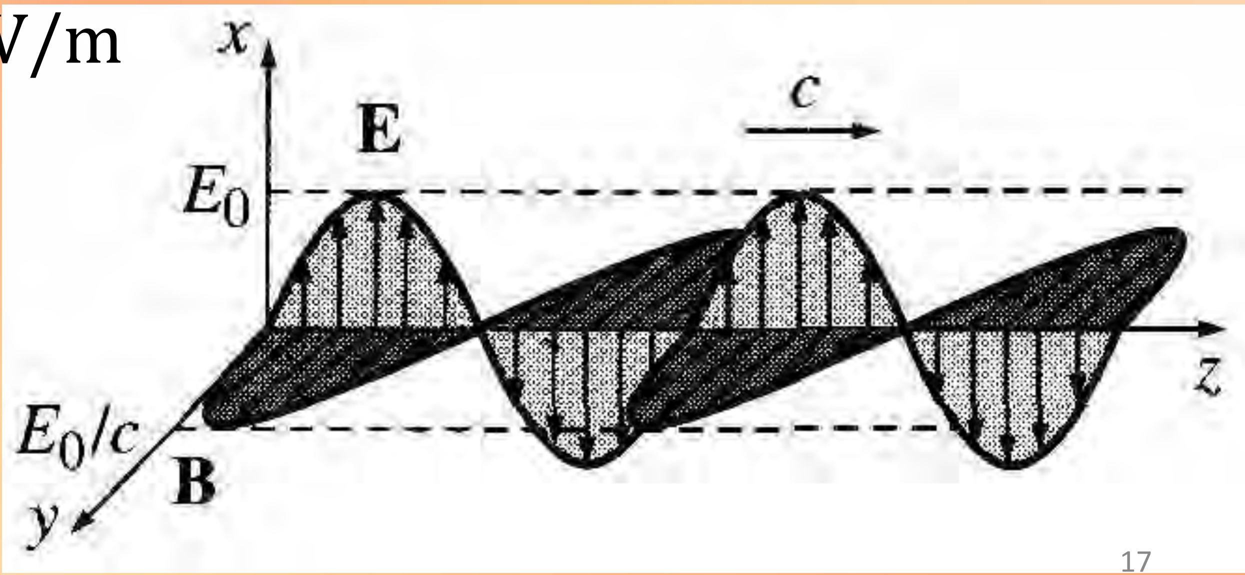
The ratio of the field magnitudes in an EM wave is always c , so e.g. if $E_0 = 300\ \mathrm{V/m}$

then $B_0 = E_0/c$

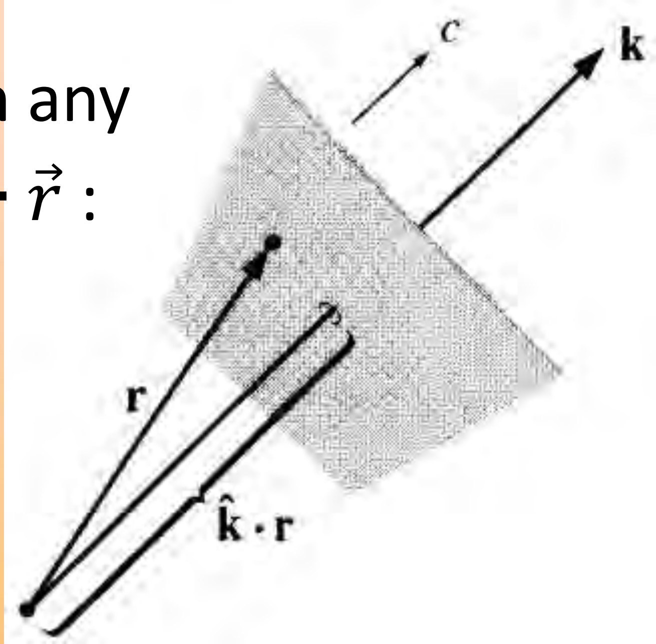
$= 3\times 10^2/3\times 10^8$

$= 1\times 10^{-6}\ \mathrm{T}$

or $1\ \mu\mathrm{T}$



Plane EM Waves in Arbitrary Direction



- In general the wave vector $\vec{k}$ can be in any direction. Then we replace kz with $\vec{k}\cdot\vec{r}$:
- The field equations for a **plane wave with wave vector $\vec{k}$ and polarized in the direction $\hat{n}$** (the direction of $\vec{E}$, which is $\perp \vec{k}$, so that $\hat{n}\cdot\hat{k}=0$) are

$$\boxed{\vec{E}\,(\vec{r},t)=E_0\hat{n}\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\ ,\ \ \vec{B}\,(\vec{r},t)=\frac{E_0}{c}\big(\hat{k}\times\hat{n}\big)\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}}$$

with the real fields again the real parts:

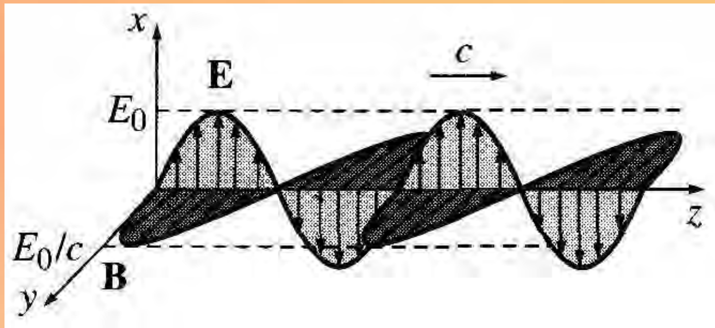
$$\vec{E}\,(\vec{r},t)=E_0\hat{n}\ \cos\big(\vec{k}.\vec{r}-\omega t\big)\ ,$$

$$\vec{B}\,(\vec{r},t)=(E_0/c)\big(\hat{k}\times\hat{n}\big)\cos\big(\vec{k}.\vec{r}-\omega t\big)=\big(\hat{k}\times\vec{E}\big)/c$$

[see Worked Examples 2 for some applications]

Energy in EM Waves

- Energy density in EM fields: $u_e = \frac{1}{2}\varepsilon_0 E^2$ and $u_m = \frac{B^2}{2\mu_0}$
- For a plane EM wave $B^2 = E^2/c^2 = \mu_0\varepsilon_0 E^2$ so $u_e = u_m$
- Total EM energy density $u = \varepsilon_0 E^2 = \varepsilon_0 {E_0}^2 \cos^2(kz - \omega t)$
- Energy per unit area per unit time transported by the fields is given by the Poynting vector $\vec{S} = \vec{E} \times \vec{H} = \left(\vec{E} \times \vec{B}\right)/\mu_0$
- For plane EM wave in z-dirn. $\vec{S} = c\, \varepsilon_0 {E_0}^2 \cos^2(kz - \omega t)\, \hat{z}$
  i.e. $\vec{S} = cu\hat{z}$ $\quad$ $[\, S = EB/\mu_0 = E^2/\mu_0 c = c\varepsilon_0 E^2 \,]$
- Fields also carry momentum: momentum density (p.u. vol.)
  $\overrightarrow{\mathcal{P}} = \vec{S}/c^2 = (u/c)\hat{z}$
  [don't worry about origin of this]



Intensity and Pressure in EM Waves

- For light, $\lambda \sim 10^{-7}$ m and $T \sim 10^{-15}$ s so we only want the average values (averaged over many cycles)
- Average of $\cos^2 = 1/2$ and we denote time avg. by $\langle \quad \rangle$
- Then $\langle u \rangle = \frac{1}{2}\varepsilon_0 {E_0}^2$; $\langle \vec{S} \rangle = \frac{1}{2} c\,\varepsilon_0 {E_0}^2 \hat{z}$; $\langle \vec{\mathcal{P}} \rangle = \frac{1}{2c}\varepsilon_0 {E_0}^2 \hat{z}$
- Average power per unit area carried by EM wave is the **intensity** : $\boxed{I = \langle S \rangle = \frac{1}{2} c\, \varepsilon_0 {E_0}^2}$
- When EM wave falls on absorbing surface, change in momentum $\rightarrow$ force $\rightarrow$ **radiation pressure**:

$$P = \frac{F}{A} = \frac{\Delta p/\Delta t}{A} = \frac{\langle \mathcal{P} \rangle A c \Delta t}{A \Delta t} = \frac{1}{2c}\varepsilon_0 {E_0}^2$$

or $P = I/c$ For a perfectly reflecting surface the pressure is $2 \times$ this.