Time-Varying Systems; Maxwell's Equations

1. Faraday's law in differential form
2. Scalar and vector potentials; the Lorenz condition
3. Ampere's law with displacement current
4. Maxwell's equations
5. The Helmholtz theorem
6. Maxwell's equations in terms of potentials
7. Gauge transformations

Faraday's Law

- Familiar form : $\mathcal{E} = -\frac{d\Phi_m}{dt}$
- Integral form : $\oint_c \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B} \cdot d\vec{A}$
- Apply Stokes's theorem to the LHS and assume integration path C bounding surface S is fixed in space, so the RHS differentiation and integration can be interchanged:

$\int_S \left(\nabla \times \vec{E}\right) \cdot d\vec{a} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$ for any S, thus:

- Differential form : $\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$
- This is **induced electric field** for which $\nabla \cdot \vec{E} = 0$ (no charges), i.e. lines of induced $\vec{E}$ (sometimes called "non-electrostatic electric field" $\vec{E}_n$) are continuous.

Scalar and Vector Potentials (1)

- Faraday's law: $\nabla \times \vec{E} = -\partial\vec{B}/\partial t$ with $\vec{B} = \nabla \times \vec{A}$:

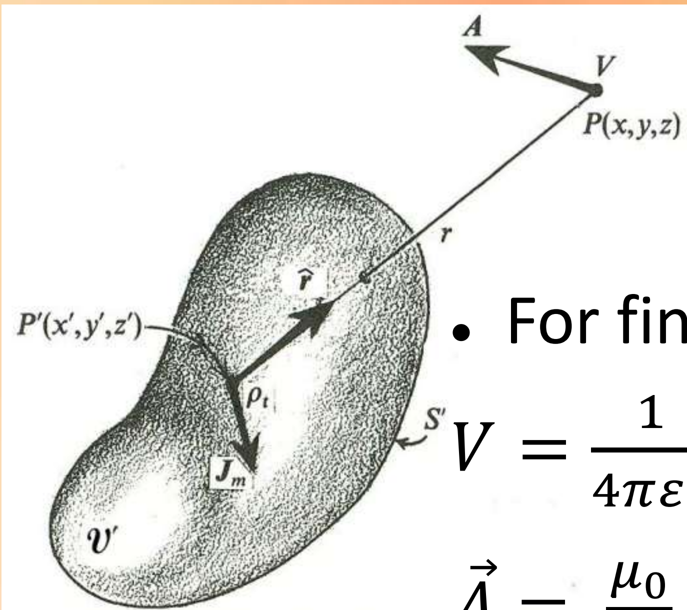
$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right) = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

i.e. $\nabla \times \left(\vec{E} + \partial\vec{A}/\partial t\right) = 0$

- ( ) must be a quantity whose curl = 0, viz. a grad of a scalar; call this $-\nabla V$, then

$$\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V}$$

- $\Rightarrow$ Electric field can arise from (1) changing magnetic fields and (2) potential gradients due to static charges.
- Above can also be derived from Faraday's law in the form $\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\oint \vec{A} \cdot d\vec{l}$ (use Stokes's theorem...)



Scalar and Vector Potentials (2)

- For finite charge and current distributions,

$$V = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \frac{\rho_f - \nabla.\vec{P}}{r} d\mathcal{V}' + \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma_f - \vec{P}.\hat{n}}{r} da'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}_f + \nabla\times\vec{M}}{r} d\mathcal{V}' + \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{\lambda}_f + \vec{M}\times\hat{n}}{r} da'$$

(free + bound charges and currents over volume + surface)

- In l.i.h. materials $\nabla.\vec{P} = 0$; $\nabla\times\vec{M} = 0$; $\varepsilon_0 \to \varepsilon$
- Differentiate these potentials for field vectors :

$\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla\times\vec{A}$

- Equations for V and $\vec{A}$ are very similar, and ρ_f and $\vec{J}_f$ are related through the continuity eqn.: $\partial\rho_f/\partial t + \nabla\cdot\vec{J} = 0$, so is there a relation between V and $\vec{A}$? Yes, indeed...

The Lorenz* Condition

- For free current distribution $\vec{J}_f$, $\quad \vec{A} = \frac{\mu}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}_f}{r} d\mathcal{V}'$
- Obtain div $\vec{A}$; distinguish derivatives at source point and field pt.: $\nabla \cdot \vec{A} = \frac{\mu}{4\pi}\int_{\mathcal{V}'} \frac{\nabla' . \vec{J}_f}{r} d\mathcal{V}' = -\frac{\mu}{4\pi}\int_{\mathcal{V}'} \frac{(\partial \rho_f / \partial t)}{r} d\mathcal{V}'$ [by continuity eqn.] $= -\varepsilon\mu \frac{\partial}{\partial t}\left(\int_{\mathcal{V}'} \frac{\rho_f}{r} d\mathcal{V}'\right) = -\varepsilon\mu\, \partial V / \partial t$

  i.e. $$\boxed{\nabla \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t} = 0}$$ [$\varepsilon_0\mu_0$ in vacuum]
- With this condition, we can compute 3 components of $\vec{B}$ and 3 components of $\vec{E}$ from only the 3 components of $\vec{A}$.

* Note: this eqn. is due to Ludvig V. Lorenz (Danish physicist, 1867), **not** Hendrik Antoon Lorentz (Dutch physicist, ca. 1900). Many textbooks (including Griffiths) propagate this error.

Ampere's Law Revisited (à la Maxwell)

- In defining $\vec{H}$, we obtained a version of "Ampere's law for free currents": $\nabla \times \vec{H} = \vec{J}_f$
- Now, as for any vector, $\nabla \cdot (\nabla \times \vec{H}) = 0$
  However, we have a **problem**: by the continuity eqn.,
  $\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}_f = -\partial\rho_f/\partial t \neq 0$ in general
- Maxwell: "something is missing" from Ampere's law.
  Add "**displacement current density**" $\vec{J}_d$,
  then $\nabla \times \vec{H} = \vec{J} = \vec{J}_f + \vec{J}_d$ and we have
  $\nabla \cdot \vec{J} = \nabla \cdot \vec{J}_f + \nabla \cdot \vec{J}_d = -\partial\rho_f/\partial t + \nabla \cdot \vec{J}_d = 0$
  With Gauss's law for free charges,
  $\nabla \cdot \vec{J}_d = \partial\rho_f/\partial t = \partial/\partial t\,(\nabla \cdot \vec{D}) = \nabla \cdot (\partial\vec{D}/\partial t)$

Displacement Current

- We have $\nabla \cdot \left(\vec{J}_d - \partial\vec{D}/\partial t\right) = 0$ to which the obvious solution is $\boxed{\vec{J}_d = \partial\vec{D}/\partial t}$ (no reason to add anything) and the term "displacement current" now makes sense.
- By def. $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ and in vacuum $\vec{D} = \varepsilon_0\vec{E}$
- "**Ampere's law with displacement current**" (Maxwell's mod): $\boxed{\nabla \times \vec{H} = \vec{J}_f + \partial\vec{D}/\partial t}$

or in terms of $\vec{B}$, $\nabla \times \vec{B} = \mu_0\vec{J}_{\mathrm{mat}} + \mu_0\varepsilon_0(\partial\vec{E}/\partial t)$

where $\vec{J}_{\mathrm{mat}}$ is the total current density "in matter" (conduction + polarization + magnetization currents) $\vec{J}_{\mathrm{mat}} = \vec{J}_f + \frac{\partial\vec{P}}{\partial t} + \nabla \times \vec{M}$

while the displacement current can "flow" in free space.

Maxwell's Equations (1)

(A) In terms of $\vec{E}$ and $\vec{B}$ only:

- $\nabla \cdot \vec{E} = \dfrac{\rho_t}{\varepsilon_0}$ Gauss's Law $[\rho_t = \rho_f + \rho_b]$
- $\nabla \cdot \vec{B} = 0$ Magnetic field divergenceless
- $\nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$ Faraday's law
- $\nabla \times \vec{B} - \mu_0 \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}_{\mathrm{mat}}$ Ampere's law

$$[\ \vec{J}_{\mathrm{mat}} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times \vec{M}\ ]$$

[this version: fields on LHS, sources on RHS]

Maxwell's Equations (2)

(B) In terms of four fields $\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$:

- $\nabla \cdot \vec{D} = \rho_f$ Gauss's Law (free charges)
- $\nabla \cdot \vec{B} = 0$ Magnetic field divergenceless
- $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ Faraday's law
- $\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f$ Ampere's law

[no constants, free charges and currents only]

Remember Maxwell's equations are PDEs with space and time derivatives of the field vectors. To find the fields we need to integrate and apply boundary conditions...

The Helmholtz Theorem

- Maxwell's equations specify the div and curl of $\vec{E}$ and $\vec{B}$:

$$\nabla \cdot \vec{E} = \frac{\rho_t}{\varepsilon_0} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{B} = \mu_0 \vec{J}_{\mathrm{mat}} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

- If we require the fields to $\rightarrow 0$ at infinity, then $\vec{E}$ and $\vec{B}$ are uniquely determined by these equations.
- This is the **Helmholtz theorem**: If the divergence $D(\vec{r})$ and the curl $\vec{C}(\vec{r})$ of a vector function $\vec{F}(\vec{r})$ are both specified, and if they both go to zero faster than $1/r^2$ as $r \rightarrow \infty$, and if $\vec{F}(\vec{r})$ goes to zero as $r \rightarrow \infty$, then $\vec{F}$ is given uniquely by ...

The Helmholtz Theorem continued

- $\vec{F}$ is given uniquely by: $\quad \vec{F} = -\nabla U + \nabla \times \overrightarrow{W}$

where $U(\vec{r}) \equiv \frac{1}{4\pi}\int \frac{D(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\mathcal{V}'$ and $\overrightarrow{W}(\vec{r}) \equiv \frac{1}{4\pi}\int \frac{\vec{C}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\mathcal{V}'$

- We expect electric and magnetic fields to go to zero at ∞, so the above argument is valid for $\vec{E}$ and $\vec{B}$.
- **Corollary** of Helmholtz theorem: any (differentiable) vector function $\vec{F}(\vec{r})$ that $\rightarrow 0$ faster than $1/r$ as $r \rightarrow \infty$ can be expressed as grad(scalar) plus curl (vector):

$$\vec{F}(\vec{r}) = \nabla\left(\frac{-1}{4\pi}\int \frac{\nabla' \cdot \vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\mathcal{V}'\right) + \nabla \times \left(\frac{1}{4\pi}\int \frac{\nabla' \times \vec{F}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\mathcal{V}'\right)$$

[check this for static $\vec{E}$ and $\vec{B}$ (using Gauss and Ampere)]

Scalar and Vector Potentials again

- Potential formulation: $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V$
- This satisfies $\nabla \cdot \vec{B} = 0$ and $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$

  (the eqn. for $\vec{E}$ was derived from Faraday's law)
- Substitute in Gauss's law $\nabla \cdot \vec{E} = \rho/\varepsilon_0$:

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\rho/\varepsilon_0}$$

- Substitute in Ampere's law $\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$:

$$\nabla \times \left(\nabla \times \vec{A}\right) = \mu_0 \vec{J} - \mu_0 \varepsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right) - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2} \quad \text{or ... (\#)}$$

$$\boxed{\left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}}$$

- Maxwell's equations in terms of potentials

Gauge Transformations (1)

- The equations $\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\rho/\varepsilon_0$
  and $\left(\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \vec{J}$
  are cumbersome. But V and $\vec{A}$ are not uniquely defined.
- Consider two sets of potentials $(V, \vec{A})$ and $(V', \vec{A}')$ which correspond to the same fields.
- Write $\vec{A}' = \vec{A} + \vec{\alpha}$; $V' = V + \beta$
  Since $\nabla \times \vec{A}' = \nabla \times \vec{A} = \vec{B}$, we must have $\nabla \times \vec{\alpha} = 0$
  and so $\vec{\alpha} = \nabla\lambda$, for some scalar λ.
  Also $-\partial\vec{A}'/\partial t - \nabla V' = -\partial\vec{A}/\partial t - \nabla V = \vec{E}$ so
  $\partial\vec{\alpha}/dt + \nabla\beta = 0$ or $\nabla(\partial\lambda/\partial t + \beta) = 0$
  ( ) indep. of position, but could depend on t, say $k(t)$,
  then $\beta = -\partial\lambda/\partial t + k(t)$

Gauge Transformations: Coulomb

- Redfine $\lambda = \lambda + \int_0^t k(t')dt'$; $\nabla\lambda$ unchanged
- Thus for any scalar function λ we can add $\nabla\lambda$ to $\vec{A}$ and subtract $\partial\lambda/\partial t$ from V without changing $\vec{E}$ and $\vec{B}$. This is a **gauge transformation**.

(a) The Coulomb Gauge :

- As for magnetostatics, set $\nabla \cdot \vec{A} = 0$ then $\nabla^2 V = -\rho/\varepsilon_0$ (Poisson's eqn.) with soln. $V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\prime,t)}{|\vec{r}-\vec{r}\prime|} d\mathcal{V}'$
- But for $\vec{E}$ we also require $\vec{A}$ where [from (#) with $\nabla.\vec{A} = 0$] $\nabla^2\vec{A} - \mu_0\varepsilon_0\,\partial^2\vec{A}/\partial t^2 = -\mu_0\vec{J} + \mu_0\varepsilon_0\nabla(\partial V/\partial t)$
- For the Coulomb Gauge, while V is easy, $\vec{A}$ is very difficult to calculate – a major disadvantage. Also no 'symmetry'.

Gauge Transformations: Lorenz

(b) The Lorenz Gauge :

- Here put (see slide 6) $\boxed{\nabla \cdot \vec{A} = -\mu_0 \varepsilon_0 \, \partial V / \partial t}$

  (the Lorenz condition). Then eqn. (#) (slide 13) becomes

  [$\mu_0 \varepsilon_0 = 1/c^2$] $\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \, \partial^2 \vec{A} / \partial t^2 = -\mu_0 \vec{J}$

- The eqn. for V is $\nabla^2 V - \mu_0 \varepsilon_0 \, \partial^2 V / \partial t^2 = -\rho / \varepsilon_0$

  i.e. the equations are identical; there is 'symmetry' between $\vec{A}$ and V, a major advantage of the Lorenz gauge.

- The same differential operator, the **d'Alembertian**

  $\boxed{\Box^2 \equiv \nabla^2 - (1/c^2) \, \partial^2 / \partial t^2}$ operates on both:

  $\Box^2 V = -\rho / \varepsilon_0$ and $\Box^2 \vec{A} = -\mu_0 \vec{J}$

Gauge Transformations

The Lorenz Gauge (continued) $\nabla \cdot \vec{A} = -\mu_0 \varepsilon_0 \, \partial V / \partial t$

$$\Box^2 V = -\rho/\varepsilon_0 \quad \text{and} \quad \Box^2 \vec{A} = -\mu_0 \vec{J}$$

- In special relativity, the d'Alembertian $\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is the natural extension of the Laplacian ∇^2 to 4-D ; above eqns. are 4-D equivalents of Poisson's equation.
- In the Lorenz gauge, V and $\vec{A}$ satisfy the **inhomogeneous wave equation**. The problem of finding the potentials for given sources then becomes a matter of solving the inhomogeneous wave eqn. for these sources.

$$\boxed{\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \quad , \quad \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}}$$

Gauge Invariance

- If the fields are invariant under a gauge transformation, then we have **gauge invariance**.
- The Lorenz gauge is most commonly used
  (i) because of its equal treatment of V and $\vec{A}$ leading to the inhomogeneous wave equations, and
  (ii) because it is independent of the coordinate system and so fits naturally with special relativity.
- The Lorenz condition $\nabla \cdot \vec{A} = -\mu_0 \varepsilon_0 \dfrac{\partial V}{\partial t}$ and wave eqns.
  $\nabla^2 \vec{A} - \mu_0 \varepsilon_0 \dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$ and $\nabla^2 V - \mu_0 \varepsilon_0 \dfrac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon_0$
  unite 'electricity' and 'magnetism', and show $\mu_0 \varepsilon_0 = 1/c^2$ just as Maxwell's equations do, but in terms of potentials.