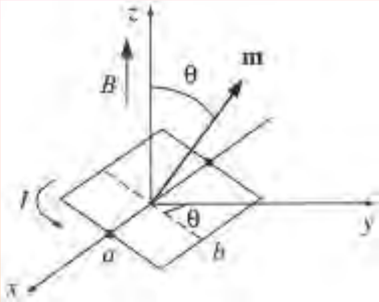
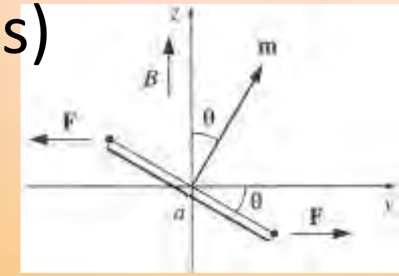
Magnetic Materials

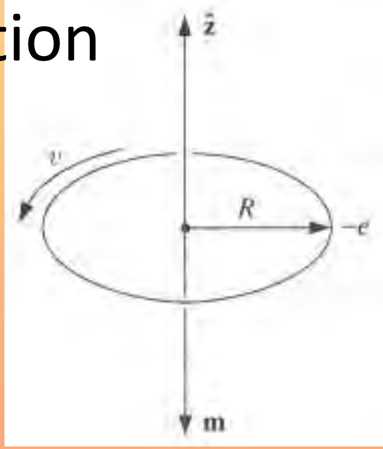
1. Magnetization
2. Potential and field of a magnetized object
3. H-field
4. Susceptibility and permeability
5. Boundary conditions
6. Magnetic field energy and magnetic pressure

Magnetic Materials (1)

- Magnetic dipole with moment $\vec{m}$ in uniform $\vec{B}$ experiences torque $\vec{\tau} = \vec{m} \times \vec{B}$ tending to align $\vec{m}$ with $\vec{B}$:
- In non-uniform $\vec{B}$, net force $\vec{F} = \nabla\left(\vec{m} \cdot \vec{B}\right)$
- Analogous to $\vec{\tau} = \vec{p} \times \vec{E}$ and $\vec{F} = \nabla\left(\vec{p} \cdot \vec{E}\right)$ for electric dipole (above for permanent dipoles)
- Also: orbiting electron: current $I = \frac{ev}{2\pi R}$, dipole moment $\vec{m} = -\frac{1}{2}evR\hat{z}$
- In $\vec{B}$ field, tilting of $\vec{m}$ minimal (orbital contribution to paramagnetism very small), but orbit speed increases and we get $\Delta\vec{m} = -\frac{e^2R^2}{4m_e}\vec{B}$
- This is mechanism of **diamagnetism**: induced effect in opposite direction to $\vec{B}$.

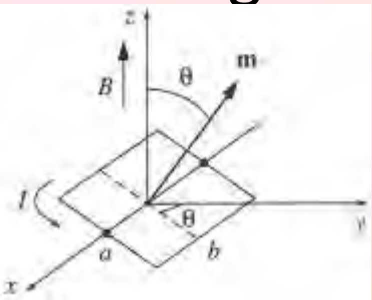






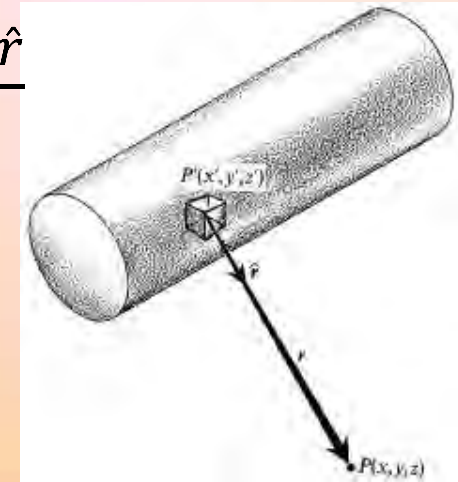
Magnetic Materials (2)

- Magnetic dipole moment due to electron **spin** shows slight tendency to align with $\vec{B}$; this is mechanism of **paramagnetism** (effect in same direction as $\vec{B}$).



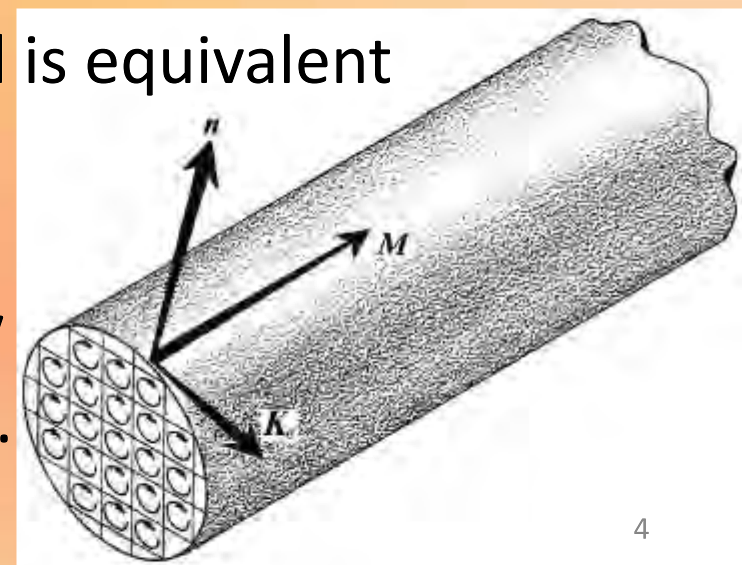
- Pauli exclusion principle $\rightarrow$ opposite spins $\rightarrow$ cancellation for even numbers; paramagnetism usually in atoms or molecules with odd number of electrons. Effect stronger than diamagnetism; latter mainly for even nos.
- Ferromagnetism: strong paramagnetic alignment with $\vec{B}$ due to coupling of spins (QM).
- Define **magnetization** $\boxed{\vec{M} = n\vec{m}}$ (n atoms/unit vol., average atomic magnetic dipole moment $\vec{m}$) $=$ magnetic dipole moment per unit volume (whether dia-, para- or ferro-magnetic) [units: A/m]. Analogous to polarization $\vec{P} = n\vec{p}$.

Potential of a Magnetized Object



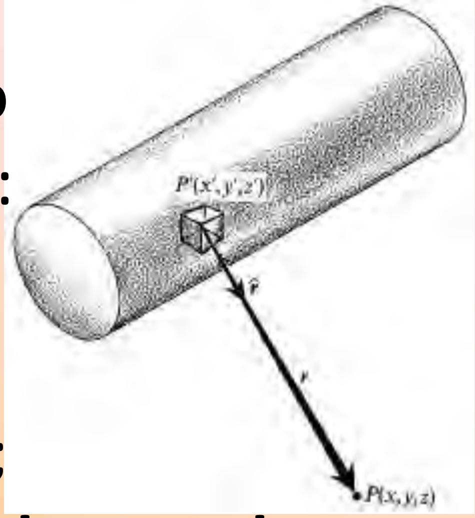
- Vector potential of current loop is $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$
- Volume element $d\mathcal{V}'$ has $\vec{m} = \vec{M}d\mathcal{V}'$ so
  $\vec{A} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}'}\frac{\vec{M}\times\hat{r}}{r^2}d\mathcal{V}'$ for whole body.
- This can be written as

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int_{S'}\frac{\vec{M}\times\hat{n}}{r}da' + \frac{\mu_0}{4\pi}\int_{\mathcal{V}'}\frac{\nabla'\times\vec{M}}{r}d\mathcal{V}'}$$



- Shows $\vec{A}$ due to magnetic material is equivalent to $\vec{A}$ of **equivalent surface current density** $\boxed{\vec{K}_e = \vec{M}\times\hat{n}}$ plus $\vec{A}$ of **equivalent volume current density** $\boxed{\vec{J}_e = \nabla\times\vec{M}}$ ("Amperian currents").
  [If material is uniform $\vec{J}_e = 0$.]

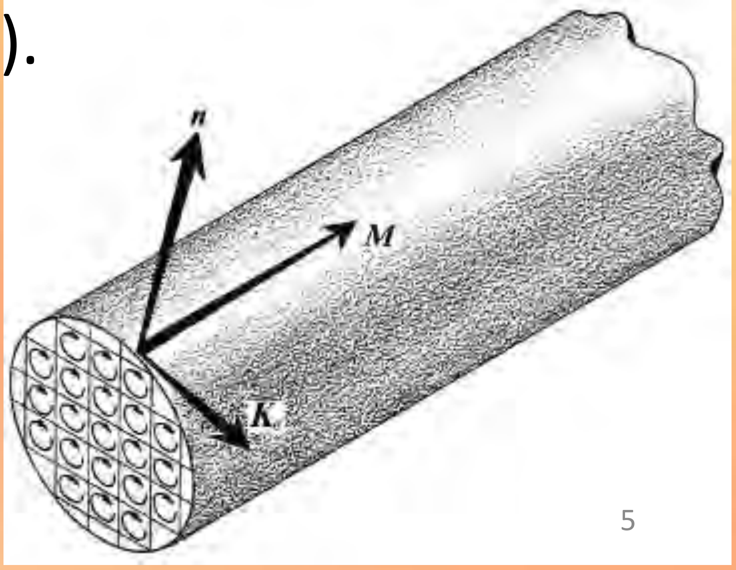
Field of a Magnetized Object



- Since equivalence works for $\vec{A}$, also for $\vec{B}$, so use Biot-Savart-law for surface, vol. currents:

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{K}_e \times \hat{r}}{r^2} da' + \frac{\mu_0}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}_e \times \hat{r}}{r^2} d\mathcal{V}'}$$

- This eqn. is valid outside and inside material; we can **always** replace material by **equivalent currents** $\vec{K}_e = \vec{M} \times \hat{n}$ and (if non-uniform) $\vec{J}_e = \nabla \times \vec{M}$ then calculate $\vec{B}$ as if in vacuum (note μ_0).
- So even with materials, $\nabla \cdot \vec{B} = 0$ always.
- Note also that $\nabla \cdot \vec{J}_e = 0$ (div of curl of any vector is zero), i.e. no accumulation of bound charges.



Magnetic Field Intensity $\vec{H}$ ("H-field")

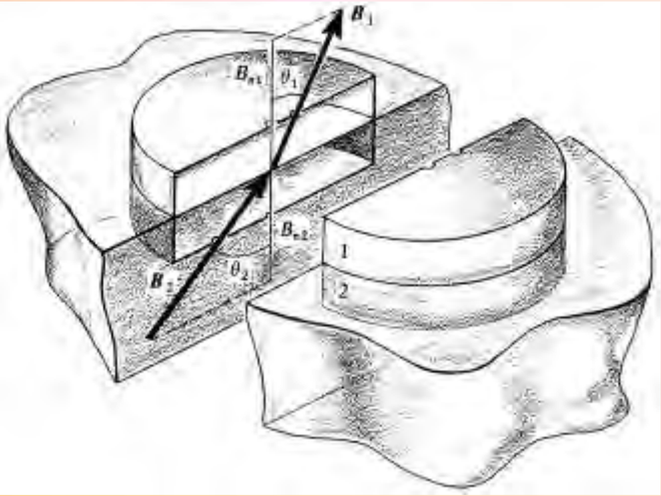
- Ampere's law with magnetic materials:
  $\boxed{\nabla \times \vec{B} = \mu_0\left(\vec{J}_f + \vec{J}_e\right)}$ (free currents + bound currents)
- Substitute $\vec{J}_e = \nabla \times \vec{M}$ then RHS $= \mu_0\left(\vec{J}_f + \nabla \times \vec{M}\right)$
  or $\nabla \times \left(\vec{B}/\mu_0 - \vec{M}\right) = \vec{J}_f$ (free current density only)
- Define "**magnetic field intensity**" $\boxed{\vec{H} = \vec{B}/\mu_0 - \vec{M}}$
  then $\boxed{\nabla \times \vec{H} = \vec{J}_f}$ **Ampere's law for free currents**.
- Analogous to $\nabla \cdot \vec{D} = \rho_f$ (Gauss's law for free charges)
  where electric displacement $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$.
- Integrate over surface S and apply Stokes's theorem to obtain the integral form of Ampere's law for free currents:

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J}_f \cdot d\vec{a} = I_f$$ (curve C bounds surface S)

Magnetic Susceptibility χ_m , Permeability μ

- For non-ferromagnetic materials, magnetization $\vec{M} \propto \vec{H}$: $\boxed{\vec{M} = \chi_m \vec{H}}$ (linear) where $\chi_m =$ **magnetic susceptibility**
- χ_m is dimensionless ($\vec{M}$ and $\vec{H}$ both have units $\mathrm{A/m}$), positive for paramagnetics, negative for diamagnetics, typical values $\sim 10^{-5}$
- For linear materials also $\vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) = \mu_0(1 + \chi_m)\vec{H}$ written as $\boxed{\vec{B} = \mu \vec{H}}$ where **permeability** $\mu = \mu_0(1 + \chi_m)$
- In vacuum $\chi_m = 0$ and $\mu = \mu_0$ **permeability of free space** so that $\vec{B} = \mu_0 \vec{H}$
- Also **relative permeability** $\mu_r = 1 + \chi_m = \mu/\mu_0$
- Since χ_m is very small for non-ferromagnetics, we can often assume to a reasonable approximation $\mu_r \approx 1$ or $\mu = \mu_0$, whereas we cannot do this for ε_r and permittivity ε .

Boundary Conditions (1)



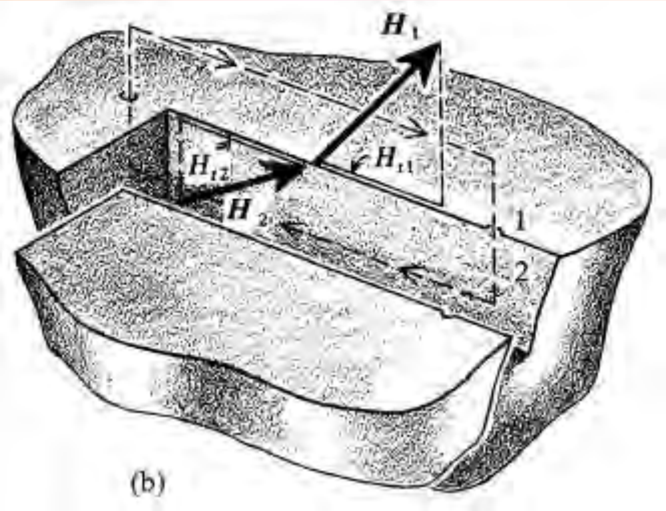
- Interface between two media, with relative permeabilities μ_{r1} and μ_{r2}
- Assume no free currents on interface
- Short 'Gaussian' cylinder across boundary: $\oint_S \vec{B}\cdot d\vec{a} = 0$ always

  Let length of cylinder $\to 0$, then zero flux through sides and

  $\int_{S_1} \vec{B}\cdot d\vec{a} + \int_{S_2} \vec{B}\cdot d\vec{a} = 0$ (ends S_1 and S_2 in 1 and 2)

  i.e. $\int_{S_1} B_{n1} da - \int_{S_2} B_{n2} da = 0$ ($\vec{B}_2$ inwards, so negative)
- Hence $\boxed{B_{n1} = B_{n2}}$ i.e. **normal component of $\vec{B}$ is continuous across the boundary**

Boundary Conditions (2)



- Rectangular path across boundary:
  $\oint_C \vec{H} \cdot d\vec{l} = 0$ if no free currents
- Sides perpendicular to boundary $\rightarrow 0$
  then $\int_{C_1} \vec{H} \cdot d\vec{l} + \int_{C_2} \vec{H} \cdot d\vec{l} = 0$
  (sides C_1 and C_2 in media 1 and 2)
  i.e. $\int_{C_1} H_{t1} dl - \int_{C_2} H_{t2} dl = 0$ ($\vec{H}_2$ component opp. to $d\vec{l}$)
- Hence $\boxed{H_{t1} = H_{t2}}$ i.e. **tangential component of $\vec{H}$ is continuous across the boundary**
- $\Rightarrow$ 'refraction' of magnetic fieldlines at boundary:
  $\mu_{r1} \cot\theta_1 = \mu_{r2} \cot\theta_2$ i.e. $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\mu_{r1}}{\mu_{r2}}$

Magnetic Field Energy

- For current density $\vec{J}$ in a conductor, power delivered to vol. element $d\mathcal{V}$ is $-\nabla V . d\vec{l}\, J_f da$ with $-\nabla V = \vec{E} + \partial\vec{A}/\partial t$
- Total power from source is (1st term Joule heating)

$$\frac{dU}{dt} = \int_{\mathcal{V}'} \vec{E} . \vec{J}_f \, d\mathcal{V}' + \int_{\mathcal{V}'} \frac{\partial\vec{A}}{\partial t} . \vec{J}_f \, d\mathcal{V}'$$

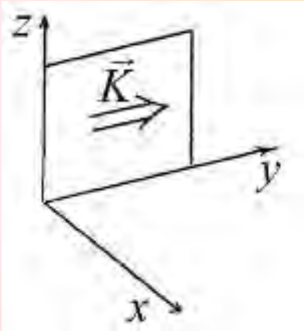
- In terms of $\vec{B}$ use Ampere's law: $\nabla \times \vec{B} = \mu_0 \vec{J}_f$
- Similar arguments to $\vec{E}$ and take $\int dt$:

$\Rightarrow U_m = \frac{1}{2\mu_0} \int_{\mathcal{V}'} B^2 d\mathcal{V}'$;energy density $\boxed{u_m = \frac{dU_m}{d\mathcal{V}} = \frac{B^2}{2\mu_0}}$

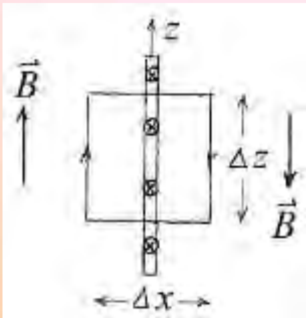
- In magnetic materials $\frac{dU_m}{d\mathcal{V}} = \frac{1}{2} \vec{B} \cdot \vec{H}$;

  and in l.i.h. materials $\frac{dU_m}{d\mathcal{V}} = \frac{B^2}{2\mu}$

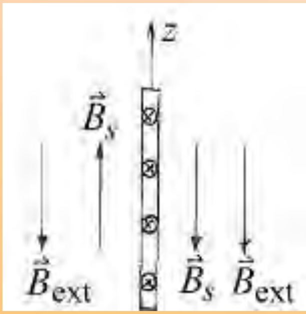
Magnetic Pressure (1)



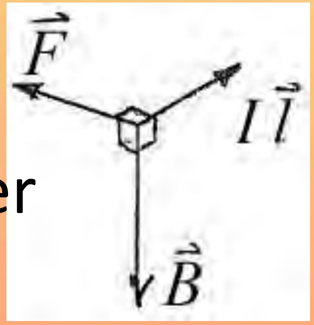
- Consider surface current density $\vec{K}$ in yz-plane:
  $\vec{K} = K\hat{y}$ and $K = dI/dz$ A/m



- By the right hand rule and symmetry,
  $\vec{B} = +B\hat{z}$ for $x < 0$ and $\vec{B} = -B\hat{z}$ for $x > 0$
- Ampere's law for loop Δx by Δz :
  $\oint \vec{B} \cdot d\vec{l} = 2\,B\Delta z = \mu_0 I_{\text{enc}} = \mu_0 K \Delta z$
  $\Longrightarrow\ \vec{B} = \pm\,(\mu_0 K/2)\,\hat{z}$ on the two sides



- Now place sheet in external field $\vec{B}_{\text{ext}} = -\dfrac{\mu_0 K}{2}\hat{z}$
  Fields cancel for $x < 0$, add for $x > 0$.
- If we rename B_{ext} as $B/2$, then if $K = B/\mu_0$,
  we will have field $\vec{B}$ on one side, zero on the other



- Force $\vec{F} = I\vec{l} \times \vec{B}$ is in $-x$ direction by RH rule

Magnetic Pressure (2)

- Here we have surface current density $\vec{K}$ so $I = K\Delta z$ and since $\vec{l} = \Delta y\hat{y}$ here, we have $\vec{F} = K\Delta z\,\Delta y\,\hat{y} \times (B/2)(-\hat{z})$
- But $\Delta y \Delta z$ is an element of area of the current sheet, so we get $\vec{F}/A = K(B/2)(-\hat{x})$, i.e. since $K = B/\mu_0$ we have a **magnetic pressure** $\boxed{\dfrac{F}{A} = \dfrac{B^2}{2\mu_0}}$
- This is the same as the energy density [$N/m^2 = N.m/m^3 = J/m^3$]
- Pressure on current sheet on side with field $\vec{B}$. Important in space physics.