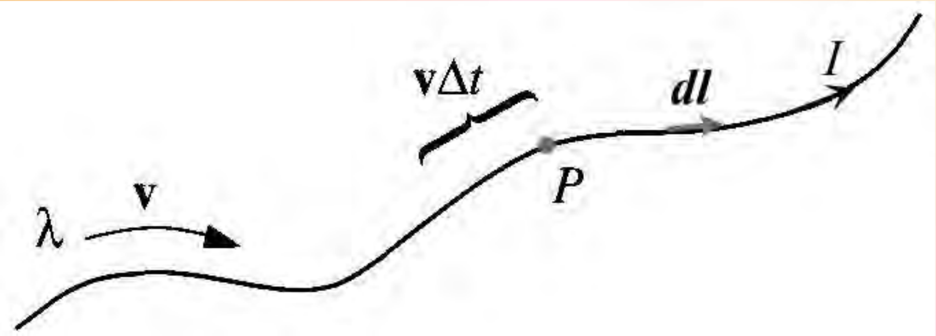
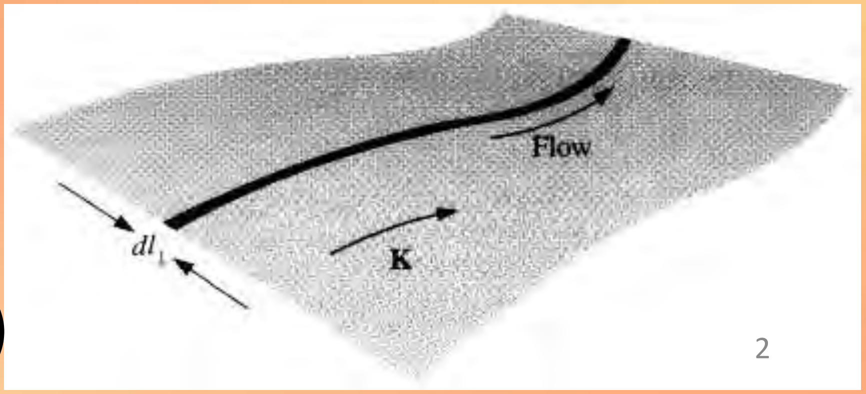
Magnetostatics

1. Currents
2. Relativistic origin of magnetic field
3. Biot-Savart law
4. Magnetic force between currents
5. Applications of Biot-Savart law
6. Ampere's law in differential form
7. Magnetic vector potential

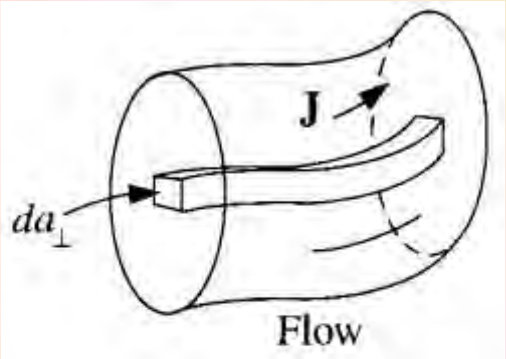
Currents (1)



- Line charge λ (C/m) with velocity $\vec{v}$: in time Δt , segment with length $v\Delta t$, charge $\lambda v \Delta t$ passes point P. This constitutes a **current** $\vec{I} = \lambda\vec{v}$ (vector).
- **Magnetic force** on a segment of length $d\vec{l}$ is
  $$\vec{F}_{\text{mag}} = \int\left(\vec{v} \times \vec{B}\right)dq = \int\left(\vec{v} \times \vec{B}\right)\lambda dl = \int\left(\vec{I} \times \vec{B}\right)dl$$
  But $\vec{I}$ and $d\vec{l}$ are in the same direction so we write
  $$\boxed{\vec{F}_{\text{mag}} = I\int\left(d\vec{l} \times \vec{B}\right)}$$ (usually I constant around circuit)
- When charge flows over a surface:
  if current $d\vec{I}$ in width $dl_{\perp}$ then **surface current density** $\vec{K} = d\vec{I}/dl_{\perp} = \sigma\vec{v}$ (surface charge density σ at velocity $\vec{v}$)

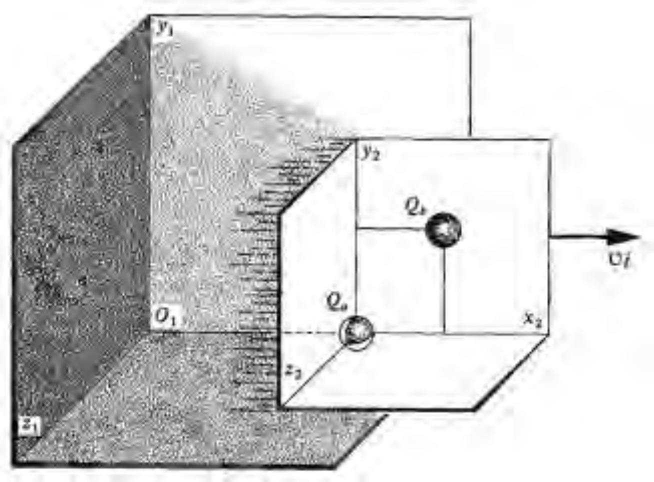


Currents (2)



- Magnetic force on a surface current:
  $\vec{F}_{\text{mag}} = \int (\vec{v} \times \vec{B}) \sigma \, da = \int (\vec{K} \times \vec{B}) \, da$
- When charge flows through a volume, we define **volume current density** $\vec{J} = d\vec{I}/da_\perp$ ('current per unit area-⊥-flow')
  or $\vec{J} = \rho \vec{v}$ (vol. charge density ρ moving at velocity $\vec{v}$)
- Magnetic force on a volume current:
  $\vec{F}_{\text{mag}} = \int (\vec{v} \times \vec{B}) \, \rho \, d\mathcal{V} = \int (\vec{J} \times \vec{B}) \, d\mathcal{V}$
- Current crossing surface S is $I = \int_S J da_\perp = \int_S \vec{J} . d\vec{a}$
- **Conservation of charge**: for closed surface S, volume $\mathcal{V}$,
  $\oint_S \vec{J} . d\vec{a} = \int_{\mathcal{V}} (\nabla . \vec{J}) d\mathcal{V} = -\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\mathcal{V} = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}\right) d\mathcal{V}$
- Hence: **Continuity equation**: $\boxed{\nabla . \vec{J} = -\frac{\partial \rho}{\partial t}}$

Origin of the Magnetic Field (1)



- Consider charges Q_a and Q_b fixed in frame 2, which is moving at constant velocity $v\hat{\imath}$ relative to frame 1.
- Assume charge invariance.
- Coulomb force $\vec{F}_{b2}$ exerted by Q_a on Q_b in frame 2 has components :

$$X_{b2} = \frac{Q_a Q_b x_2}{4\pi\varepsilon_0 {r_2}^3};\ Y_{b2} = \frac{Q_a Q_b y_2}{4\pi\varepsilon_0 {r_2}^3};\ Z_{b2} = 0$$

- Transform $\vec{F}_b$ to frame 1 at time $t_1 = t_2 = 0$:

$$X_b = \frac{\gamma Q_a Q_b x}{4\pi\varepsilon_0(\gamma^2 x^2 + y^2)^{3/2}};\ Y_b = \frac{Q_a Q_b y}{4\pi\varepsilon_0 \gamma(\gamma^2 x^2 + y^2)^{3/2}};\ Z_b = 0$$

- Rewrite $\vec{F}_b$ to obtain

$$\vec{F}_b = \frac{\gamma Q_a Q_b \vec{r}}{4\pi\varepsilon_0(\gamma^2 x^2 + y^2)^{3/2}} - \frac{\gamma Q_a Q_b v^2 y \hat{\jmath}}{4\pi\varepsilon_0 c^2(\gamma^2 x^2 + y^2)^{3/2}}$$ which yields...

Origin of the Magnetic Field (2)

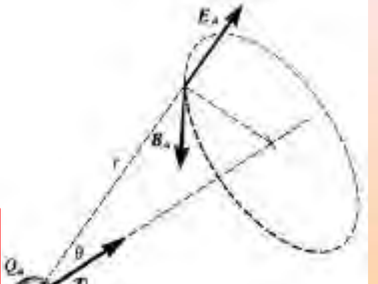
$$\vec{F}_b = Q_b\left\{\frac{\gamma Q_a \vec{r}}{4\pi\varepsilon_0(\gamma^2x^2+y^2)^{3/2}} + \left[\vec{v}\times\frac{\gamma Q_a v y\hat{k}}{4\pi\varepsilon_0 c^2(\gamma^2x^2+y^2)^{3/2}}\right]\right\}$$

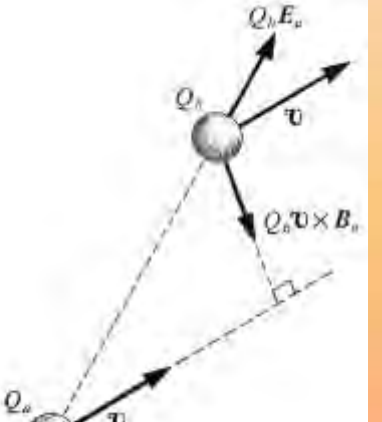
Define $4\pi\varepsilon_0 c^2 = 10^7$ or $\boxed{1/\varepsilon_0 c^2 = \mu_0 = 4\pi\times 10^{-7}\ \mathrm{H/m}}$, the **permeability of free space**. Write this as

$\boxed{\vec{F}_b = Q_b\{\vec{E}_a + \vec{v}\times\vec{B}_a\}}$ (**Lorentz force**) where

$$\boxed{\vec{E}_a = \frac{\gamma Q_a \vec{r}}{4\pi\varepsilon_0(\gamma^2x^2+y^2)^{3/2}} \text{ and } \vec{B}_a = \frac{\mu_0\gamma Q_a v y\hat{k}}{4\pi(\gamma^2x^2+y^2)^{3/2}}}$$

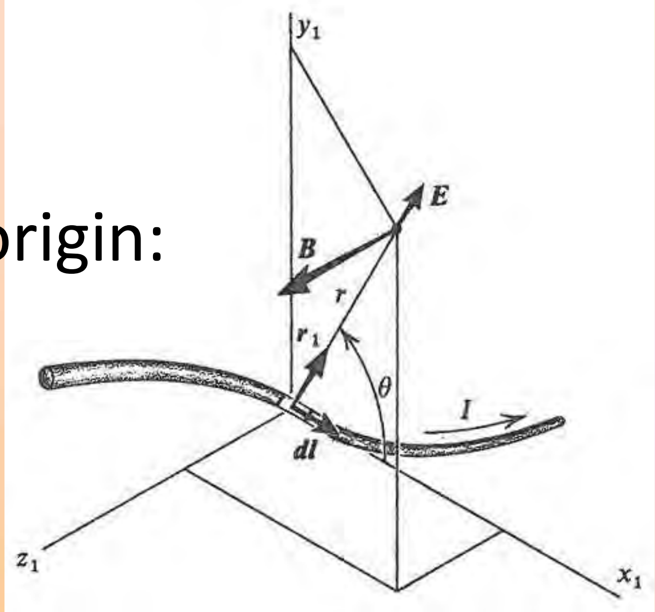
are the **electric & magnetic fields** of Q_a in frame 1. Note direction of $\vec{B}$ (the "magnetic induction").





- *The magnetic field of frame 1 has appeared as a result of the application of a relativistic transformation to the electric force in frame 2.*

Biot-Savart Law (1)



- Consider short length of wire $d\vec{l}$ at origin:
- Positive charge $\lambda_p dl$ stationary
- Negative charge $\lambda_n dl$, velocity $-\vec{v}$
- Electric field at $\vec{r}$ due to $+$ and $-$ is

$$\vec{E} = \frac{\lambda_p dl}{4\pi\varepsilon_0 r^2}\left\{1 - \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta)^{3/2}}\right\}\hat{r}$$ (note: non-zero)

- Magnetic field due to moving $-$ charges is

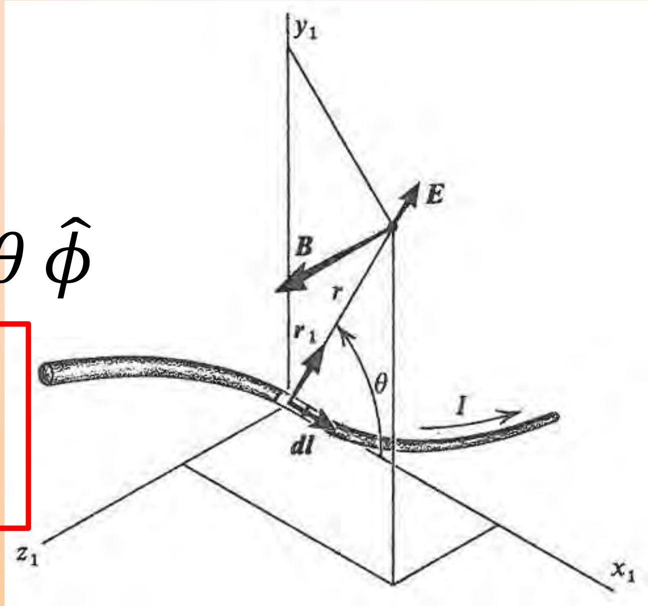
$$\vec{B} = \frac{\mu_0(1-\beta^2)\, v\, \lambda_p\, dl}{4\pi r^2(1-\beta^2(\sin\theta)^2)^{3/2}}\hat{\phi}$$

- Approximation: $\beta^2 \ll 1$

- Using $I = \lambda_n(-\vec{v}) = \lambda_p v$ and $1/(\varepsilon_0 c^2) = \mu_0$ we obtain

$$\vec{E} \approx \frac{\mu_0 I dl}{4\pi r^2} v\left\{1 - \frac{3}{2}\sin^2\theta\right\}\hat{r}$$ for the electric field and ...

Biot-Savart Law (2)

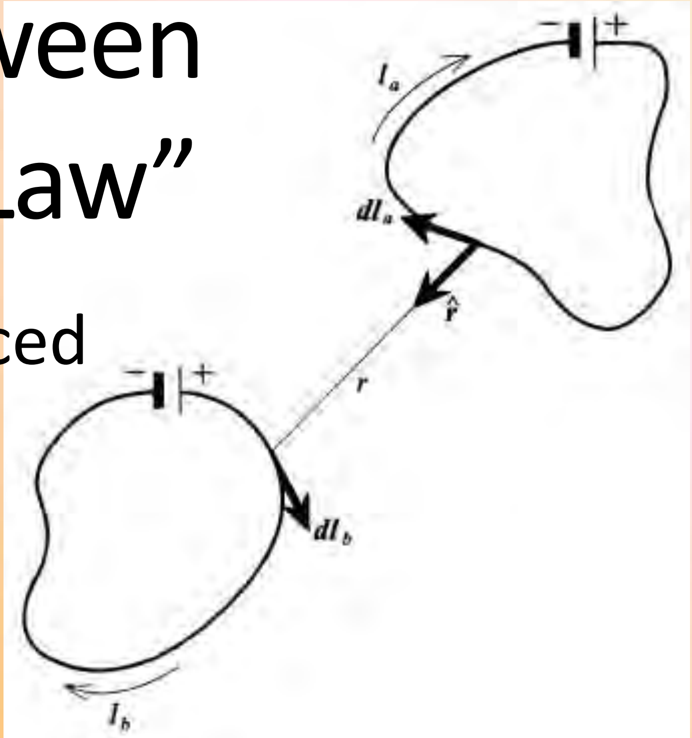


$$\vec{B} = \frac{\mu_0 I dl}{4\pi r^2}(1-\beta^2)\left\{1+\tfrac{3}{2}\beta^2 \sin^2\theta\right\}\sin\theta\,\hat{\phi}$$

$$\approx \frac{\mu_0 I dl \sin\theta}{4\pi r^2}\hat{\phi} \quad \text{i.e.} \quad \boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l}\times\hat{r}}{r^2}}$$

- The Biot-Savart law can be derived directly from the relativistic transformations.
- We usually call the above $d\vec{B}$ due to current element $Id\vec{l}$ so that for a complete circuit we have $\vec{B} = \frac{\mu_0}{4\pi} I \oint \frac{d\vec{l}\times\hat{r}}{r^2}$
- Recall that $\vec{r}$ is always from the source point (in this case the current element) to the field point where we determine $\vec{B}$, and that $\vec{B}$ is in the azimuthal direction.
- "Magnetic field strength" $\vec{B}$: unit tesla (T) = weber/m^2

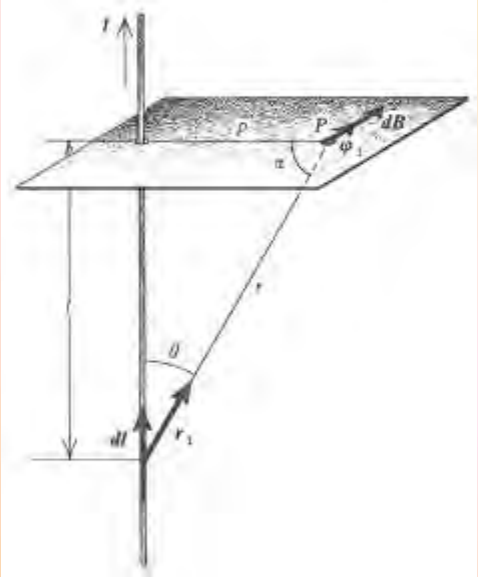
Magnetic Forces Between Currents: "Ampere's Law"

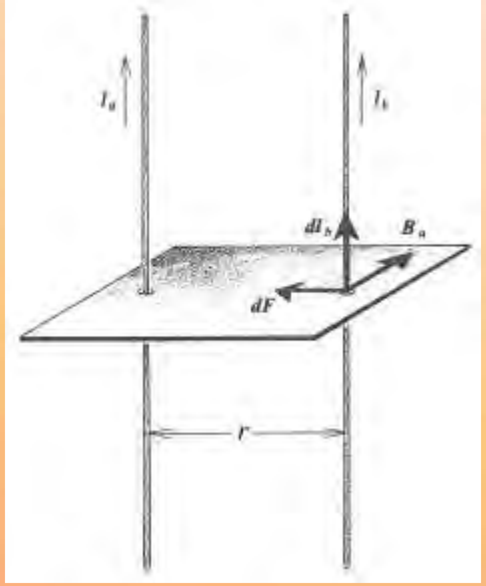


- Biot-Savart law gives $\vec{B}$-field produced by current I_a: $\vec{B}_a = \frac{\mu_0 I_a}{4\pi} \oint_a \frac{d\vec{l}_a \times \hat{r}}{r^2}$
- If this is at position of current I_b, there is force $d\vec{F}_{ab} = I_b d\vec{l}_b \times \vec{B}_a$
- For circuit $\boxed{\vec{F}_{ab} = \frac{\mu_0}{4\pi} I_a I_b \oint_a \oint_b \frac{d\vec{l}_b \times (d\vec{l}_a \times \hat{r})}{r^2}}$
- This is often called the "**magnetic force law**", derived by Ampere. It is the magnetic equivalent of Coulomb's law, with the product of the currents in place of charges and an inverse square dependence (just a bit more geometry!)

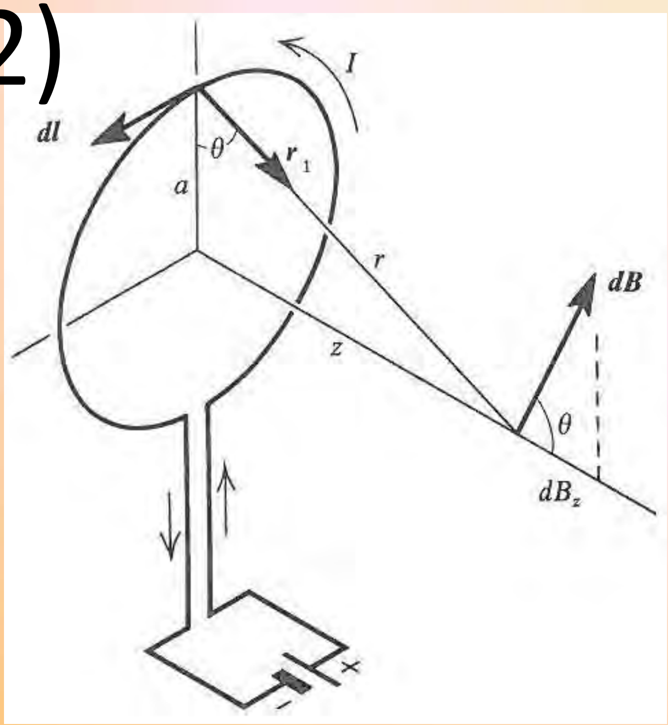
Biot-Savart Law Applied (1)

- Use Biot-Savart law to obtain $\vec{B}$ at distance r from a **long straight wire**: $\boxed{\vec{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\phi}}$
- Then force between two parallel wires is: on element $I_b d\vec{l}_b$: $d\vec{F} = I_b\left(d\vec{l}_b \times \vec{B}_a\right)$ and for **two long parallel wires** (distance r apart as in the figure), we get force per unit length $\boxed{\frac{dF}{dl} = \frac{\mu_0 I_a I_b}{2\pi r}}$
- This is used for the definition of the unit of current, the ampere (A) and hence the unit of charge, the coulomb (C).





Biot-Savart Law Applied(2)



- Use Biot-Savart law to obtain $\vec{B}$ at distance z from the centre, along the axis, of a circular current loop, radius a : $$\boxed{\vec{B} = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}}\,\hat{k}}$$
- This is a 'physical' magnetic dipole, with dipole moment $\vec{m} = I\vec{A} = I\pi a^2\hat{k}$ here, and for $z \gg a$ we get 'far field' $\vec{B} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{z^3}$ on axis.
- Generalised concept of magnetic dipole later...

Generalised Biot-Savart Law

- Replace $Id\vec{l}$ by $\vec{J}_f d\mathcal{V}$ where free current density $\vec{J}_f$ has magnitude $J_f = dI/da$, then generalised Biot-Savart:

$$\vec{B} = \frac{\mu_0}{4\pi}\int_{\mathcal{V}'} \frac{\vec{J}_f \times \hat{r}}{r^2} d\mathcal{V}'$$

- Assume for now that the currents due to the motion of free charges are constant and there are no materials.

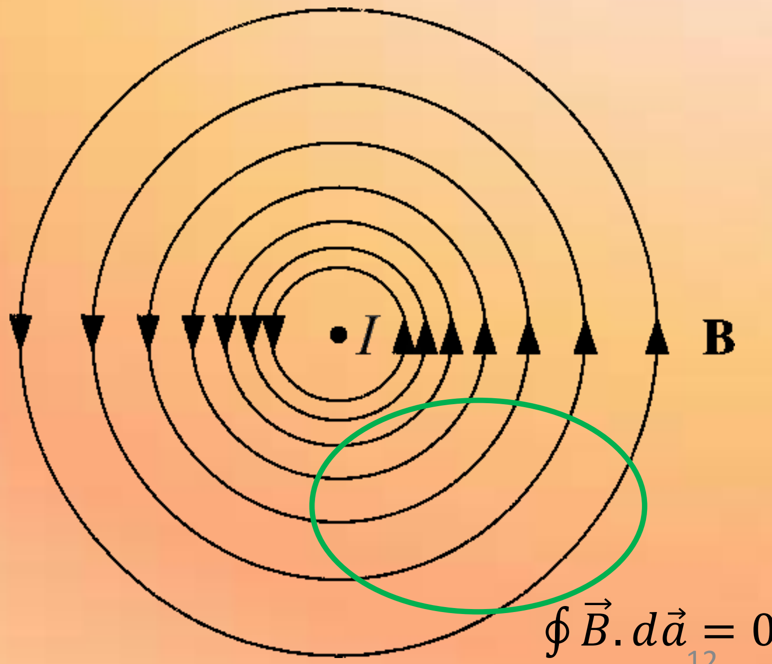
Magnetic Flux

- $\vec{B}$ is a vector field like $\vec{E}$ and we use the concept of magnetic field lines and **magnetic flux** [in webers (Wb)]

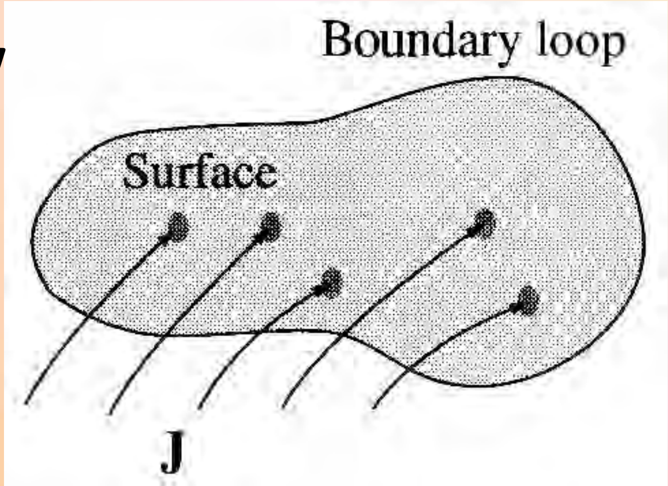
$$\Phi_m = \int_S \vec{B}.d\vec{a}$$

Magnetic Field is Divergenceless

- One can show by direct integration of the generalized Biot-Savart law that $\boxed{\nabla . \vec{B} = 0}$
- In integral form $\oint \vec{B} . d\vec{a} = 0$ for any closed surface, which is of course equivalent to saying that the lines of $\vec{B}$ are closed loops.
- [This is sometimes (incorrectly) called "Gauss's law for magnetic fields".]



Ampere's Circuital Law



- For static fields Ampere's law in integral form is: $\oint \vec{B}.d\vec{l} = \mu_0 I_{\text{enc}}$ or more generally,

$$\boxed{\oint \vec{B}.d\vec{l} = \mu_0 \int \vec{J}.d\vec{a}}$$

with $\vec{J}$ integrated over the surface bounded by the loop.

- Apply Stokes' theorem to LHS: $\int \left(\nabla \times \vec{B}\right).d\vec{a} = \mu_0 \int \vec{J}.d\vec{a}$
- This applies to any surface , so integrands are equal:

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

– **Ampere's circuital law in differential form**

Magnetic Vector Potential (1)

- In electrostatics, $\nabla \times \vec{E} = 0$ means we can have $\vec{E} = -\nabla V$.
- In magnetostatics, $\nabla . \vec{B} = 0$ likewise means we can define a **magnetic vector potential** $\vec{A}$ such that $\boxed{\vec{B} = \nabla \times \vec{A}}$ (div of curl of any vector is zero).
- We can put $\nabla . \vec{A} = 0$ (can be proved: add $\nabla \lambda$ to $\vec{A}$ ...)
- Ampere's law in terms of $\vec{A}$:

$$\nabla \times \vec{B} = \nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla . \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

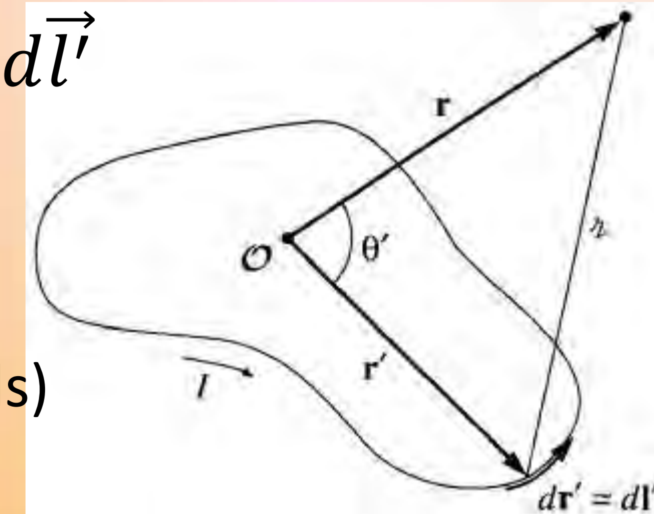
i.e. since $\nabla . \vec{A} = 0$, $\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$

- This is Poisson's equation again (for each component).

Assuming $\vec{J} \rightarrow 0$ at ∞ , solution is $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}')}{(\vec{r} - \vec{r}')} d\mathcal{V}'$

- [Note $\vec{A}$ is in the direction of the current.]

Magnetic Vector Potential (2)



- For **line currents** $\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{(\vec{r}-\vec{r}')}\, d\vec{l'}$
- **Multipole expansion** of $\vec{A}$ (like V):

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{|\vec{r}-\vec{r}'|}\, d\vec{l'} =$$

(Legendre Polynomials)

$$\frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta')\, d\vec{l'} =$$

$$\frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l'} + \frac{1}{r^2}\oint r' \cos\theta'\, d\vec{l'} + \frac{1}{r^3}\oint (r')^2 \left(\tfrac{3}{2}\cos^2\theta' - \tfrac{1}{2}\right) d\vec{l'} + \cdots\right]$$

monopole term dipole term quadrupole term ...

- But: no magnetic monopoles, so first term = 0
- Dominant term: **dipole** $\vec{A}_{\mathrm{dip}}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\theta'\, d\vec{l'}$ or

$\vec{A}_{\mathrm{dip}}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$ with dipole moment $\vec{m} = I\int d\vec{a}$