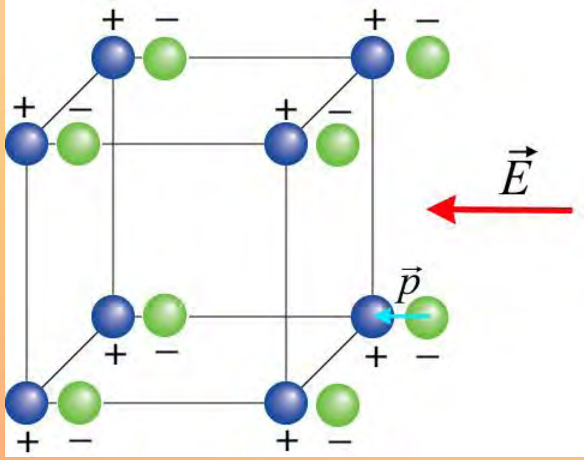
Polarization

- Apply an external $\vec{E}$-field to a non-conducting material, i.e. a “dielectric”. We get $\vec{E}$ inside the material.
- The atomic charges in the material are not free to move, but the $\vec{E}$-field can cause relative displacement of the positive and negative charges = “**polarization**”
- Each atom has an **induced dipole moment** $\vec{p} = q\vec{s}$ (s = separation)
- If the material has n atoms per unit volume, we define the **polarization** $\vec{P}$ as $\boxed{\vec{P} = n\vec{p}}$ (units C.m^{-2})

  i.e. $\vec{P}$ is the “dipole moment per unit volume”.



- This is an induced effect, but some molecules also have permanent dipole moments (“polar dielectrics”).

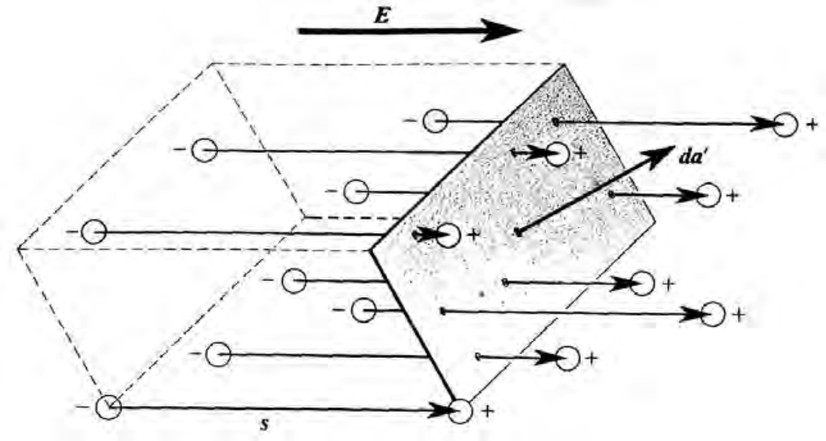
Bound Charge Densities (1)

- Suppose $\vec{E}$ causes separation $\vec{s}$ of $+$ and $-$
- Charge crossing surface element $d\vec{a}'$ is charge in volume of parallelepiped $d\mathcal{V} = \vec{s}.d\vec{a}'$:

  $dQ = n\,Q\,\vec{s}.d\vec{a}'$ (charge per unit volume × volume)

  $= n\,\vec{p}.d\vec{a}' = \vec{P}.d\vec{a}'$ (dipole moment $\vec{p} = Q\vec{s}$)
- Separation $s \ll$ size of molecule, so surface charge with density $\boxed{\sigma_b = \frac{dQ}{da} = \vec{P}.\hat{n}}$ ($\hat{n}$ = normal to surface)

  i.e. "**bound surface charge density = normal component of the polarization vector**"



Bound Charge Densities (2)

- Similarly, by considering the charge remaining within the parallelepiped $\vec{s}.d\vec{a}'$, we find a volume charge density

$$\boxed{\rho_b = -\nabla.\vec{P}}$$

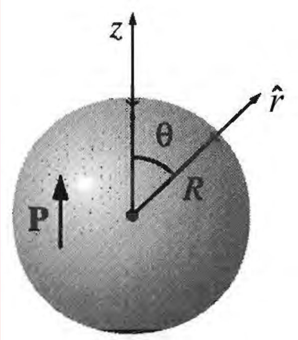
i.e. “**bound volume charge density = negative of the divergence of the polarization vector**”.

- Then the potential due to a slab of dielectric material is

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\vec{P}\cdot d\vec{a}'}{r} - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'} \frac{\nabla.\vec{P}}{r} d\mathcal{V}'}$$

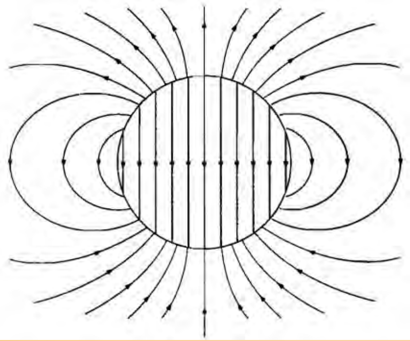
i.e. that due to all the bound surface and volume charges. In a uniform material $\nabla.\vec{P} = 0$, so there are only surface charges.

Uniformly Polarized Sphere



- Useful result for model of atom within material
- Field inside: $\vec{E} = -\frac{1}{3\varepsilon_0}\vec{P}$ (uniform)
- Potential outside: $V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ (dipole at origin)

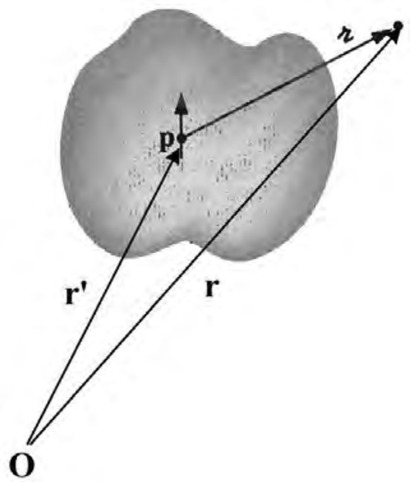
  where $\vec{p} = \frac{4}{3}\pi R^3\vec{P}$ (total dipole moment)



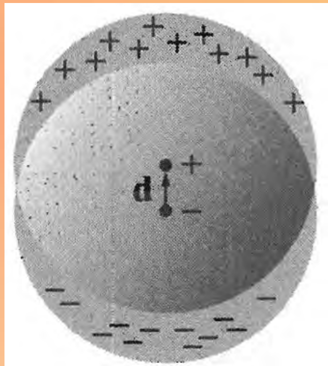
- Potential $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'}\frac{\hat{\mathscr{r}}\,.\,\vec{P}(\vec{r}')}{\mathscr{r}^2}d\mathcal{V}'$

  (Griffiths' notation: "script r" $\vec{\mathscr{r}} = \vec{r} - \vec{r}'$)

  for all points inside and outside dielectric.



- Another (exaggerated) view: separation of "centres of charge" (+ and –) → bound surface charge density σ_b



Polarization Current Density

- If E-field applied to dielectric varies with time, bound charges produce **polarization current**, current density $\vec{J}_b$
- Apply conservation of charge $\int_S \vec{J}_b \cdot d\vec{a} = -\frac{\partial}{\partial t}\int_{\mathcal{V}} \rho_b \, d\mathcal{V}$ and the divergence theorem:

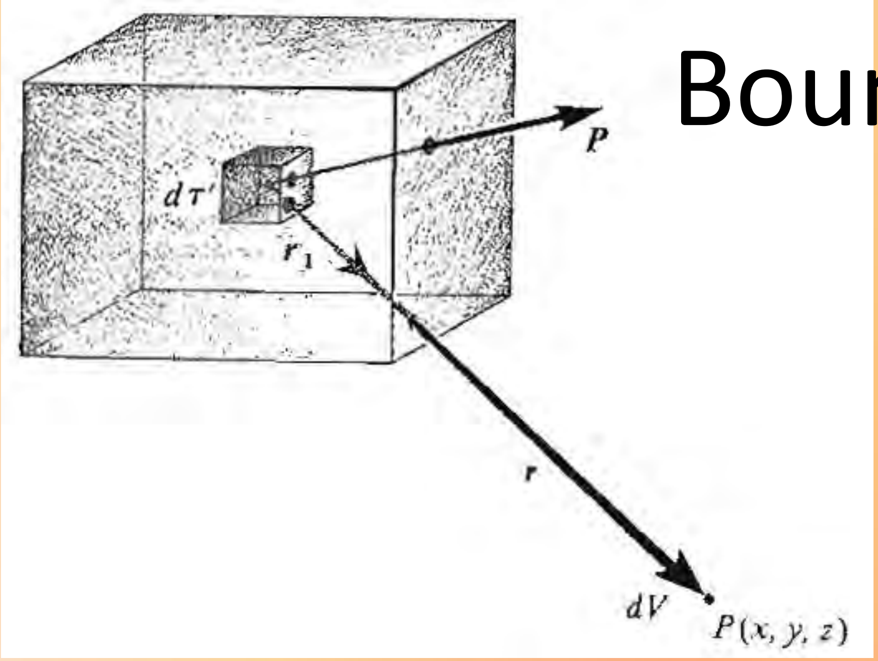
$$\int_{\mathcal{V}} \nabla \cdot \vec{J}_b \, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\mathcal{V}} \nabla \cdot \vec{P} \, d\mathcal{V} = \int_{\mathcal{V}} \nabla \cdot \frac{\partial \vec{P}}{\partial t} d\mathcal{V}$$

- This holds for any arbitrary $\mathcal{V}$, so integrands are equal:

$$\boxed{\vec{J}_b = \frac{\partial \vec{P}}{\partial t}}$$

[i.e. polarization current density or bound current density (in A.m^{-2}) is simply the **rate of change of polarization**]

Bound Charge Densities



Standard notation: primed quantities denote the **source** region and unprimed quantities denote the **field** region (i.e. where we find $\vec{E}$ or V , etc.).

$$V = \frac{1}{4\pi\varepsilon_0}\int_{S'}\frac{\vec{P}\cdot d\vec{a}'}{r} - \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}'}\frac{\nabla\,.\,\vec{P}}{r}\,d\mathcal{V}'$$

“Bound charge densities”, also called “polarization charge densities” or “induced charge densities”, distinguished from “free charge densities”, i.e. “moveable charges”.

Bound charges and free charges must both be considered in applying e.g. Gauss’s law.

Gauss's Law with Dielectrics

- Total charge density (free + bound charges): $\rho = \rho_f + \rho_b$
- Gauss's law: $\nabla . \vec{E} = \frac{\rho}{\varepsilon_0}$ or $\varepsilon_0 \nabla . \vec{E} = \rho = \rho_f + \rho_b$

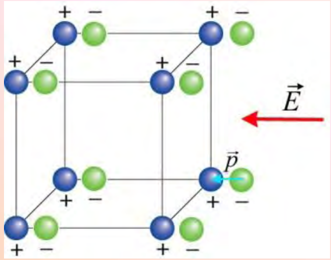
  but $\rho_b = -\nabla . \vec{P}$ so $\varepsilon_0 \nabla . \vec{E} = \rho_f - \nabla . \vec{P}$ or

  $\nabla . (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f$ (**free charge** density only)
- Define **electric displacement** $\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$
- Then Gauss's law: $\boxed{\nabla . \vec{D} = \rho_f}$

  "divergence of electric displacement = free charge density"
- In integral form: $\oint \vec{D} . d\vec{a} = Q_{f\,\text{enc}} = \oint \rho_f \, d\mathcal{V}$

  (total free charge enclosed)
- Outside the dielectric material $\vec{P} = 0$ and so $\vec{D} = \varepsilon_0 \vec{E}$

“L.I.H.” Dielectric Materials



- For many materials, induced dipole moment $\vec{p} \propto \vec{E}$
- On macroscopic scale, polarization $\vec{P} \propto \vec{E}$
- Define **electric susceptibility** χ_e of material by

  $\vec{P} = \chi_e \varepsilon_0 \vec{E}$ (χ_e is dimensionless constant)

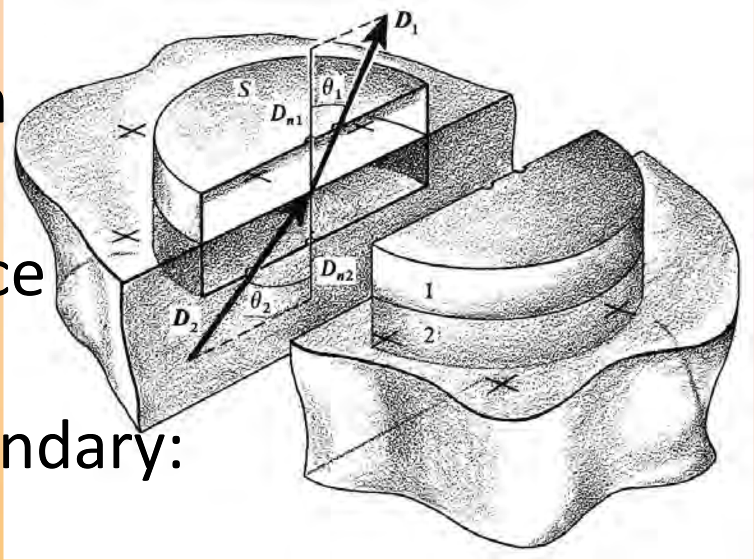
  Materials which obey this eqn. are **linear dielectrics**.
- If material is also isotropic (same in all directions) and homogeneous (uniform), we call it an **l.i.h. dielectric** (linear, isotropic, homogeneous) or “Class A dielectric”.
- In an l.i.h. dielectric $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E}$

  i.e. $\vec{D} = (1 + \chi_e)\varepsilon_0 \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$ or $\boxed{\vec{D} = \varepsilon \vec{E}}$

  where **relative permittivity** $\varepsilon_r = 1 + \chi_e$ (dimensionless)

  (= **dielectric constant** K) and **permittivity** $\boxed{\varepsilon = \varepsilon_r \varepsilon_0}$

Boundary Conditions (1)



- Interface between two media, with relative permittivities ε_{r1} and ε_{r2}
- Assume no free charges on interface
- Potential V must be continuous
- Short Gaussian cylinder across boundary:

  $\oint_S \vec{D} \cdot d\vec{a} = 0$ if no free charges

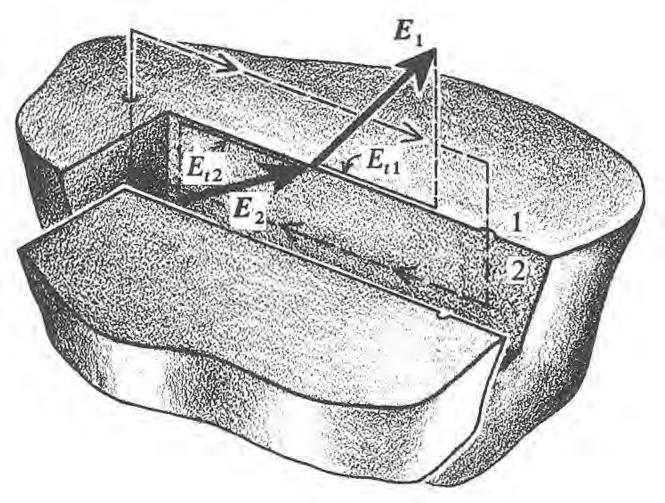
  Let length of cylinder → 0, then zero flux through sides and

  $\int_{S_1} \vec{D} \cdot d\vec{a} + \int_{S_2} \vec{D} \cdot d\vec{a} = 0$ (ends S_1 and S_2 in 1 and 2)

  i.e. $\int_{S_1} D_{n1} da - \int_{S_2} D_{n2} da = 0$ ($\vec{D}_2$ inwards, so negative)

- Hence $\boxed{D_{n1} = D_{n2}}$ i.e. **normal component of $\vec{D}$ is continuous across the boundary**

Boundary Conditions (2)



- Rectangular path across boundary:

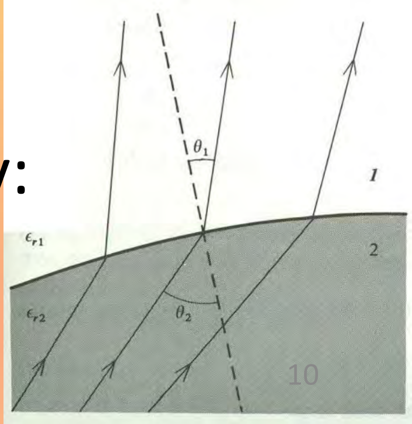
  $\oint_C \vec{E} \cdot d\vec{l} = 0$ for electrostatic field
- Sides perpendicular to boundary → 0

  then $\int_{C_1} \vec{E} \cdot d\vec{l} + \int_{C_2} \vec{E} \cdot d\vec{l} = 0$

  (sides C_1 and C_2 in media 1 and 2)

  i.e. $\int_{C_1} E_{t1} dl - \int_{C_2} E_{t2} dl = 0$ ($\vec{E}_2$ // component opp. to $d\vec{l}$)
- Hence $\boxed{E_{t1} = E_{t2}}$ i.e. **tangential component of $\vec{E}$ is continuous across the boundary**
- $\Rightarrow$ 'refraction' of electric fieldlines at boundary:

  $\varepsilon_{r1} \cot\theta_1 = \varepsilon_{r2} \cot\theta_2$ i.e. $\dfrac{\tan\theta_1}{\tan\theta_2} = \dfrac{\varepsilon_{r1}}{\varepsilon_{r2}} = \dfrac{K_1}{K_2}$



Electric Field Energy

- For N point charges Q_i the electric potential energy is $U_e = \frac{1}{2}\sum_{i=1}^{n} V_i Q_i$ where V_i is potential at position of Q_i
- For continuous charge distribution: $U_e = \frac{1}{2}\int_{\mathcal{V}'} V\rho\, d\mathcal{V}'$
- By Poisson's eqn. $\rho = -\varepsilon_0 \nabla^2 V$; substitute in above: $U_e = \frac{\varepsilon_0}{2}\int_{\mathcal{V}'} V\, \nabla^2 V\, d\mathcal{V}'$; use vector identity for $\nabla.(V\nabla V)$ and divergence theorem ...

$$U_e = -\frac{\varepsilon_0}{2}\left[\int_{S'} (V\nabla V)\cdot d\vec{a} - \int_{\mathcal{V}'} (\nabla V)^2 d\mathcal{V}'\right]$$

- Surface int. at least as $r^{-3}.r^2$, choose distant S then → 0

$\Rightarrow U_e = \frac{\varepsilon_0}{2}\int_{\mathcal{V}'} E^2 d\mathcal{V}'$ or energy density $\boxed{\frac{dU_e}{d\mathcal{V}} = \frac{1}{2}\varepsilon_0 E^2}$

- In dielectrics $\frac{dU_e}{d\mathcal{V}} = \frac{1}{2}\vec{D}\cdot\vec{E}$; in l.i.h. dielectrics $\frac{dU_e}{d\mathcal{V}} = \frac{1}{2}\varepsilon E^2$