Electrostatic Field and Potential

- The **electrostatic** field is **conservative** i.e.

$$\oint_P \vec{E} \cdot d\vec{l} = 0$$

  for any closed path P.

- This allows us to define the **potential** V as a single-valued function at a point, and

$$-\int_a^b \vec{E} \cdot d\vec{l} = V_b - V_a \qquad (\text{“p.d.”})$$
$$= 0 \quad \text{if } a = b$$

- Note only **potential difference** is defined; zero of potential is arbitrary.

Conservative Nature of the Electrostatic Field (1)

- Stokes' theorem gives

$$\oint_c \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E}\right). d\vec{a}$$

Since LHS = 0 for **any** curve C the RHS must = 0.

- This means that for **any** surface S we must have for all points in space

$$\boxed{\nabla \times \vec{E} = 0}$$

for an **electrostatic** field.

("An electrostatic field is curl-free.")

Conservative Nature of the Electrostatic Field (2)

Conversely we can argue that if $\nabla \times \vec{E} = 0$

this implies that $\vec{E} = -\nabla V$ where V is potential.

- Why? $\nabla \times \nabla f = 0$ for any f

  (The curl of the grad of any scalar function is zero.)

- Thus $\nabla \times \vec{E} = 0 \implies \vec{E} = \nabla f$ for some f.

- We call f $-V$ so that $\vec{E} = -\nabla V$; the direction of $\vec{E}$ is from high to low V.

- For a non-static field $\nabla \times \vec{E} \neq 0$. Thus we can determine if a given $\vec{E}$ field is electrostatic or not.

Gauss's Law in Differential Form

- In integral form: $\oint_S \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int_{\mathcal{V}} \rho \, d\mathcal{V}$ where surface S bounds volume $\mathcal{V}$.
- Apply divergence theorem to the LHS:

$$\oint_S \vec{E} \cdot d\vec{A} = \int_{\mathcal{V}} \nabla . \vec{E} \, d\mathcal{V} = \frac{1}{\varepsilon_0}\int_{\mathcal{V}} \rho \, d\mathcal{V}$$

- This is true for **any** volume $\mathcal{V}$, so the integrands must be equal:

$$\boxed{\nabla . \vec{E} = \frac{\rho}{\varepsilon_0}}$$

- This is **Gauss's law in differential form**.

Poisson's Equation

- Consider potential V : the Laplacian of V is

$$\nabla^2 V = \nabla \cdot (\nabla V) = \nabla \cdot \left(-\vec{E}\right) = -\nabla \cdot \vec{E} = -\frac{\rho}{\varepsilon_0}$$

(last step by Gauss's law). This is **Poisson's eqn**.:

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

- In **charge-free** regions ($\rho = 0$), this becomes **Laplace's equation**.:

$$\boxed{\nabla^2 V = 0}$$

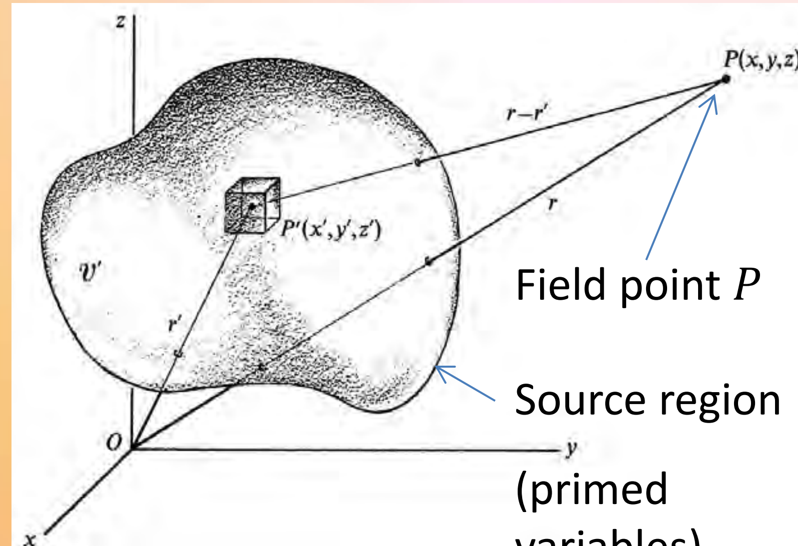
- Many electrostatic problems involve solutions of these eqns. with appropriate boundary conditions.

Multipole Expansion of Potential (1)

- Potential of an arbitrary charge distribution:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d\mathcal{V}'$$

(Solution of Poisson's equation)



- Using Taylor series for $1/|\vec{r}-\vec{r}\,'|$:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

(P_n are Legendre polynomials)

we get the **multipole expansion** of V:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')\,\rho(\vec{r}\,')d\mathcal{V}' \quad \text{or}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\begin{array}{l}\frac{1}{r}\int \rho(\vec{r}\,')d\mathcal{V}' + \frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}\,')d\mathcal{V}' \\ + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}\,')d\mathcal{V}' + \cdots\end{array}\right]$$

Multipole Expansion of Potential (2)

- **Monopole term**: $V_{\text{mon}}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r}$ $[Q = \int \rho(\vec{r}\,')d\mathcal{V}']$
- **Dipole term**: $V_{\text{dip}}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}\,')d\mathcal{V}'$

  or $V_{\text{dip}}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{\vec{p}\,.\,\hat{r}}{r^2}$ where $\vec{p} = \int \vec{r}\,'\rho(\vec{r}\,')\,d\mathcal{V}'$

  is the **dipole moment** of the charge distribution.
- **Quadrupole term**:

  $V_{\text{quad}}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r^3}\int (r')^2\left(\tfrac{3}{2}\cos^2\theta' - \tfrac{1}{2}\right)\rho(\vec{r}\,')d\mathcal{V}'$ etc.
- Note r dependence: $1/r\,;1/r^2\,;1/r^3\,\ldots$
- Dipole electric field: $[\,\vec{E} = -\nabla V\,]$

  $\vec{E}_{\text{dip}}(r,\theta) = \dfrac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$