Electrodynamics Test

Q1. Show that the phase and group speeds, v_ϕ and v_g, are related by [6]

$$v_\mathrm{g} = v_\phi - \lambda \frac{\partial v_\phi}{\partial \lambda}.$$

Q2. A 2 GHz wave with electric field amplitude 1 V/m is propagating through benzene. Find

a. the phase speed, [2]

b. the wavelength, [2]

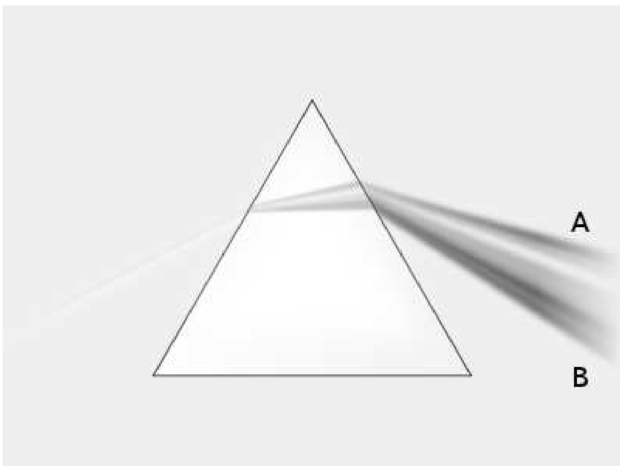
c. the magnitude of the wave $\mathbf{H}$. [2]

Q3. The dispersion relation for electromagnetic waves in a collisionless, unmagnetised plasma is

$$\omega^2 = c^2 k^2 + \Pi^2.$$

Discuss the dispersion relation in the limits of low and high k. How can this be understood in terms of the wavelength and the typical distance between electrons in the plasma? What happens when $\omega < \Pi$? [5]

Q4. White light is refracted through a prism as indicated in the image below.



a. Describe the progression of the spectrum from A to B. Red to violet or *vice versa*? [2]

b. Does this represent "normal" or "anomalous" dispersion? Explain. [3]

Q5. The two orthogonal components of a monochromatic plane wave can be represented by

$$E_x = E\cos(kz - \omega t)$$
$$E_y = E\cos(kz - \omega t + \delta),$$

where $-\pi \leq \delta \leq \pi$.

a. What value of δ would result in right-handed circular polarisation? [2]

b. Illustrate using a table and sketch the evolution of the wave electric field vector over a full cycle for right-handed circular polarisation. [4]

Extra information:

- the classical electron radius is $\mathrm{r_e} = \dfrac{\mathrm{e}^2}{4\pi\varepsilon_0 \mathrm{m_e c^2}} = 2.82 \times 10^{-15}\,\mathrm{m}$
- the Larmor formula is $P = \dfrac{\mu_0 e^2 a^2}{6\pi c}$
- the mean radius of the Earth is $\mathrm{R_E} = 6371\,\mathrm{km}$
- rest mass energy: electron 0.511 MeV, neutron 939.566 MeV, proton 938.272 MeV
- refractive indices:

	n
water	1.333
ice	1.309
quartz	1.544
benzene	1.501

- in spherical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} \\
\nabla \cdot \mathbf{f} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}r^2\sin\theta f_r + \frac{\partial}{\partial\theta}r\sin\theta f_\theta + \frac{\partial}{\partial\phi}r f_\phi\right] \\
\nabla \times \mathbf{f} &= \begin{vmatrix} \dfrac{\hat{\mathbf{r}}}{r^2\sin\theta} & \dfrac{\hat{\boldsymbol{\theta}}}{r\sin\theta} & \dfrac{\hat{\boldsymbol{\phi}}}{r} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ f_r & r f_\theta & r\sin\theta f_\phi \end{vmatrix} \\
\nabla^2 f &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial f}{\partial r} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\sin\phi\frac{\partial f}{\partial\phi}
\end{aligned}$$

- in cylindrical coordinates:

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \\
\nabla \cdot \mathbf{f} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho f_\rho + \frac{1}{\rho}\frac{\partial f_\phi}{\partial\phi} + \frac{\partial f_z}{\partial z} \\
\nabla \times \mathbf{f} &= \begin{vmatrix} \dfrac{\hat{\boldsymbol{\rho}}}{\rho} & \hat{\boldsymbol{\phi}} & \dfrac{\hat{\mathbf{k}}}{\rho} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ f_\rho & \rho f_\phi & f_z \end{vmatrix} \\
\nabla^2 f &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

- some vector identities:

$$\begin{aligned}
\nabla \times \nabla f &= 0 \\
\nabla \times (\nabla \times \mathbf{u}) &= \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u} \\
\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) &= (\mathbf{u}\cdot\mathbf{v})\mathbf{u} - |\mathbf{u}|^2\mathbf{v}
\end{aligned}$$

  for scalar field f and vector fields $\mathbf{u}$ and $\mathbf{v}$.

- covariant derivative

$$v^k_{;j} = \frac{\partial v^k}{\partial x^j} + \Gamma^k_{ji}v^i$$