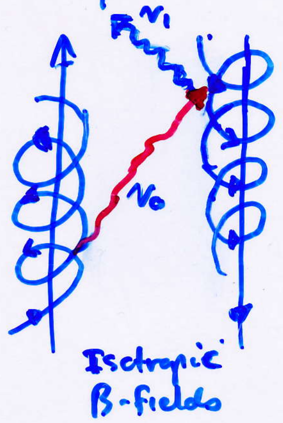
Synchrotron Self-Compton

Consider a population of relativistic electrons in a magnetized region. Typically radio-bright spots in AGN jets or Supernova Remnants. These relativistic electrons will fill the region with photons through the synchrotron process. However these photons may in turn interact with the relativistic relativistic electrons and be upscattered to high energies. This is the so-called synchrotron self-Compton process



In this process the electrons do work twice. First they produce the synchrotron photons and secondly the scatter these photons up to higher energies.

SSC emissivity

The importance of this process will obviously be high for systems with i.) high electron density (relativistic), ii) photon density and iii) high magnetic field energy density. If the relativistic electron distribution is a power-law

$$N(\gamma) = K\gamma^{-p}$$

we expect the SSC flux to scale as K^2 (electrons work twice), i.e quadratic in the electron density.

Let's investigate this:

we showed earlier that

$$\varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi}\frac{(\frac{4}{3})^{\alpha}}{2}\frac{\tau_e}{R/c}\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{U_{ph}(\nu)}{\nu}\nu^{\alpha}\,d\nu$$

If the photon field is produced by synchrotron emitting electrons we should substitute the appropriate expression for the synchrotron radiation energy density

$$U_{syn}(\nu) = \frac{L_\nu\,(\mathrm{erg\,s^{-1}\,Hz^{-1}})}{V} \times t_{esc}$$

$\uparrow\ t_{esc} = \underbrace{\frac{3}{4}}_{4\pi < 1}\frac{R}{c}$

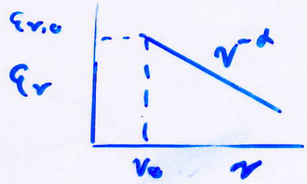
$$= \frac{3}{4}\frac{R}{c}\frac{L_\nu}{V}$$

$$= 4\pi\,\frac{3}{4}\frac{R}{c}\left[\frac{1}{4\pi}\frac{L_\nu}{V}\right]$$

$$= 4\pi\,\frac{3R}{4c}\,\varepsilon_\nu(syn)\ \left[\mathrm{erg\,cm^{-3}\,s^{-1}\,Hz^{-1}\,sr^{-1}}\right]$$

For a simple power-law we can write

$$\varepsilon_\nu(syn) = \varepsilon_{\nu,0}(syn)\,\nu^{-\alpha}$$



Then

$$\varepsilon_{IC}(\nu_{IC}) = \frac{1}{4\pi}\frac{(\frac{4}{3})^{\alpha}}{2}\frac{\tau_e}{R/c}\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{4\pi\frac{3R}{4c}\varepsilon_\nu(syn)}{\nu}\nu^{\alpha}\,d\nu$$

$$= \frac{1}{2}\left(\frac{4}{3}\right)^{\alpha}\frac{3R}{4c}\frac{\tau_e}{R/c}\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{\varepsilon_{\nu,0}(syn)\,\nu^{-\alpha}}{\nu}\nu^{\alpha}\,d\nu$$

$$= \frac{(\frac{4}{3})^{\alpha}}{2\,(\frac{4}{3})}\tau_e\,\varepsilon_{\nu,0}(syn)\,\nu_{IC}^{-\alpha}\int_{\nu_{min}}^{\nu_{max}}\frac{d\nu}{\nu}$$

$$= \frac{(\frac{4}{3})^{\alpha-1}}{2}\tau_e\,\varepsilon_{\nu,0}(syn)\,\nu_{IC}^{-\alpha}\times\int_{\nu_{min}}^{\nu_{max}}\frac{d\nu}{\nu}$$

Notice that

$$\underbrace{\varepsilon_{\nu,0}(syn)\,\nu_{IC}^{-\alpha}} = \varepsilon_{syn}(\nu_{IC})$$

Synchrotron emissivity at Compton scattered frequency.

Also

$$\varepsilon_{IC}(\nu_{IC}) = \frac{\left(\frac{4}{3}\right)^{\alpha-1}}{2}\,\tau_c\,\varepsilon_{\nu,0}(syn)\,\nu_{IC}^{-\alpha}\,\ln\Lambda$$

$$\int_{\nu_{min}}^{\nu_{max}} \frac{d\nu}{\nu} = \ln\left[\frac{\nu_{max}}{\nu_{min}}\right]$$

$$\varepsilon_{SSC}(\nu_{IC}) = \frac{\left(\frac{4}{3}\right)^{\alpha-1}}{2}\,\tau_c\,\varepsilon_{syn}(\nu_{IC})\,\ln\Lambda$$

In this form the ratio between the synchrotron and SSC flux is clear

$$\frac{\varepsilon_{SSC}(\nu_{IC})}{\varepsilon_{syn}(\nu_{IC})} = \frac{\left(\frac{4}{3}\right)^{\alpha-1}}{2}\,\tau_c\,\ln\Lambda$$

$$\sim \tau_c \ln\Lambda$$

$$\Rightarrow \varepsilon_{SSC}(\nu_{IC}) \propto \tau_c \ln\Lambda\; \varepsilon_{syn}(\nu_{IC})$$

<u>But</u>: $\tau_c = R\,\sigma_T\,K$

$\varepsilon_{syn}(\nu_{IC}) \propto K\,B^{1+\alpha}\,\nu_{IC}^{-\alpha}$ [see synchrotron notes]

$$\varepsilon_{SSC}(\nu_{IC}) \propto \tau_c\,\varepsilon_{syn}(\nu_{IC})$$

$$\propto K \times K\,\nu^{-\alpha}$$

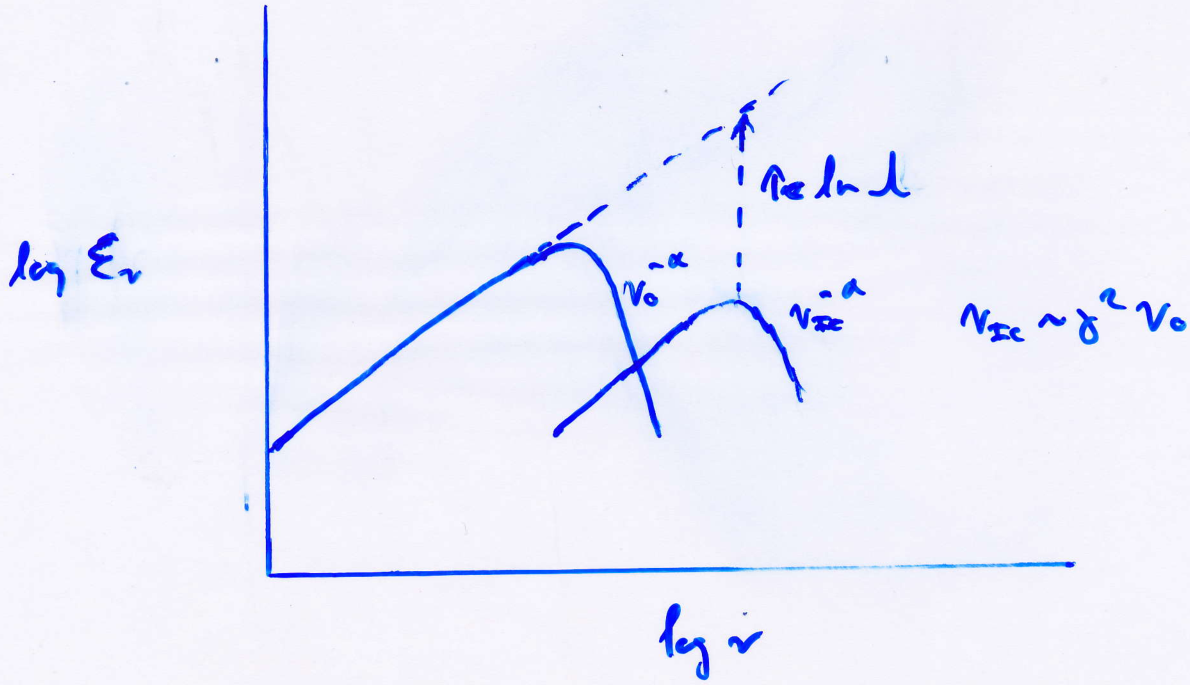
$$\propto K^2\,\nu^{-\alpha}$$ (Electrons work twice)

The following figure summorizes the main result. One can see that for an optically thin scenario

$$\tau_c < 1$$

$$\varepsilon_{ssc}(\nu_{IC}) \sim \tau_c \ln \Lambda \ \varepsilon_{syn}(\nu_{IC})$$

$$\varepsilon_{syn}(\nu_{IC}) \sim \frac{\varepsilon_{ssc}(\nu_{IC})}{\underset{\tau < 1}{\tau_c} \ln \Lambda}$$



$$\tau_c \ln \Lambda = \frac{\varepsilon_{ssc}(\nu_{IC})}{\varepsilon_{syn}(\nu_{IC})}$$

The smaller τ_c the smaller the ratio of

$$\Rightarrow \quad \tau_c \to 0 : \quad \frac{\varepsilon_{ssc}(\nu_{IC})}{\varepsilon_{syn}(\nu_{IC})} \to 0$$

If $\tau_c \to 0$ less scattering occurs between relativistic electrons and synchrotron photons !!!

Notice that the SSC emissivity drops-off as

$$\varepsilon_{SSC}(\nu_{IC}) \propto \tau_c \ln \Lambda \, \nu_{IC}^{-\alpha}$$

$$\nu_{IC} \sim \gamma^2 \nu_0$$

$\uparrow$ synchrotron seed photon frequency.

Thermal Comptonization

This involves the repeated scattering between electrons and photons. In this case the electrons have a thermal or quasi-thermal distribution. There is a fundamental parameter measuring the importance of repeated scattering, i.e the so-called Comptonization parameter (y). Its definition is

$$y = [\text{average number of scatterings}] \times [\text{average fractional energy gain per scatter}]$$

For $y > 1$ the Comptonization process is important since the Comptonized spectrum has more energy than the seed photon spectrum

Average Number of Scatterings

This is calculated by imagining that a photon trapped in the source will undergo a random-walk inside the source. The scattering optical depth (in Thomson limit) is

$$\tau_c = \tau_T$$
$$= \underbrace{R}_{\text{size of source}} \sigma_T \, \overbrace{n}^{\text{electron density. The electrons are targets for photons.}}$$

For $\tau_T < 1$ most of the photons leave the source without scattering. For $\tau_T > 1$ the mean free path is

$$l_{mfp} = d = \frac{R}{\tau_T} = \frac{R}{n\sigma_T R} = \frac{1}{n\sigma_T}$$

we showed earlier that

$$\begin{aligned} I_\nu(\tau_\nu) &= S_\nu(1 - e^{-\tau_\nu}) \\ &= \frac{j_\nu}{\alpha_\nu} \\ &= \frac{j_\nu R}{\alpha_\nu R} \qquad (\tau_\nu > 1) \\ &= \frac{j_\nu R}{\tau_\nu} \\ &= j_\nu \left(\frac{R}{\tau_\nu}\right) \end{aligned}$$

For our purposes

emissivity where optical depth $\tau_T \to 0$.

$$\begin{aligned} I_\nu(\tau_T) &= j_\nu \underbrace{\left(\frac{R}{\tau_T}\right)}_{d = \frac{R}{\tau_T}} \\ &= j_\nu(\tau_T = 0) \frac{R}{n\sigma_T R} \\ &= \left(j_\nu(\tau_T = 0) R\right) \times \frac{1}{n\sigma_T R} \\ &= I_\nu(\tau_T = 0) \times \left(\frac{d}{R}\right) \\ &= \underbrace{\left(\frac{l_{mfp}}{R}\right)}_{\ll 1 \text{ for } \tau_T > 1} I_\nu(\tau_T = 0) \end{aligned}$$

The brightness is then

$$I_\nu(\tau_T) = \frac{I_\nu(\tau_T = 0)}{\tau_T}$$

For $\tau_T > 1$ the photons will undergo a random walk process. The number of scatterings for a R.W process is

$$N = \left(\frac{R_s}{l}\right)^2 \qquad l = l_{mfp}$$

$$\approx \tau_T^{\,2}$$

The total travel time for a photon from its birth until it leaves the source is

$$t_{RW} = N \times \frac{l}{c}$$

$$= \left(\frac{R_s}{l}\right)^2 \times \frac{l}{c}$$

$$= \tau_T^{\,2} \frac{l}{c}$$

The total distance traversed is

$$c\, t_{RW} = \tau_T^{\,2}\, l$$

$$= \tau_T^{\,2} \times \frac{R}{\tau_T}$$

$$= \tau_T \times R$$

$$= \tau_T \times R_s \qquad R = R_s$$

$\uparrow$ source size

One can see that for $\tau_T \gg 1$ the photon travels in total a distance much longer than the typical source size

Average gain per scattering

Relativistic Case:

If the scattering electrons are relativistic with a thermal distribution the photons will gain energies of the order of $\langle \frac{\Delta \varepsilon}{\varepsilon} \rangle \approx \frac{4}{3}\gamma^2$ the original energy. How would this translate to the scattering between a distribution of relativistic electrons and photons, if the electrons have a Relativistic Maxwellian distribution?

The relativistic Maxwellian is (shown earlier)

$$\boxed{N(\gamma) \propto \gamma^2 e^{-\frac{\gamma}{\Theta}} \qquad \left(\Theta = \frac{kT}{m_0 c^2}\right)}$$

The average of a photon with initial energy x_0 after a single scattering with a relativistic electron belonging to this Maxwellian is

$$\langle x_1 \rangle = \frac{4}{3}\langle \gamma^2 \rangle x_0$$

$$= \frac{4}{3} x_0 \times \underbrace{\left\{ \frac{\int_1^\infty \gamma^2 N(\gamma)\, d\gamma}{\int_1^\infty N(\gamma)\, d\gamma} \right\}}_{\langle \gamma^2 \rangle}$$

The lower limit $\gamma = 1$ corresponds to a thermal electron with $\varepsilon \sim m_0 c^2$

The integrals can be solved in the following way.

$$\langle \gamma^2 \rangle = \frac{\int_1^\infty \gamma^2 N(\gamma)\, d\gamma}{\int_1^\infty N(\gamma)\, d\gamma}$$

$$= \frac{\int_1^\infty \gamma^4 e^{-\frac{\gamma}{\Theta}}\, d\gamma}{\int_1^\infty \gamma^2 e^{-\frac{\gamma}{\Theta}}\, d\gamma}$$

These integrals are of the type

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

Hence

$$\langle \gamma^2 \rangle = \frac{\Gamma(4+1) / \left(\frac{1}{\Theta}\right)^{4+1}}{\Gamma(2+1) / \left(\frac{1}{\Theta}\right)^{2+1}}$$

$$= \frac{\Theta^5}{\Theta^3} \frac{\Gamma(5)}{\Gamma(3)}$$

$$= \Theta^2 \frac{\Gamma(5)}{\Gamma(3)}$$

$$= \left(\frac{kT}{M_0 c^2}\right)^2 \times \frac{4!}{2!} \qquad \Gamma(n+1) = n!$$

$$= 12 \left(\frac{kT}{M_0 c^2}\right)^2$$

Therefore

$$\langle x_1 \rangle = \frac{4}{3} x_0 \langle \gamma^2 \rangle$$
$$= \frac{4}{3} x_0 \times 12 \left(\frac{kT}{m_0 c^2}\right)^2$$
$$= \frac{48}{3} \left(\frac{kT}{m_0 c^2}\right)^2 x_0$$
$$= 16 \left(\frac{kT}{m_0 c^2}\right)^2 x_0$$

Non-Relativistic Case:

In this case the average gain is proportional to the electron energy and not the square

In this case we must remember that in any Maxwellian, especially one at low temperatures, there will be electrons with energies lower than that of the photons. In this case the photons will give energy to the electrons. In this case the scattered photon will have less energy than the incoming one. Averaging over a Maxwellian distribution, we will have

Photon gains energy

$$\frac{\Delta x}{x} = \frac{x_1 - x_0}{x_0} = \alpha\Theta - x$$

Direct Compton scatter where photons give energy to electrons !!

$$x = \frac{h\nu}{m_0 c^2}$$

$$x_0 = \frac{h\nu_0}{m_0 c^2}$$

$$\Theta = \frac{kT}{m_0 c^2}$$

To determine [α] we make use of the following argument. We know that for multiple scattering (photon number is conserved during scattering) the photons will follow a Wien distribution

$$I_w(x) \propto \underbrace{x^3 e^{-\frac{x}{\Theta}}}_{[\text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{sr}^{-1}]} \quad \text{(Brightness)}$$

But $$I_w(x) = x \underbrace{N_w(x)}_{[\text{cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{sr}^{-1}]}$$

$$N_w(x) \propto \frac{I_w(x)}{x}$$

$$\propto x^2 e^{-\frac{x}{\Theta}}$$

When a Wien spectrum is established then it means that the photons and electrons have reached an equilibrium and $\langle \Delta x \rangle \rightarrow 0$. No net energy transfer from electrons to photons and vice versa!! Therefore

$$\left\langle \frac{\Delta x}{x} \right\rangle = \langle \alpha \Theta - x \rangle$$

$$\langle \Delta x \rangle = \alpha \Theta \langle x \rangle - \langle x^2 \rangle \Rightarrow 0$$

Now we can calculate $\langle x \rangle$ and $\langle x^2 \rangle$ for the photon Wien distributions!!

$$\langle x \rangle = \frac{\int_0^\infty x\, N_w(x)\, dx}{\int_0^\infty N_w(x)\, dx}$$

$$= \frac{\int_0^\infty x^3 e^{-\frac{x}{\Theta}}\, dx}{\int_0^\infty x^2 e^{-\frac{x}{\Theta}}\, dx}$$

$$\int x^n e^{-ax}\, dx = \frac{\Gamma(n+1)}{a^{n+1}}$$

$$= \Theta \; \frac{\Gamma(4)}{\Gamma(3)}$$

$$\Gamma(n+1) = n!$$

$$= \frac{3\times 2\times 1}{2\times 1} \left(\frac{kT}{m_e c^2}\right)$$

$$= 3\left(\frac{kT}{m_e c^2}\right)$$

Similarly

$$\langle x^2 \rangle = \frac{\int_0^\infty x^2\, N_w(x)\, dx}{\int_0^\infty N_w(x)\, dx}$$

$$= \frac{\int_0^\infty x^4 e^{-\frac{x}{\Theta}}\, dx}{\int_0^\infty x^2 e^{-\frac{x}{\Theta}}\, dx}$$

$$= \frac{\Gamma(5)}{\Gamma(3)}\, \Theta^2$$

$$= \frac{4\times 3\times 2\times 1}{2\times 1}\, \Theta^2 = 12\left(\frac{kT}{m_e c^2}\right)^2$$

This implies that

$$\langle \Delta x \rangle = \alpha \Theta \langle x \rangle - \langle x^2 \rangle = 0$$

$$\alpha \Theta (3\Theta) - 12 \Theta^2 = 0$$

$$3\alpha \Theta^2 - 12 \Theta^2 = 0$$

$$3\alpha \Theta^2 = 12 \Theta^2$$

$$\alpha = \frac{12}{3} = 4$$

This result can now be combined with the relativistic case to give us a uniform formulation. We showed that in the relativistic case

$$\langle x_1 \rangle = 16 \Theta^2 x_0$$

$$\frac{\langle x_1 \rangle - x_0}{x_0} = \frac{16\Theta^2 x_0 - x_0}{x_0}$$

$$= \frac{x_0 (16\Theta^2 - 1)}{x_0}$$

$$= 16 \Theta^2 - 1$$

$$\approx 16 \Theta^2 \qquad \left(\frac{kT}{mc^2}\right) \gg 1$$

$$\left(\frac{\Delta x}{x}\right)_{rel} \approx 16 \Theta^2$$

Hence the combined result:

$$\left(\frac{\Delta x}{x}\right) = \left(\frac{\Delta x}{x}\right)_{rel} + \left(\frac{\Delta x}{x}\right)_{non.rel}$$

$$= 16\Theta^2 + \alpha \Theta - x$$

$$= 16\Theta^2 + 4\Theta - x \qquad \alpha = 4$$

Now we can go back to the Compton y-parameter:

$$y = [\text{Average number of scatters}] \times [\text{average fractional gain per scatter}]$$

$$= \max\left[\tau_T \,;\, \tau_T^2\right] \times \left[\left\langle \frac{\Delta x}{x} \right\rangle\right]$$

$$= \max[\underbrace{\overset{\tau_T<1}{\uparrow}\!\tau_T \,;\, \overset{\tau_T>1}{\uparrow}\!\tau_T^2}_{\text{Number of scatters!!}}] \times \left[16\,\Theta^2 + 4\,\Theta - x\right]$$

If we ignore down scattering then the average energy gain per scatter is

$$\frac{\Delta x}{x} = 16\,\Theta^2 + 4\,\Theta$$

The total energy gain in k-scatters is:

$$\left(\frac{dx}{x}\right)_k = \left(16\,\Theta^2 + 4\,\Theta\right) dk$$

$$\int_{x_0}^{x} \frac{dx}{x} = 16\,\Theta^2 + 4\,\Theta \int_0^k dk$$

$$\left[\ln x\right]_{x_0}^{x} = \left(16\,\Theta^2 + 4\,\Theta\right) k$$

$$\ln\left(\frac{x_f}{x_0}\right) = \left(16\,\Theta^2 + 4\,\Theta\right) k$$

$$\left(\frac{x_f}{x_0}\right) = e^{(16\Theta^2 + 4\Theta)k}$$

$$= e^{y}$$

<u>Remember:</u> For $\tau_T \ll 1 \Rightarrow k = \tau_T$

$$y = \left(16\,\Theta^2 + 4\,\Theta\right) \tau_T$$
$$= \left(16\,\Theta^2 + 4\,\Theta\right) k$$

[ignoring down scattering]

If we subtract the initial photon energy and consider that this eqn is valid for all x_0 of the seed photon distribution of luminosity L_0, we have

$$x_f = x_0 e^{y}$$

$$\frac{L_{ot}}{L_0} = \frac{L_f + L_0}{L_0} = e^{y}$$

$$\frac{L_f}{L_0} = e^{y} - 1$$

can produce a significant boost in the high energy end of the spectrum !!

This explains the importance of y, especially in the limit $y > 1$.

Comptonization Spectra

We shall illustrate that even a thermal Maxwellian distribution of electrons can produce a power-law photon spectrum. The basic idea is that the net spectrum is a superposition of many orders of Compton scattering. If they are not too much separated in frequency and summed-up, the resultant spectrum will be a power-law. We can distinguish 4 regimes according to the values of τ_T and y. As usual we set $x = \frac{h\nu}{m_e c^2}$ and $\Theta = \frac{kT}{m_e c^2}$.

Case 1: $\tau_T < 1$

Let's again neglect downscattering. The fractional energy gain is

$$\frac{\Delta x}{x} = 16\Theta^2 + 4\Theta$$

$$\frac{\Delta x}{x} = \frac{x_1 - x}{x} = 16\Theta^2 + 4\Theta$$

$$\frac{x_1}{x} = 16\Theta^2 + 4\Theta + 1$$

Remember that for $\tau_T < 1$

$$y = \tau_T \times (16\Theta^2 + 4\Theta) \quad \text{ignoring downscatter}$$

$$\frac{y}{\tau_T} = 16\Theta^2 + 4\Theta$$

Let

$$A = \frac{x_1}{x} \approx 16\Theta^2 + 4\Theta$$
$$= \frac{y}{\tau_T}$$

The probability that a photon will survive a region of optical depth τ_T without scattering is:

$$P_{ns} = \frac{N}{N_0} = e^{-\tau_T} \qquad [ns \Rightarrow \text{no scatter}]$$

The probability that a seed photon can escape the medium with at least one scatter before escaping.

$$\Rightarrow P_s = (1 - P_{ns}) = 1 - e^{-\tau_T}$$
$$= 1 - (1 - \tau_T + \text{HOT}) \qquad \tau_T < 1$$
$$\approx \tau_T$$

Therefore

$$P_{esc}(1\text{-scatt.}) \simeq \tau_T$$

We can expand this result to include higher scattering orders. Remember that since $\tau_T < 1$ the fraction of photons undergoing multiple scattering before they escape will in fact decrease. The reason is the following:

If a statistical experiment has a certain outcome with a small probability p, then to get a certain outcome in N- experiments is

$$P_{Tot}(N) = p^N$$

Therefore

$$P_{esc}(1\text{-scat.}) = \tau_T$$

$$P_{esc}(2\text{-scat.}) = \tau_T^2$$

$$P_{esc}(3\text{-scat.}) = \tau_T^3$$

$$\vdots$$

$$P_{esc}(N\text{-scat.}) = \tau_T^N$$

Then one can determine the average gain per scatter

No-scatter : $\langle x \rangle = x_0$ no gain

Then we showed earlier

$$\frac{x_1}{x_0} = A = \frac{y}{\tau_T} \Rightarrow y = A\,\tau_T$$

Now we need to find a suitable description for

$$A = \frac{x_1}{x_0}$$

during a single scattering, which can help us determine the expected energy transfer over N-scatterings.

It has been shown earlier that the average energy transfer in IC scattering for an isotropic photon field is

$$\left\langle \frac{\Delta \varepsilon}{\varepsilon} \right\rangle = \frac{4}{3} \gamma^2 \beta^2$$

$$\approx \frac{4}{3} \beta^2 \qquad \gamma \to 1$$

For a M-B distribution of thermal electrons

$$\langle \beta^2 \rangle = \left(\frac{3kT}{m_0 c^2} \right)$$ [mean square speed of the electrons]

Then

$$\left\langle \frac{\Delta \varepsilon}{\varepsilon_0} \right\rangle = \frac{4}{3} \langle \beta^2 \rangle$$

$$= \frac{4}{3} * \frac{3kT}{m_0 c^2}$$

$$= \frac{4kT}{m_0 c^2}$$

$$\frac{\Delta \varepsilon}{\varepsilon_0} = \frac{\varepsilon_1 - \varepsilon_0}{\varepsilon_0} = \frac{4kT}{m_0 c^2}$$

$$\frac{\varepsilon_1}{\varepsilon_0} = 1 + \frac{4kT}{m_0 c^2}$$

$$\frac{x_1}{x_0} = 1 + \frac{4kT}{m_0 c^2}$$

$$= 1 + 4\Theta$$

Therefore

$$\frac{x_1}{x_0} = 1 + 4\Theta$$

for $\Theta \ll 1$ ($kT \ll m_0c^2$) we can make the following approximation

$$e^x \approx 1 + x + \text{H.O.T}$$

$$\approx 1 + x$$

Let $\quad 1 + x = 1 + 4\Theta$

$$\Rightarrow \quad 1 + 4\Theta \approx e^{4\Theta}$$

$$\Rightarrow \quad \frac{x_1}{x_0} = 1 + 4\Theta$$

$$\approx e^{4\Theta}$$

$$\Rightarrow \quad \frac{x_1}{x_0} \approx e^{4\Theta} = A$$

$$\Rightarrow \quad \left(\frac{x_1}{x_0}\right)_N \approx \left(e^{4\Theta}\right)^N \approx A^N$$

Number of scatterings

Hence

$$\left(\frac{x_1}{x_0}\right)_1 = \left(e^{4\Theta}\right)^1 = A$$

$$\left(\frac{x_1}{x_0}\right)_2 = \left(e^{4\Theta}\right)^2 = A^2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left(\frac{x_1}{x_0}\right)_N = \left(e^{4\Theta}\right)^N = A^N$$

But for $\tau_T < 1 \Rightarrow N = \tau_T$

Therefore

$$\left(\frac{x_1}{x_0}\right)_N = \left(e^{4\Theta}\right)^N = \left(e^{4\Theta}\right)^{\tau_T} = A^N$$

Therefore

$x_1 = x_0$ no-scatter

$x_1 = A x_0$ 1-scatter

$x_2 = A^2 x_0$ 2-scatter

$\vdots \qquad \vdots$

$x_N = A^N x_0$ N-scatters

In a medium with small optical depth and even smaller absorption depth the probability of a photon scattering N-times before escaping the medium is

$$P_N(\tau_T) = \tau_T^{\,N} \qquad \text{(shown earlier)}$$

The intensity of emergent radiation at energy ε_N must be proportional to

$$\Rightarrow \tau_T^{\,N}$$

Therefore the emergent intensity at energy ε_N has the form

$$I(\varepsilon_N) \sim I(\varepsilon_i)\, P_N(\tau_T)$$

$$\sim I(\varepsilon_i)\, \tau_T^{\,N}$$

This is equivalent to

$$I(\xi_N) \sim I(\xi_i)\, \uparrow_T^{\,N} \qquad \uparrow_T = \frac{y}{A}$$

$$\sim I(\xi_i) \left(\frac{y}{A}\right)^N \qquad = \frac{y}{(x_1/x_0)}$$

$$\sim I(\xi_i) \left(\frac{y}{x_1/x_0}\right)^N$$

$$\sim I(\xi_i)\, y^N x_0^N \; x_1^{-N}$$

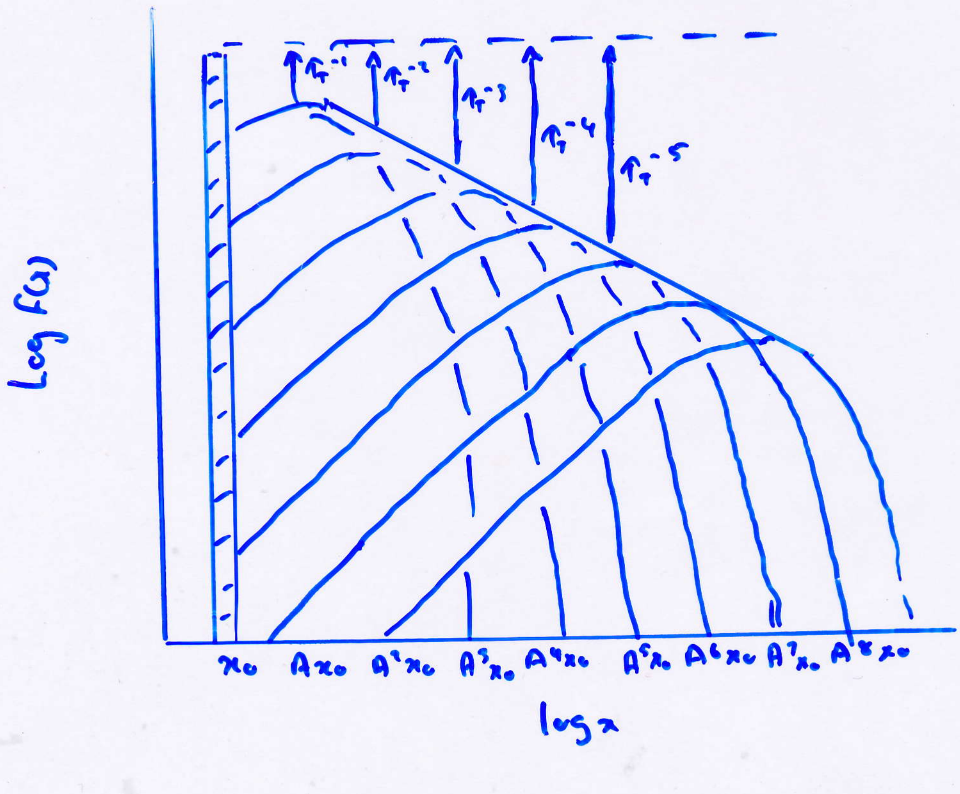
$$\sim (y x_0)^N \; I(\xi_i)\, x_1^{-N}$$

From the expression

$$I(\xi_N) \sim I(\xi_i)\, \uparrow_T^{\,N}$$

$$\uparrow_T^{\,N} = \frac{I(\xi_N)}{I(\xi_i)}$$

$$\uparrow_T^{\,-N} = \frac{I(\xi_i)}{I(\xi_N)}$$



From fig 5.10 one can see that since it is a Log. Log plot the slope is just

$$m = \frac{\Delta y}{\Delta x}$$

Then from

$$I(x) \propto x^{-\alpha}$$

$$\log I(x) \approx \log x^{-\alpha}$$

$$\approx -\alpha \log x$$

$$\alpha \approx -\frac{\log I(x)}{\log x}$$

$$\approx -\frac{\log(\Upsilon_T{}^N)}{\log(A^N x_0)} \qquad I(x) \propto \Upsilon_T{}^N, \quad x = A^N x_0$$

$$= \frac{-\log \Upsilon_T{}^N}{[\log A^N + \log x_0]}$$

$$\approx \frac{-\log(\Upsilon_T)^N}{\log(A)^N}$$

$$\approx \frac{-N \log \Upsilon_T}{N \log A}$$

$$\approx -\frac{\log \Upsilon_T}{\log A} \qquad A = \frac{y}{\Upsilon_T}$$

$$= -\left[\frac{\log \Upsilon_T}{\log y - \log \Upsilon_T}\right]$$

Then we have the following:

$$\alpha = -\left[\frac{\log \uparrow_T}{\log y - \log \uparrow_T}\right]$$

For $y \sim 1 \Rightarrow \log y \to 0$ and

$\alpha = 1 \quad \Rightarrow I(x) \propto x^{-1}$

1.) For $y > 1 \Rightarrow \alpha < 1$ Flat or Hard

2.) For $y < 1 \Rightarrow \alpha > 1$ Steep or Soft

Note that

$$y = A \uparrow_T$$
$$= \underbrace{\frac{x_1}{x_0}}_{\text{energy transfer in scatter}} \times \uparrow_T$$

Check:

① $y = 1.5 \qquad \uparrow_T = 0.5$

$$\alpha = -\frac{\ln 0.5}{[\ln 1.5 - \ln 0.5]}$$
$$= -\frac{(-0.69)}{(0.79)}$$
$$= \frac{0.69}{0.79} \simeq 0.88 \Rightarrow I(x) \propto x^{-0.88}$$

$\underbrace{\text{Hard}}$

② $y \sim 0.8 \qquad \uparrow_T \sim 0.5$

$$\alpha = \frac{-\ln 0.5}{[\ln 0.8 - \ln 0.5]} = -\frac{(-0.69)}{0.4} = 1.7$$

$I(x) \propto x^{-1.7}$ (Soft)

The case $\tau_T > 1$:

This is the most difficult case requiring the so-called Komponeets eqn. The result is still a power-law with spectral index given by

$$\alpha = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{4}{y}}$$

$\tau_T \gg 1$: Saturation

In this case the interaction between photons and electrons is so intense that they reach an equilibrium at some temperature. Instead of a BB spectrum a Wien spectrum is reached. The Wien spectrum

$$\boxed{I(x) \propto x^3 e^{-\frac{x}{\Theta}}}$$

At low frequencies it is harder than a BB, i.e it has a steeper slope towards higher frequencies (see Fig 2.3, p.35 Ghisellini).

BB : $h\nu \ll kT$: $I^{BB}(x) \propto x^2$

Wien : $h\nu \ll kT$: $I^{W}(x) \propto x^3 e^{-\frac{x}{\Theta}}$

$$\propto x^3 \left(1 - \frac{x}{\Theta} + \text{h.o.t.}\right)$$

$$\propto x^3 \quad \left(\frac{x}{\Theta} \ll 1\right)$$

The case $\tau_T > 1$, $y > 1$: quasi saturation

Suppose that in a source characterized by a large τ_T the source of soft photons is spread throughout the source. In this case the photons produced close to the surface, in a skin of optical depth $\tau_T = 1$ leave the source without any scattering. The remaining fraction, i.e $(1 - \frac{1}{\tau_T})$ are trapped inside the source. Remember we said earlier

$$I(\tau_T) = S_\nu (1 - e^{-\tau_T})$$

$$= \frac{j_\nu}{\alpha_T} (1 - e^{-\tau_T})$$

$$= \frac{j_\nu R}{\alpha_T R} (1 - e^{-\tau_T})$$

$$\approx \frac{I_0}{\tau_T} \qquad (\tau_T > 1)$$

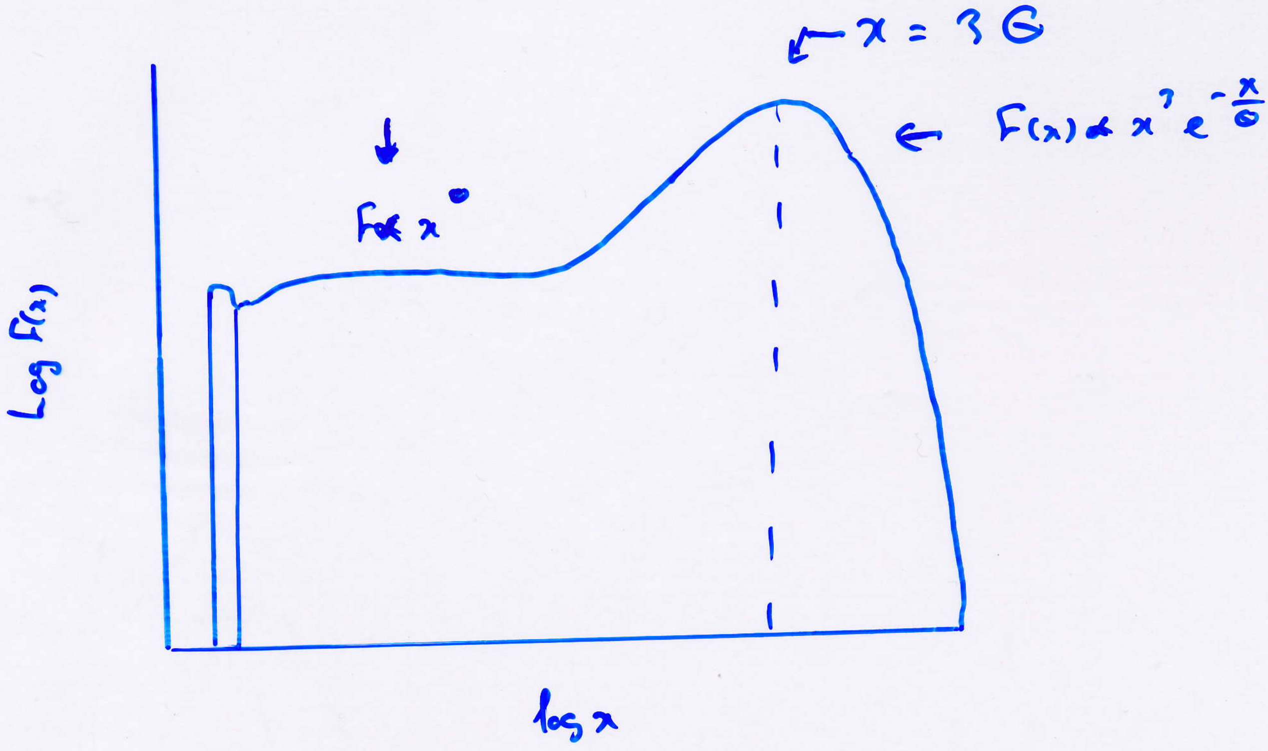
$$\Rightarrow \quad \frac{I(\tau_T)}{I_0} \approx \frac{1}{\tau_T} \qquad \text{(This can escape)}$$

The fraction this trapped in the source is

$$\text{Trapped} = 1 - \frac{I(\tau_T)}{I_0}$$

$$= 1 - \frac{1}{\tau_T} \qquad (\tau_T > 1)$$

For $\tau_T \gg 1$ one can see all photons will be trapped and it will establish a Wien spectrum.



Since photons are trapped inside the source we get that the spectrum saturates at $x \sim 3\Theta$ in a Wien spectrum. This is the result of repeated scattering. This is observed in the x-ray spectra of some galactic black hole binaries like Sco X-1 or Cen X-3.

Comptonization Spectra - Different approach

The different Comptonization spectra can be modelled with the so-called **Kompaneets eqn**:

$$\frac{\partial n}{\partial y} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left[\frac{\partial n}{\partial x} + n^2 + n\right]\right]$$

Here $n = n(x) = f(x)$ represents the phase space density of photons

$$x = \frac{h\nu}{kT}$$

$$\frac{\partial n}{\partial y} = \frac{m_e c^2}{kT}\frac{1}{n_e \sigma_T c}\frac{\partial n}{\partial t}$$

This becomes

$$\frac{m_e c^2}{kT}\frac{1}{n_e \sigma_T c}\frac{\partial n}{\partial t} = \frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left[\frac{\partial n}{\partial x} + n^2 + n\right]\right]$$

$$\frac{\partial n}{\partial t} = \frac{kT n_e \sigma_T c}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left[\frac{\partial n}{\partial x} + n^2 + n\right]\right]$$

For most astrophysical scenarios $n < 1$ and hence ($n^2 \ll 1$)

$$\frac{\partial n}{\partial t} = \frac{kT_e n_e \sigma_T c}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right]$$

Hence

$$\left.\frac{\partial n}{\partial t}\right|_{comp} = \frac{kT_e n_e \sigma_T c}{m_e c^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right]$$

Let's investigate this

Let's inject photons with energy $x_i = \frac{E_i}{kT} \ll 1$ in a gas cloud with radius R and electron density n_e. Let's take the optical depth $\tau_T = n_e \sigma_T R > 1$. The number of photons per unit phase space volume escaping the cloud per unit time is

$$\left.\frac{\partial n(x)}{\partial t}\right|_{esc} = -\frac{n(x)}{t_{esc}} = -\frac{n}{t_{esc}}$$

The escape time in the limit $\tau_T > 1$

$$t_{esc} = \overset{\text{number of scatters}}{N} \times \frac{l_{mfp}}{c}$$

$$= \frac{N}{c} \times l_{mfp}$$

$$= \frac{N}{c} \times \frac{R}{(N)^{1/2}}$$

$$= \frac{R}{c} \sqrt{N}$$

$\tau \geq 1$:

$$l_{mfp} = \frac{R}{\tau_T} = \frac{R}{(N)^{1/2}}$$

$$\Rightarrow N = \tau_T^2$$

$$\Rightarrow \tau_T = \sqrt{N}$$

Remember for a R.W process the number of random scatterings for $\tau_T \gg 1$

$$N \simeq \left(\frac{R}{l_{mfp}}\right)^2$$

$$\simeq \tau^2 \qquad \tau \gg 1 \qquad \text{[Eq. 1.90(a) R\&L p.36]}$$

Therefore

$$N = (\tau_T)^2 = (n_e \sigma_T R)^2$$

For $\tau_T < 1$:

$$N = \tau_T \qquad \text{Eq 1.90b} \qquad \text{[Eq 1.90(b) R\&L p.36]}$$

Therefore for any specific τ_T

$$N \approx \underset{(\tau_T > 1)}{\tau_T^2} + \underset{(\tau_T < 1)}{\tau_T}$$

$$\approx \tau_T(\tau_T + 1)$$

For the case where $\tau_T > 1$ $\quad N \to \tau_T^2$.

The total time evolution of the photon density is then given by

$$\frac{dn(x)}{dt} = \frac{dn(x)}{dt}\Big|_{comp} + \frac{dn(x)}{dt}\Big|_{esc}$$

$$\approx \frac{kT_e n_e \sigma_T c}{m_e c^2} \frac{1}{x^2} \frac{d}{dx}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right] - \frac{n(x)}{t_{esc}}$$

$$= \frac{kT_e n_e \sigma_T c}{m_e c^2} \frac{1}{x^2} \frac{d}{dx}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right] - \frac{n(x)}{\left(\frac{R}{c}\tau_T\right)}$$

$$\frac{1}{n_e \sigma_T c} \frac{dn(x)}{dt} = \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{d}{dx}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right] - \frac{n(x)}{\frac{R}{c} n_e \sigma_T c \tau_T}$$

$$\frac{1}{n_e \sigma_T c} \frac{dn(x)}{dt} = \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{d}{dx}\left[x^4\left[\frac{\partial n}{\partial x} + n\right]\right] - \frac{n(x)}{\tau_T^2}$$

$\tau_T = n_e \sigma_T R$

Therefore, in the limit $x = \frac{\varepsilon}{kT_e} << 1$ we can approximate

$$\frac{dn(x)}{dx} \approx \frac{n(x)}{x} >> n \qquad x << 1$$

Hence

$$\frac{1}{n_e \sigma_T c} \frac{dn(x)}{dt} = \frac{kT_e}{m_e c^2} \frac{1}{x^2} \frac{d}{dx}\left[x^4 \frac{\partial n}{\partial x}\right] - \frac{n(x)}{\tau_T^2}$$

In the steady state

$$\Rightarrow \frac{dn(x)}{dt} \to 0$$

We mentioned earlier that $n(x)$ is the phase volume density of the photons. We can determine the phase-space density of the photons in the following way:

Density function in phase space!!

$$N = \int f(\bar{p},\bar{r})\, d^3r\, d^3p$$

$$n = \frac{N}{V} = \int f(\bar{p})\, d^3p$$

$$\Rightarrow dn = f(\bar{p})\, d^3p$$

For photons : $p = \frac{h\nu}{c}$

$$dn = 4\pi p^2 f(\bar{p})\, dp$$

$$= 4\pi \left(\frac{h\nu}{c}\right)^2 f(\bar{p})\, d\left(\frac{h\nu}{c}\right)$$

$$\approx 4\pi \left(\frac{h\nu}{c}\right)^3 f(\bar{p})$$

Then

$$\varepsilon_{ph}\, dn = u_{ph}$$

$$= \left(\frac{u_{ph}}{4\pi}\right) 4\pi$$

$$= u(\Omega)\, 4\pi$$

$$= 4\pi \frac{u(\Omega)}{h\nu} h\nu$$

$$= \frac{4\pi}{h} \frac{u(\Omega)}{\nu} h\nu$$

$$= \frac{4\pi}{h} u_\nu(\Omega)\, h\nu \qquad \left[u_\nu(\Omega) = \frac{I_\nu}{c}\right]$$

$$= \frac{4\pi}{h} \frac{I_\nu}{c} h\nu$$

$$\Rightarrow h\nu\, dn = \frac{4\pi}{h} \frac{I_\nu}{c} h\nu$$

$$dn = \frac{4\pi}{h} \frac{I_\nu}{c}$$

Therefore

$$dn = 4\pi \left(\frac{h\nu}{c}\right)^3 f(\hat{p}) \approx \frac{4\pi}{h} \quad \frac{I_\nu}{c}$$

$$f(\hat{p}) = \frac{1}{hc} \frac{I_\nu}{\left(\frac{h\nu}{c}\right)^3}$$

$$= \frac{c^2}{h} \quad \frac{I_\nu}{(h\nu)^3}$$

$$f\left(\frac{h\nu}{c}\right) \propto c^2 \frac{(I_\nu/h)}{(h\nu)^3}$$

$$\propto c^2 \frac{I_{h\nu}}{(h\nu)^3} \qquad (I_\nu/h) = I_{h\nu} = I(x)$$

$$f(x) \propto \frac{I(x)}{x^3}$$

$$n(x) = f(x) \propto \frac{I(x)}{x^3}$$

Therefore if the emergent spectrum is a power-law:

$$I(x) \propto x^{-\alpha}$$

$$n(x) \propto \frac{I(x)}{x^3}$$

$$\propto \frac{x^{-\alpha}}{x^3}$$

$$\propto x^{-(\alpha+3)}$$

Then the spectral index can be determined in the following way, using the Kompaneets eqn.

In the steady state

$$\frac{1}{x^2}\frac{d}{dx}\left[x^4 \frac{dn(x)}{dx}\right] - \frac{m_e c^2}{kT_e}\frac{n(x)}{\tau_T^2} = 0$$

$$\frac{1}{x^2}\frac{d}{dx}\left[x^4 \frac{d}{dx} x^{-(\alpha+3)}\right] - \frac{m_e c^2}{kT_e}\frac{x^{-(\alpha+3)}}{\tau_T^2} = 0$$

$$\frac{1}{x^2}\frac{d}{dx}\left[x^4 (-(\alpha+3))\, x^{-(\alpha+3)-1}\right] - \frac{m_e c^2}{kT_e}\frac{x^{-(\alpha+3)}}{\tau_T^2} = 0$$

$$\frac{-(\alpha+3)}{x^2}\frac{d}{dx}\left(x^{-\alpha}\right] - \frac{m_e c^2}{kT_e}\frac{x^{-(\alpha+3)}}{\tau_T^2} = 0$$

$$\frac{-(\alpha+3)}{x^2}\frac{d}{dx}\left(x^{-\alpha}\right) - \frac{x^{-(\alpha+3)}}{\left(\frac{kT_e}{m_e c^2}\right)\tau_T^2} = 0$$

$$\frac{-(\alpha+3)(-\alpha x^{-\alpha-1})}{x^2} - \frac{x^{-(\alpha+3)}}{\left(\frac{kT_e}{m_e c^2}\right)\tau_T^2} = 0$$

$$\alpha(\alpha+3)\, x^{-(\alpha+3)} - \frac{x^{-(\alpha+3)}}{\left(\frac{kT_e}{m_e c^2}\right)\tau_T^2} = 0$$

$$x^{-(\alpha+3)}\left[\alpha(\alpha+3) - \frac{1}{\left(\frac{kT_e}{m_e c^2}\right)\tau_T^2}\right] = 0$$

Remember earlier we defined

$4\Theta \Rightarrow$ energy transfer in a single scatter in non-rel. case.

$$y = 4\Theta N$$

$$= \left(\frac{4 kT_e}{m_e c^2}\right)\tau_T^2$$

$N = \tau_T^2 \qquad \tau_T > 1$

$$x^{-(\alpha+3)}\left[\alpha(\alpha+3) - \frac{4}{\left(\frac{4kT}{m_e c^2}\right)\tau_T^2}\right] = 0$$

$$x^{-(\alpha+3)}\left\{\alpha(\alpha+3) - \frac{4}{y}\right] = 0$$

$$\Rightarrow \quad \alpha(\alpha+3) - \frac{4}{y} = 0$$

$$\Rightarrow \quad \alpha^2 + 3\alpha - \frac{4}{y} = 0$$

This is a quadratic eqn of type

$$ax^2 + bx + c = 0$$

$$\alpha^2 + 3y - \frac{4}{y} = 0$$

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{(3)^2 - 4(1)(-\frac{4}{y})}}{2}$$

$$= -\frac{3}{2} \pm \frac{\sqrt{9 + \frac{16}{y}}}{2}$$

$$= -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}$$

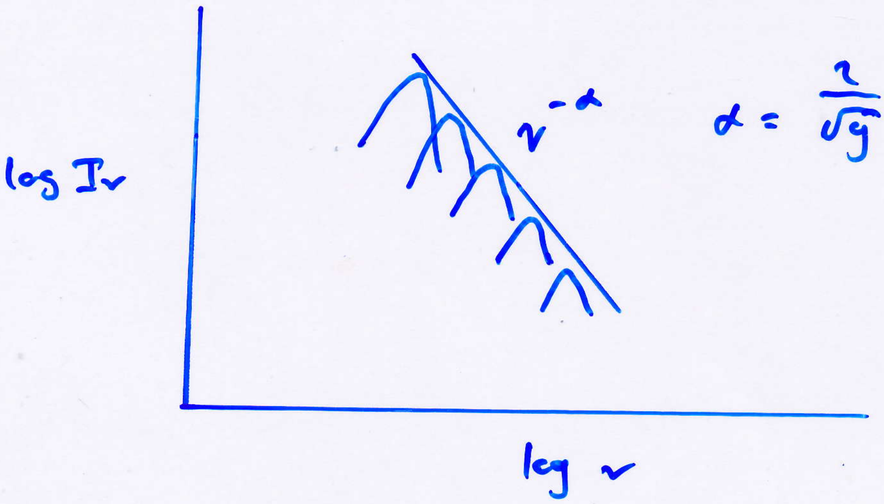
There are three cases:

a.) <u>Very unsaturated $(y \ll 1)$</u>

$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{4}{y}}$$

$$= -\frac{3}{2} \pm \frac{2}{\sqrt{y}}$$

$$\approx \frac{2}{\sqrt{y}} \quad (\gg 1) \quad \text{Steep spectrum}$$



$$\alpha = \frac{2}{\sqrt{y}} \gg 1$$

b.) Unsaturated : $y \sim 1$

$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}$$

$$\simeq -\frac{3}{2} \pm \sqrt{\frac{9}{4} + 4} \qquad y \sim 1$$

$$\simeq -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{16}{4}}$$

$$\approx -\frac{3}{2} \pm \sqrt{\frac{25}{4}}$$

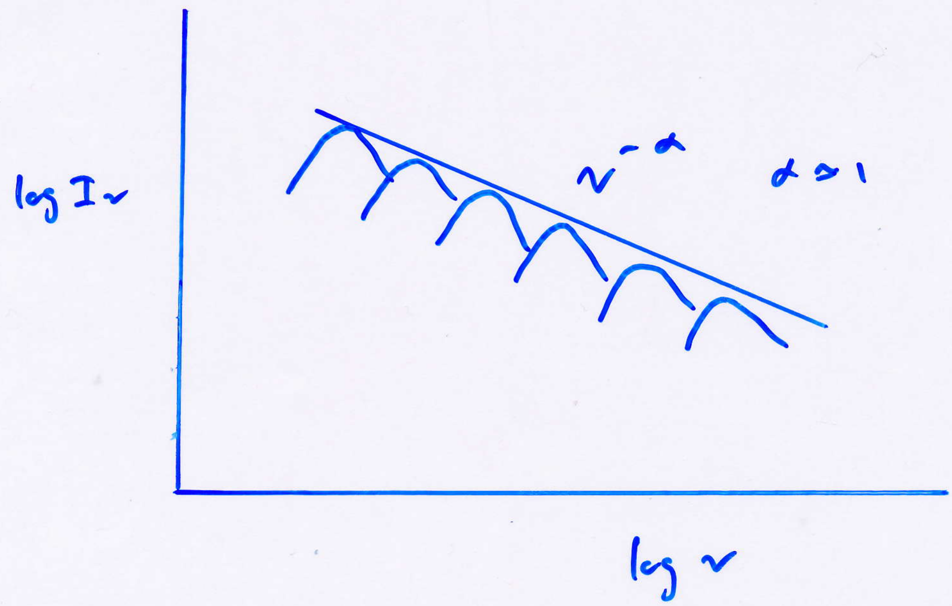
$$\approx -\frac{3}{2} \pm \frac{5}{2}$$

$$\approx -\frac{3}{2} + \frac{5}{2} \qquad \text{(pos. solution)}$$

$$\approx \frac{2}{2}$$

$$\approx 1$$

$y \approx 1$



c.) Saturated $y \gg 1$:

$$\alpha = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{4}{y}}$$

$$= -\frac{3}{2} \pm \sqrt{\frac{9}{4}}$$

$$= -\frac{3}{2} \pm \frac{3}{2}$$

$\alpha_1 = 0$

$\alpha_2 = -3$

This is represented graphically as follows: